Elements of Power Electronics
PART I: Bases

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Goal and expectations

The goal of the course is to provide a toolbox that allows you to:

▶ understand power electronics concepts and topologies,
▶ to model a switching converter,
▶ to build it (including its magnetic components) and,
▶ to control it (with digital control).

Power Electronics is a huge area and the correct approach is to focus on **understanding** concepts.
The course is divided in three parts:

- PART I: Bases
- PART II: Topologies and applications
- PART III: Digital control

In PART I and PART II, chapters are numbered according to the reference book [?].
In PART III, chapters are numbered according to the reference book [?].
PART I: Bases

▶ Chapter 1: Introduction
▶ Chapter 2: Principles of Steady-State Converter Analysis
▶ Chapter 3: Steady-State Equivalent Circuit Modeling, Losses, and Efficiency
▶ Chapter 13: Basic Magnetics Theory
▶ Chapter 4: Switch Realization
▶ Chapter 5: The Discontinuous Conduction Mode
Chapter 1: Introduction
Fundamentals of Power Electronics
Second edition

Robert W. Erickson
Dragan Maksimovic
University of Colorado, Boulder
1.1 Introduction to Power Processing

- **Dc-dc conversion**: Change and control voltage magnitude
- **Ac-dc rectification**: Possibly control dc voltage, ac current
- **Dc-ac inversion**: Produce sinusoid of controllable magnitude and frequency
- **Ac-ac cycloconversion**: Change and control voltage magnitude and frequency
Control is invariably required

Switching converter

Power input

Control input

Controller

Reference

Feedback

Feedforward

Power output
High efficiency is essential

\[ \eta = \frac{P_{\text{out}}}{P_{\text{in}}} \]

\[ P_{\text{loss}} = P_{\text{in}} - P_{\text{out}} = P_{\text{out}} \left( \frac{1}{\eta} - 1 \right) \]

High efficiency leads to low power loss within converter.
Small size and reliable operation is then feasible.
Efficiency is a good measure of converter performance.
A high-efficiency converter

A goal of current converter technology is to construct converters of small size and weight, which process substantial power at high efficiency.
Devices available to the circuit designer

<table>
<thead>
<tr>
<th>Resistors</th>
<th>Capacitors</th>
<th>Magnetics</th>
<th>Linear-mode</th>
<th>Switched-mode</th>
</tr>
</thead>
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DTs Ts

+ –

Semiconductor devices
### Devices available to the circuit designer

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</table>

**Signal processing: avoid magnetics**
Devices available to the circuit designer

Power processing: avoid lossy elements
Power loss in an ideal switch

Switch closed: \( v(t) = 0 \)

Switch open: \( i(t) = 0 \)

In either event: \( p(t) = v(t) \, i(t) = 0 \)

Ideal switch consumes zero power
A simple dc-dc converter example

Input source: 100V
Output load: 50V, 10A, 500W
How can this converter be realized?
Dissipative realization

Resistive voltage divider

\[ V_g = 100\text{V} \]

\[ P_{\text{in}} = 1000\text{W} \]

\[ V = 50\text{V} \]

\[ P_{\text{out}} = 500\text{W} \]

\[ R = 5\Omega \]

\[ P_{\text{loss}} = 500\text{W} \]
Dissipative realization

Series pass regulator: transistor operates in active region

\[ \begin{align*}
V_g & = 100V \\
P_{\text{in}} & \approx 1000W \\
P_{\text{loss}} & \approx 500W \\
P_{\text{out}} & = 500W \\
P_{\text{in}} & \approx 1000W \\
R & = 5\Omega \\
V & = 50V \\
I & = 10A \\
\end{align*} \]
Use of a SPDT switch

\[ v_s(t) = DV_g \]

- **Switch position:**
  1. Position 1: 0 ≤ t ≤ DT_s
  2. Position 2: DT_s ≤ t ≤ (1 – D) T_s

- **Waveforms:**
  - \( v_g(t) \) is the input voltage
  - \( v_s(t) \) is the output voltage
  - \( I \) is the output current
  - \( R \) is the load resistance

\[ I = 10 \text{ A} \]
\[ R = 50 \text{ V} \]
The switch changes the dc voltage level

\[ D = \text{switch duty cycle} \quad 0 \leq D \leq 1 \]

\[ T_s = \text{switching period} \]

\[ f_s = \text{switching frequency} = \frac{1}{T_s} \]

DC component of \( v_s(t) \) = average value:

\[ V_s = \frac{1}{T_s} \int_0^{T_s} v_s(t) \, dt = DV_g \]
Addition of low pass filter

Addition of (ideally lossless) \( L-C \) low-pass filter, for removal of switching harmonics:

Choose filter cutoff frequency \( f_0 \) much smaller than switching frequency \( f_s \)

This circuit is known as the “buck converter”
Addition of control system for regulation of output voltage

\[\delta(t)\]

\[T_{sd}T_s t\]

Power input

Switching converter

Load

\[v_g\]

\[\oplus\]

\[-\]

\[\delta(t)\]

Transistor gate driver

Pulse-width modulator

\[v_c\]

\[G_c(s)\]

Compensator

\[v_e\]

Error signal

Sensor gain

Reference input

\[v_{ref}\]

\[H(s)\]

\[i\]

\[v\]

Load

\[Hv\]
The boost converter

\[ V_g \]

\[ V \]

\[ D \]

\[ 0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \]

\[ 0 \quad 2V_g \quad 3V_g \quad 4V_g \quad 5V_g \]

\[ V_g \quad V \]

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Chapter 1: Introduction
A single-phase inverter

Modulate switch duty cycles to obtain sinusoidal low-frequency component

“H-bridge”
1.2 Several applications of power electronics

Power levels encountered in high-efficiency converters

- less than 1 W in battery-operated portable equipment
- tens, hundreds, or thousands of watts in power supplies for computers or office equipment
- kW to MW in variable-speed motor drives
- 1000 MW in rectifiers and inverters for utility dc transmission lines
Chapter 2
Principles of Steady-State Converter Analysis

2.1. Introduction

2.2. Inductor volt-second balance, capacitor charge balance, and the small ripple approximation

2.3. Boost converter example

2.4. Cuk converter example

2.5. Estimating the ripple in converters containing two-pole low-pass filters

2.6. Summary of key points
2.1 Introduction
Buck converter

SPDT switch changes dc component

Switch output voltage waveform

Duty cycle \( D \):
\[ 0 \leq D \leq 1 \]

complement \( D' \):
\[ D' = 1 - D \]
**Dc component of switch output voltage**

\[ v_s(t) \]

\[ V_g \]

\[ \text{area} = DT_s V_g \]

\[ \langle v_s \rangle = DV_g \]

**Fourier analysis:** Dc component = average value

\[ \langle v_s \rangle = \frac{1}{T_s} \int_0^{T_s} v_s(t) \, dt \]

\[ \langle v_s \rangle = \frac{1}{T_s} (DT_s V_g) = DV_g \]
Insertion of low-pass filter to remove switching harmonics and pass only dc component

\[ v \approx \langle v_s \rangle = DV_g \]

\[ v(t) \]

\[ V(t) \]

Fundamentals of Power Electronics

Chapter 2: Principles of steady-state converter analysis
Three basic dc-dc converters

**Buck**

![Buck Converter Diagram](image)

\[
M(D) = D
\]

**Boost**

![Boost Converter Diagram](image)

\[
M(D) = \frac{1}{1-D}
\]

**Buck-boost**

![Buck-boost Converter Diagram](image)

\[
M(D) = \frac{D}{1-D}
\]
Objectives of this chapter

- Develop techniques for easily determining output voltage of an arbitrary converter circuit
- Derive the principles of *inductor volt-second balance* and *capacitor charge (amp-second) balance*
- Introduce the key *small ripple approximation*
- Develop simple methods for selecting filter element values
- Illustrate via examples
2.2. Inductor volt-second balance, capacitor charge balance, and the small ripple approximation

Actual output voltage waveform, buck converter

**Buck converter containing practical low-pass filter**

**Actual output voltage waveform**

\[ v(t) = V + v_{\text{ripple}}(t) \]

**Actual waveform**

\[ v(t) = V + v_{\text{ripple}}(t) \]

**dc component V**
The small ripple approximation

In a well-designed converter, the output voltage ripple is small. Hence, the waveforms can be easily determined by ignoring the ripple:

\[ v(t) = V + v_{\text{ripple}}(t) \]

\[ v(t) \approx V \]

Actual waveform: \[ v(t) = V + v_{\text{ripple}}(t) \]

\[ dc \text{ component } V \]

\[ v_{\text{ripple}} \ll V \]
Buck converter analysis: inductor current waveform

\[ i_L(t) = \begin{cases} +V_g & \text{switch in position 1} \\ -V_g & \text{switch in position 2} \end{cases} \]
Inductor voltage and current
Subinterval 1: switch in position 1

Inductor voltage

\[ v_L = V_g - v(t) \]

Small ripple approximation:

\[ v_L \approx V_g - V \]

Knowing the inductor voltage, we can now find the inductor current via

\[ v_L(t) = L \frac{di_L(t)}{dt} \]

Solve for the slope:

\[ \frac{di_L(t)}{dt} = \frac{v_L(t)}{L} \approx \frac{V_g - V}{L} \]

⇒ The inductor current changes with an essentially constant slope
Inductor voltage and current
Subinterval 2: switch in position 2

Inductor voltage
\[ v_L(t) = -v(t) \]

Small ripple approximation:
\[ v_L(t) \approx -V \]

Knowing the inductor voltage, we can again find the inductor current via
\[ v_L(t) = L \frac{di_L(t)}{dt} \]

Solve for the slope:
\[ \frac{di_L(t)}{dt} \approx -\frac{V}{L} \]

⇒ The inductor current changes with an essentially constant slope
Inductor voltage and current waveforms

\[ v_L(t) = L \frac{di_L(t)}{dt} \]

Switch position:

\[ \begin{array}{c}
0 & 1 & DT_s & 2 & DT_s & 1 \\
V_g - V & -V & V_L & V_L & 0 & 0
\end{array} \]

\[ i_L(0) \quad i_L(DT_s) \quad i_L(T_s) \]

\[ I \quad \frac{V_g - V}{L} \quad -\frac{V}{L} \]

\[ \Delta i_L \]
Determination of inductor current ripple magnitude

\[ \Delta i_L = \frac{V_g - V}{2L} DT_s \]

\[ L = \frac{V_g - V}{2\Delta i_L} DT_s \]
Inductor current waveform during turn-on transient

When the converter operates in equilibrium:

\[ i_L((n + 1)T_s) = i_L(nT_s) \]
The principle of inductor volt-second balance: Derivation

Inductor defining relation:

\[ v_L(t) = L \frac{di_L(t)}{dt} \]

Integrate over one complete switching period:

\[ i_L(T_s) - i_L(0) = \frac{1}{L} \int_0^{T_s} v_L(t) \, dt \]

In periodic steady state, the net change in inductor current is zero:

\[ 0 = \int_0^{T_s} v_L(t) \, dt \]

Hence, the total area (or volt-seconds) under the inductor voltage waveform is zero whenever the converter operates in steady state. An equivalent form:

\[ 0 = \frac{1}{T_s} \int_0^{T_s} v_L(t) \, dt = \langle v_L \rangle \]

The average inductor voltage is zero in steady state.
Inductor volt-second balance: Buck converter example

Inductor voltage waveform, previously derived:

Integral of voltage waveform is area of rectangles:

\[ \lambda = \int_0^{T_s} v_L(t) \, dt = (V_g - V)(DT_s) + (-V)(D'T_s) \]

Average voltage is

\[ \left\langle v_L \right\rangle = \frac{\lambda}{T_s} = D(V_g - V) + D'(-V) \]

Equate to zero and solve for \( V \):

\[ 0 = DV_g - (D + D')V = DV_g - V \implies V = DV_g \]
The principle of capacitor charge balance: Derivation

Capacitor defining relation:

\[ i_C(t) = C \frac{dv_C(t)}{dt} \]

Integrate over one complete switching period:

\[ v_C(T_s) - v_C(0) = \frac{1}{C} \int_0^{T_s} i_C(t) \, dt \]

In periodic steady state, the net change in capacitor voltage is zero:

\[ 0 = \frac{1}{T_s} \int_0^{T_s} i_C(t) \, dt = \langle i_C \rangle \]

Hence, the total area (or charge) under the capacitor current waveform is zero whenever the converter operates in steady state. The average capacitor current is then zero.
2.3 Boost converter example

Boost converter with ideal switch

Realization using power MOSFET and diode
Boost converter analysis

original converter

switch in position 1

switch in position 2

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Chapter 2: Principles of steady-state converter analysis

ELEC0055: Elements of Power Electronics - Fall 2018
Subinterval 1: switch in position 1

**Inductor voltage and capacitor current**

\[ v_L = V_g \]
\[ i_C = -\frac{v}{R} \]

**Small ripple approximation:**

\[ v_L = V_g \]
\[ i_C = -\frac{V}{R} \]
Subinterval 2: switch in position 2

Inductor voltage and capacitor current

\[ v_L = V_g - v \]
\[ i_C = i_L - v / R \]

Small ripple approximation:

\[ v_L = V_g - V \]
\[ i_C = I - V / R \]
Inductor voltage and capacitor current waveforms

\[ v_L(t) = V_g - V \]

\[ i_C(t) = I - V/R \]

\[ DT_s \quad D'T_s \]

\[ V_g - V \]

\[ -V/R \]
Inductor volt-second balance

Net volt-seconds applied to inductor over one switching period:

\[ \int_0^{T_s} v_L(t) \, dt = (V_g) \, DT_s + (V_g - V) \, D'T_s \]

Equate to zero and collect terms:

\[ V_g (D + D') - V \, D' = 0 \]

Solve for \( V \):

\[ V = \frac{V_g}{D'} \]

The voltage conversion ratio is therefore

\[ M(D) = \frac{V}{V_g} = \frac{1}{D'} = \frac{1}{1 - D} \]
Conversion ratio $M(D)$ of the boost converter

$$M(D) = \frac{1}{D'} = \frac{1}{1 - D}$$

![Diagram showing conversion ratio $M(D)$ of the boost converter.](image-url)
Determination of inductor current dc component

Capacitor charge balance:

\[ \int_{0}^{T_s} i_C(t) \, dt = (-\frac{V}{R}) \, DT_s + (I - \frac{V}{R}) \, D'T_s \]

Collect terms and equate to zero:

\[-\frac{V}{R} (D + D') + I \, D' = 0\]

Solve for \( I \):

\[ I = \frac{V}{D' \, R} \]

Eliminate \( V \) to express in terms of \( V_g \):

\[ I = \frac{V_g}{D'^2 \, R} \]
Determination of inductor current ripple

Inductor current slope during subinterval 1:
\[
\frac{di_L(t)}{dt} = \frac{v_L(t)}{L} = \frac{V_g}{L}
\]

Inductor current slope during subinterval 2:
\[
\frac{di_L(t)}{dt} = \frac{v_L(t)}{L} = \frac{V_g - V}{L}
\]

Change in inductor current during subinterval 1 is *(slope) (length of subinterval)*:
\[
2\Delta i_L = \frac{V_g}{L} DT_s
\]

Solve for peak ripple:
\[
\Delta i_L = \frac{V_g}{2L} DT_s
\]

- Choose \( L \) such that desired ripple magnitude is obtained

*Fundamentals of Power Electronics*
Determination of capacitor voltage ripple

Capacitor voltage slope during subinterval 1:
\[ \frac{dv_c(t)}{dt} = \frac{i_c(t)}{C} = -\frac{V}{RC} \]

Capacitor voltage slope during subinterval 2:
\[ \frac{dv_c(t)}{dt} = \frac{i_c(t)}{C} = \frac{I}{C} - \frac{V}{RC} \]

Change in capacitor voltage during subinterval 1 is (slope) (length of subinterval):
\[ -2\Delta v = -\frac{V}{RC} DT_s \]

Solve for peak ripple:
\[ \Delta v = \frac{V}{2RC} DT_s \]

- Choose \( C \) such that desired voltage ripple magnitude is obtained
- In practice, capacitor equivalent series resistance (esr) leads to increased voltage ripple

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2.4 Cuk converter example

Cuk converter, with ideal switch

Cuk converter: practical realization using MOSFET and diode
Cuk converter circuit
with switch in positions 1 and 2

Switch in position 1:
MOSFET conducts
Capacitor $C_1$ releases
energy to output

Switch in position 2:
diode conducts
Capacitor $C_1$ is
charged from input
Waveforms during subinterval 1
MOSFET conduction interval

Inductor voltages and capacitor currents:

\[ v_{L1} = V_g \]
\[ v_{L2} = -v_1 - v_2 \]
\[ i_{C1} = i_2 \]
\[ i_{C2} = i_2 - \frac{v_2}{R} \]

Small ripple approximation for subinterval 1:

\[ v_{L1} = V_g \]
\[ v_{L2} = -V_1 - V_2 \]
\[ i_{C1} = I_2 \]
\[ i_{C2} = I_2 - \frac{V_2}{R} \]
Waveforms during subinterval 2
Diode conduction interval

Inductor voltages and capacitor currents:

\[ v_{L1} = V_g - v_1 \]
\[ v_{L2} = -v_2 \]
\[ i_{C1} = i_1 \]
\[ i_{C2} = i_2 - \frac{v_2}{R} \]

Small ripple approximation for subinterval 2:

\[ v_{L1} = V_g - V_1 \]
\[ v_{L2} = -V_2 \]
\[ i_{C1} = I_1 \]
\[ i_{C2} = I_2 - \frac{V_2}{R} \]
Equate average values to zero

The principles of inductor volt-second and capacitor charge balance state that the average values of the periodic inductor voltage and capacitor current waveforms are zero, when the converter operates in steady state. Hence, to determine the steady-state conditions in the converter, let us sketch the inductor voltage and capacitor current waveforms, and equate their average values to zero.

**Waveforms:**

Inductor voltage $v_{L1}(t)$

![Inductor voltage waveform](image)

Volt-second balance on $L_1$:

$$\langle v_{L1} \rangle = DV_g + D'(V_g - V_1) = 0$$
Equate average values to zero

**Inductor $L_2$ voltage**

$$v_{L2}(t) = -V_2$$

Average the waveforms:

$$\langle v_{L2} \rangle = D(-V_1 - V_2)$$

**Capacitor $C_1$ current**

$$i_{C1}(t) = I_1$$

Average the waveforms:

$$\langle i_{C1} \rangle = DI_2 + D'I_1 = 0$$
Equate average values to zero

Capacitor current $i_{C2}(t)$ waveform

Note: during both subintervals, the capacitor current $i_{C2}$ is equal to the difference between the inductor current $i_2$ and the load current $V_2/R$. When ripple is neglected, $i_{C2}$ is constant and equal to zero.

$$i_{C2} = I_2 - \frac{V_2}{R}$$
Cuk converter conversion ratio $M = \frac{V}{V_g}$

$$M(D) = \frac{V_2}{V_g} = -\frac{D}{1-D}$$
Inductor current waveforms

Interval 1 slopes, using small ripple approximation:
\[
\frac{di_1(t)}{dt} = \frac{v_{L1}(t)}{L_1} = \frac{V_g}{L_1}
\]
\[
\frac{di_2(t)}{dt} = \frac{v_{L2}(t)}{L_2} = -\frac{V_1 - V_2}{L_2}
\]

Interval 2 slopes:
\[
\frac{di_1(t)}{dt} = \frac{v_{L1}(t)}{L_1} = \frac{V_g - V_1}{L_1}
\]
\[
\frac{di_2(t)}{dt} = \frac{v_{L2}(t)}{L_2} = -\frac{V_2}{L_2}
\]
Capacitor $C_1$ waveform

Subinterval 1:
\[
\frac{dv_1(t)}{dt} = \frac{i_{C1}(t)}{C_1} = \frac{I_2}{C_1}
\]

Subinterval 2:
\[
\frac{dv_1(t)}{dt} = \frac{i_{C1}(t)}{C_1} = \frac{I_1}{C_1}
\]
Ripple magnitudes

Analysis results

\[
\begin{align*}
\Delta i_1 &= \frac{V_g D T_s}{2L_1} \\
\Delta i_2 &= \frac{V_1 + V_2}{2L_2} DT_s \\
\Delta v_1 &= -\frac{I_2 D T_s}{2C_1}
\end{align*}
\]

Use dc converter solution to simplify:

\[
\begin{align*}
\Delta i_1 &= \frac{V_g D T_s}{2L_1} \\
\Delta i_2 &= \frac{V_g D T_s}{2L_2} \\
\Delta v_1 &= \frac{V_g D^2 T_s}{2D'RC_1}
\end{align*}
\]

Q: How large is the output voltage ripple?
2.5 Estimating ripple in converters containing two-pole low-pass filters

Buck converter example: Determine output voltage ripple

Inductor current waveform.

What is the capacitor current?
Capacitor current and voltage, buck example

**Must not neglect inductor current ripple!**

*If the capacitor voltage ripple is small, then essentially all of the ac component of inductor current flows through the capacitor.*
Estimating capacitor voltage ripple $\Delta v$

Current $i_C(t)$ is positive for half of the switching period. This positive current causes the capacitor voltage $v_C(t)$ to increase between its minimum and maximum extrema. During this time, the total charge $q$ is deposited on the capacitor plates, where

$$q = C (2\Delta v)$$

$$\text{(change in charge)} = C \text{(change in voltage)}$$
Estimating capacitor voltage ripple $\Delta v$

The total charge $q$ is the area of the triangle, as shown:

$$q = \frac{1}{2} \Delta i_L \frac{T_s}{2}$$

Eliminate $q$ and solve for $\Delta v$:

$$\Delta v = \frac{\Delta i_L T_s}{8C}$$

Note: in practice, capacitor equivalent series resistance (esr) further increases $\Delta v$. 
Inductor current ripple in two-pole filters

Example: problem 2.9

\[ \lambda = L (2\Delta i) \]

\[ \lambda = \text{inductor flux linkages} \]

\[ \lambda = \text{inductor volt-seconds} \]
2.6 Summary of Key Points

1. The dc component of a converter waveform is given by its average value, or the integral over one switching period, divided by the switching period. Solution of a dc-dc converter to find its dc, or steady-state, voltages and currents therefore involves averaging the waveforms.

2. The linear ripple approximation greatly simplifies the analysis. In a well-designed converter, the switching ripples in the inductor currents and capacitor voltages are small compared to the respective dc components, and can be neglected.

3. The principle of inductor volt-second balance allows determination of the dc voltage components in any switching converter. In steady-state, the average voltage applied to an inductor must be zero.
Summary of Chapter 2

4. The principle of capacitor charge balance allows determination of the dc components of the inductor currents in a switching converter. In steady-state, the average current applied to a capacitor must be zero.

5. By knowledge of the slopes of the inductor current and capacitor voltage waveforms, the ac switching ripple magnitudes may be computed. Inductance and capacitance values can then be chosen to obtain desired ripple magnitudes.

6. In converters containing multiple-pole filters, continuous (nonpulsating) voltages and currents are applied to one or more of the inductors or capacitors. Computation of the ac switching ripple in these elements can be done using capacitor charge and/or inductor flux-linkage arguments, without use of the small-ripple approximation.

7. Converters capable of increasing (boost), decreasing (buck), and inverting the voltage polarity (buck-boost and Cuk) have been described. Converter circuits are explored more fully in a later chapter.
Chapter 3: Steady-State Equivalent Circuit Modeling, Losses, and Efficiency
Chapter 3. Steady-State Equivalent Circuit Modeling, Losses, and Efficiency

3.1. The dc transformer model
3.2. Inclusion of inductor copper loss
3.3. Construction of equivalent circuit model
3.4. How to obtain the input port of the model
3.5. Example: inclusion of semiconductor conduction losses in the boost converter model
3.6. Summary of key points
3.1. The dc transformer model

Basic equations of an ideal dc-dc converter:

\[ P_{in} = P_{out} \]  \hspace{1cm} (\eta = 100\%)

\[ V_g I_g = V I \]

\[ V = M(D) V_g \]  \hspace{1cm} (ideal conversion ratio)

\[ I_g = M(D) I \]

These equations are valid in steady-state. During transients, energy storage within filter elements may cause \( P_{in} \neq P_{out} \).
Equivalent circuits corresponding to ideal dc-dc converter equations

\[ P_{in} = P_{out} \quad V_g I_g = V I \quad V = M(D) V_g \quad I_g = M(D) I \]

**Dependent sources**

- Power input: \( V_g \) \( I_g \)
- Power output: \( M(D)I \) \( M(D)V_g \)

**DC transformer**

- Power input: \( V_g \)
- Power output: \( V \)
- Control input: \( D \)
The DC transformer model

Models basic properties of ideal dc-dc converter:

- conversion of dc voltages and currents, ideally with 100% efficiency
- conversion ratio $M$ controllable via duty cycle

- Solid line denotes ideal transformer model, capable of passing dc voltages and currents
- Time-invariant model (no switching) which can be solved to find dc components of converter waveforms
Example: use of the DC transformer model

1. Original system

2. Insert dc transformer model

3. Push source through transformer

4. Solve circuit

\[ V = M(D) \frac{V_1}{R + M^2(D) R_1} \]
3.2. Inclusion of inductor copper loss

Dc transformer model can be extended, to include converter nonidealities.

Example: inductor copper loss (resistance of winding):

\[ L \quad R_L \]

Insert this inductor model into boost converter circuit:
Analysis of nonideal boost converter

switch in position 1

switch in position 2
Circuit equations, switch in position 1

Inductor current and capacitor voltage:

\[
\begin{align*}
    v_L(t) &= V_g - i(t) R_L \\
i_C(t) &= -v(t) / R 
\end{align*}
\]

Small ripple approximation:

\[
\begin{align*}
    v_L(t) &= V_g - I R_L \\
i_C(t) &= -V / R 
\end{align*}
\]
Circuit equations, switch in position 2

\[ v_L(t) = V_g - i(t) R_L - v(t) \approx V_g - I R_L - V \]
\[ i_C(t) = i(t) - v(t) / R \approx I - V / R \]
Inductor voltage and capacitor current waveforms

Average inductor voltage:

\[
\langle v_L(t) \rangle = \frac{1}{T_s} \int_0^{T_s} v_L(t) dt
\]

\[
= D(V_g - I R_L) + D'(V_g - I R_L - V)
\]

Inductor volt-second balance:

\[
0 = V_g - I R_L - D'V
\]

Average capacitor current:

\[
\langle i_C(t) \rangle = D (-V / R) + D' (I - V / R)
\]

Capacitor charge balance:

\[
0 = D'I - V / R
\]
Solution for output voltage

We now have two equations and two unknowns:

\[ 0 = V_g - IR_L - D'V \]
\[ 0 = D'I - V / R \]

Eliminate \( I \) and solve for \( V \):

\[ \frac{V}{V_g} = \frac{1}{D'} \frac{1}{1 + R_L / D'^2 R} \]
3.3. Construction of equivalent circuit model

Results of previous section (derived via inductor volt-sec balance and capacitor charge balance):

\[
\begin{align*}
\langle v_L \rangle &= 0 = V_g - I R_L - D'V \\
\langle i_C \rangle &= 0 = D'I - V / R
\end{align*}
\]

View these as loop and node equations of the equivalent circuit. Reconstruct an equivalent circuit satisfying these equations.
Inductor voltage equation

\[ \langle v_L \rangle = 0 = V_g - I R_L - D'V \]

- Derived via Kirchhoff’s voltage law, to find the inductor voltage during each subinterval
- Average inductor voltage then set to zero
- This is a loop equation: the dc components of voltage around a loop containing the inductor sum to zero
- \( IR_L \) term: voltage across resistor of value \( R_L \) having current \( I \)
- \( D'V \) term: for now, leave as dependent source
Capacitor current equation

\[ \langle i_c \rangle = 0 = D'I - V / R \]

- Derived via Kirchoff’s current law, to find the capacitor current during each subinterval
- Average capacitor current then set to zero
- This is a node equation: the dc components of current flowing into a node connected to the capacitor sum to zero

- **V/R** term: current through load resistor of value \( R \) having voltage \( V \)
- **D’I** term: for now, leave as dependent source
Complete equivalent circuit

The two circuits, drawn together:

The dependent sources are equivalent to a $D':1$ transformer:

- sources have same coefficient
- reciprocal voltage/current dependence
Solution of equivalent circuit

Converter equivalent circuit

Refer all elements to transformer secondary:

Solution for output voltage using voltage divider formula:

\[ V = \frac{V_g}{D'} \frac{R}{R + \frac{R_L}{D'^2}} = \frac{V_g}{D'} \frac{1}{1 + \frac{R_L}{D'^2 R}} \]
Solution for input (inductor) current

\[ I = \frac{V_g}{D' R + R_L} = \frac{V_g}{D'^2} \frac{1}{1 + \frac{R_L}{D'^2 R}} \]
Equation correction

\[ I = \frac{V_g}{D^{1/2}R + R_L} = \frac{V_g}{D^{1/2}R} \frac{1}{1 + \frac{R_L}{D^{1/2}R}} \]  \hspace{1cm} (1)
Solution for converter efficiency

\[ P_{in} = (V_g) (I) \]

\[ P_{out} = (V) (D'I) \]

\[ \eta = \frac{P_{out}}{P_{in}} = \frac{(V) (D'I)}{(V_g) (I)} = \frac{V}{V_g} D' \]

\[ \eta = \frac{1}{1 + \frac{R_L}{D'^2 R}} \]
Efficiency, for various values of $R_L$

$$\eta = \frac{1}{1 + \frac{R_L}{D^2 R}}$$

$R_L/R = 0.1$

$\eta$ vs. $D$
3.4. How to obtain the input port of the model

Buck converter example — use procedure of previous section to derive equivalent circuit

Average inductor voltage and capacitor current:

\[
\langle v_L \rangle = 0 = DV_g - I_L R_L - V_C \\
\langle i_C \rangle = 0 = I_L - V_C / R
\]
Construct equivalent circuit as usual

\[
\langle v_L \rangle = 0 = D V_g - I_L R_L - V_C \\
\langle i_C \rangle = 0 = I_L - V_C / R
\]

What happened to the transformer?
• Need another equation
Modeling the converter input port

Input current waveform $i_g(t)$:

![Input current waveform diagram]

Dc component (average value) of $i_g(t)$ is

$$I_g = \frac{1}{T_s} \int_0^{T_s} i_g(t) \, dt = DI_L$$
Input port equivalent circuit

\[ I_g = \frac{1}{T_s} \int_{0}^{T_s} i_g(t) \, dt = D I_L \]
Complete equivalent circuit, buck converter

Input and output port equivalent circuits, drawn together:

Replace dependent sources with equivalent dc transformer:
3.5. Example: inclusion of semiconductor conduction losses in the boost converter model

Boost converter example

Models of on-state semiconductor devices:

- **MOSFET**: on-resistance $R_{on}$
- **Diode**: constant forward voltage $V_D$ plus on-resistance $R_D$

Insert these models into subinterval circuits
Boost converter example: circuits during subintervals 1 and 2

\[ \text{switch in position 1} \quad \text{switch in position 2} \]
Average inductor voltage and capacitor current

\[ \langle v_L \rangle = D(V_g - IR_L - IR_{on}) + D'(V_g - IR_L - V_D - IR_D - V) = 0 \]

\[ \langle i_C \rangle = D(-V/R) + D'(I - V/R) = 0 \]
Construction of equivalent circuits

\[ V_g - IR_L - IDR_{on} - D'V_D - ID'R_D - D'\bar{V} = 0 \]

\[ D'I - \frac{V}{R} = 0 \]
Complete equivalent circuit
Solution for output voltage

\[ V = \left( \frac{1}{D'} \right) \left( V_g - D'V_D \right) \left( \frac{D'^2 R}{D'^2 R + R_L + DR_{on} + D'R_D} \right) \]

\[ \frac{V}{V_g} = \left( \frac{1}{D'} \right) \left( 1 - \frac{D'V_D}{V_g} \right) \left( 1 + \frac{1}{\frac{R_L + DR_{on} + D'R_D}{D'^2 R}} \right) \]
Solution for converter efficiency

\[ P_{in} = (V_g) (I) \]

\[ P_{out} = (V) (D'I) \]

\[ \eta = D' \frac{V}{V_g} = \frac{1 - \frac{D'V_D}{V_g}}{1 + \frac{R_L + D'R_{on} + D'R_D}{D'^2R}} \]

Conditions for high efficiency:

\[ \frac{V_g}{D'} \gg V_D \]
\[ D'^2R \gg R_L + D'R_{on} + D'R_D \]
Accuracy of the averaged equivalent circuit in prediction of losses

- Model uses average currents and voltages
- To correctly predict power loss in a resistor, use rms values
- Result is the same, provided ripple is small

\[ \text{MOSFET current waveforms, for various ripple magnitudes:} \]

\[ i(t) \]

\[ 0 \quad DT_s \quad T_s \quad t \]

\[ I \quad 1.1I \quad 2I \]

\[ \Delta i = 0 \quad \Delta i = 0.1I \quad \Delta i = I \]

<table>
<thead>
<tr>
<th>Inductor current ripple</th>
<th>MOSFET rms current</th>
<th>Average power loss in ( R_{on} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( \Delta i = 0 )</td>
<td>( I \sqrt{D} )</td>
<td>( D \hat{I}^2 R_{on} )</td>
</tr>
<tr>
<td>(b) ( \Delta i = 0.1I )</td>
<td>( 1.00167I \sqrt{D} )</td>
<td>( 1.0033D \hat{I}^2 R_{on} )</td>
</tr>
<tr>
<td>(c) ( \Delta i = I )</td>
<td>( 1.155I \sqrt{D} )</td>
<td>( 1.3333D \hat{I}^2 R_{on} )</td>
</tr>
</tbody>
</table>

Fundamentals of Power Electronics

Chapter 3: Steady-state equivalent circuit modeling, ...
Summary of chapter 3

1. The dc transformer model represents the primary functions of any dc-dc converter: transformation of dc voltage and current levels, ideally with 100% efficiency, and control of the conversion ratio $M$ via the duty cycle $D$. This model can be easily manipulated and solved using familiar techniques of conventional circuit analysis.

2. The model can be refined to account for loss elements such as inductor winding resistance and semiconductor on-resistances and forward voltage drops. The refined model predicts the voltages, currents, and efficiency of practical nonideal converters.

3. In general, the dc equivalent circuit for a converter can be derived from the inductor volt-second balance and capacitor charge balance equations. Equivalent circuits are constructed whose loop and node equations coincide with the volt-second and charge balance equations. In converters having a pulsating input current, an additional equation is needed to model the converter input port; this equation may be obtained by averaging the converter input current.
Example: a simple inductor

Faraday’s law:
For each turn of wire, we can write
\[ v_{\text{turn}}(t) = \frac{d\Phi(t)}{dt} \]

Total winding voltage is
\[ v(t) = n v_{\text{turn}}(t) = n \frac{d\Phi(t)}{dt} \]

Express in terms of the average flux density \( B(t) = \frac{\Phi(t)}{A_c} \)
\[ v(t) = nA_c \frac{dB(t)}{dt} \]
Inductor example: Ampere’s law

Choose a closed path which follows the average magnetic field line around the interior of the core. Length of this path is called the mean magnetic path length $l_m$.

For uniform field strength $H(t)$, the core MMF around the path is $H \ell_m$.

Winding contains $n$ turns of wire, each carrying current $i(t)$. The net current passing through the path interior (i.e., through the core window) is $ni(t)$.

From Ampere’s law, we have

$$H(t) \ell_m = n i(t)$$
Inductor example: core material model

Find winding current at onset of saturation: substitute $i = I_{sat}$ and $H = B_{sat}/\mu$ into equation previously derived via Ampere’s law. Result is

$$I_{sat} = \frac{B_{sat} \ell_m}{\mu n}$$
Electrical terminal characteristics

We have:

\[ v(t) = nA_c \frac{dB(t)}{dt} \quad H(t) \ell_m = n \, i(t) \quad B = \begin{cases} B_{sat} & \text{for } H \geq B_{sat}/\mu \\ \mu H & \text{for } |H| < B_{sat}/\mu \\ -B_{sat} & \text{for } H \leq -B_{sat}/\mu \end{cases} \]

Eliminate \( B \) and \( H \), and solve for relation between \( v \) and \( i \). For \( |i| < I_{sat} \),

\[ v(t) = \mu nA_c \frac{dH(t)}{dt} \quad \rightarrow \quad v(t) = \frac{\mu n^2 A_c}{\ell_m} \frac{di(t)}{dt} \]

which is of the form

\[ v(t) = L \frac{di(t)}{dt} \quad \text{with} \quad L = \frac{\mu n^2 A_c}{\ell_m} \]

—an inductor

For \( |i| > I_{sat} \) the flux density is constant and equal to \( B_{sat} \). Faraday’s law then predicts

\[ v(t) = nA_c \frac{dB_{sat}}{dt} = 0 \quad \text{—saturation leads to short circuit} \]
13.1.2 Magnetic circuits

Uniform flux and magnetic field inside a rectangular element:

MMF between ends of element is

\[ \mathcal{F} = H \ell \]

Since \( H = B / \mu \) and \( B = \Phi / A_c \), we can express \( \mathcal{F} \) as

\[ \mathcal{F} = \Phi \mathcal{R} \]

with

\[ \mathcal{R} = \frac{l}{\mu A_c} \]

A corresponding model:

\[ \Phi \quad \mathcal{R} \]

\( \mathcal{R} \) = reluctance of element
Magnetic circuits: magnetic structures composed of multiple windings and heterogeneous elements

- Represent each element with reluctance
- Windings are sources of MMF
- MMF $\rightarrow$ voltage, flux $\rightarrow$ current
- Solve magnetic circuit using Kirchoff’s laws, etc.
Magnetic analog of Kirchoff’s current law

Divergence of $\mathbf{B} = 0$

Flux lines are continuous and cannot end

Total flux entering a node must be zero
Magnetic analog of Kirchoff’s voltage law

Follows from Ampere’s law:

$$\oint_{\text{closed path}} H \cdot dl = \text{total current passing through interior of path}$$

Left-hand side: sum of MMF’s across the reluctances around the closed path

Right-hand side: currents in windings are sources of MMF’s. An \(n\)-turn winding carrying current \(i(t)\) is modeled as an MMF (voltage) source, of value \(ni(t)\).

Total MMF’s around the closed path add up to zero.
Example: inductor with air gap

\[ \Phi = n \cdot A_c \cdot i(t) \]

Core permeability \( \mu \)

Cross-sectional area \( A_c \)

Air gap \( \ell_g \)

Magnetic path length \( \ell_m \)
Magnetic circuit model

\[ \Phi_c + \Phi_g = ni \]

\[ ni = \Phi \left( R_c + R_g \right) \]

\[ R_c = \frac{\ell_c}{\mu A_c} \]

\[ R_g = \frac{\ell_g}{\mu_0 A_c} \]
Solution of model

Faraday’s law: \( v(t) = n \frac{d\Phi(t)}{dt} \)

Substitute for \( \Phi \): \( v(t) = \frac{n^2}{\mathcal{R}_c + \mathcal{R}_g} \frac{di(t)}{dt} \)

Hence inductance is \( L = \frac{n^2}{\mathcal{R}_c + \mathcal{R}_g} \)
Effect of air gap

- decrease inductance
- increase saturation current
- inductance is less dependent on core permeability

\[
n_i = \Phi \left( \mathcal{R}_c + \mathcal{R}_g \right)
\]

\[
L = \frac{n^2}{\mathcal{R}_c + \mathcal{R}_g}
\]

\[
\Phi_{sat} = B_{sat} A_c
\]

\[
I_{sat} = \frac{B_{sat} A_c}{n} \left( \mathcal{R}_c + \mathcal{R}_g \right)
\]
13.2 Transformer modeling

Two windings, no air gap:

\[ R = \frac{\ell_m}{\mu A_c} \]

\[ \mathcal{F}_c = n_1 i_1 + n_2 i_2 \]

\[ \Phi R = n_1 i_1 + n_2 i_2 \]

Magnetic circuit model:
13.2.1 The ideal transformer

In the ideal transformer, the core reluctance \( R \) approaches zero.

MMF \( \mathcal{I}_c = \Phi R \) also approaches zero. We then obtain

\[
0 = n_1 i_1 + n_2 i_2
\]

Also, by Faraday's law,

\[
v_1 = n_1 \frac{d\Phi}{dt}
\]

\[
v_2 = n_2 \frac{d\Phi}{dt}
\]

Eliminate \( \Phi \):

\[
\frac{d\Phi}{dt} = \frac{v_1}{n_1} = \frac{v_2}{n_2}
\]

Ideal transformer equations:

\[
\frac{v_1}{n_1} = \frac{v_2}{n_2} \quad \text{and} \quad n_1 i_1 + n_2 i_2 = 0
\]
13.2.2 The magnetizing inductance

For nonzero core reluctance, we obtain

$$\Phi R = n_1 i_1 + n_2 i_2 \quad \text{with} \quad v_1 = n_1 \frac{d\Phi}{dt}$$

Eliminate $\Phi$:

$$v_1 = \frac{n_1^2}{R} \frac{d}{dt} \left[ i_1 + \frac{n_2}{n_1} i_2 \right]$$

This equation is of the form

$$v_1 = L_M \frac{di_M}{dt}$$

with

$$L_M = \frac{n_1^2}{R}$$

$$i_M = i_1 + \frac{n_2}{n_1} i_2$$
Magnetizing inductance: discussion

- Models magnetization of core material
- A real, physical inductor, that exhibits saturation and hysteresis
- If the secondary winding is disconnected:
  - we are left with the primary winding on the core
  - primary winding then behaves as an inductor
  - the resulting inductor is the magnetizing inductance, referred to
    the primary winding
- Magnetizing current causes the ratio of winding currents to differ
  from the turns ratio
Transformer saturation

• Saturation occurs when core flux density $B(t)$ exceeds saturation flux density $B_{sat}$.

• When core saturates, the magnetizing current becomes large, the impedance of the magnetizing inductance becomes small, and the windings are effectively shorted out.

• Large winding currents $i_1(t)$ and $i_2(t)$ do not necessarily lead to saturation. If

$$0 = n_1i_1 + n_2i_2$$

then the magnetizing current is zero, and there is no net magnetization of the core.

• Saturation is caused by excessive applied volt-seconds

$$0 = n_1i_1 + n_2i_2$$
Saturation vs. applied volt-seconds

Magnetizing current depends on the integral of the applied winding voltage:

\[ i_M(t) = \frac{1}{L_M} \int v_1(t) dt \]

Flux density is proportional:

\[ B(t) = \frac{1}{n_1 A_c} \int v_1(t) dt \]

Flux density becomes large, and core saturates, when the applied volt-seconds \( \lambda_1 \) are too large, where

\[ \lambda_1 = \int_{t_1}^{t_2} v_1(t) dt \]

limits of integration chosen to coincide with positive portion of applied voltage waveform
13.2.3 Leakage inductances

\[ \Phi_M + v_1(t) - i_1(t) + v_2(t) - i_2(t) = \Phi_{l1} + \Phi_{l2} \]

Diagram:

- Transformer with primary and secondary windings.
- Leakage inductances \( \Phi_{l1} \) and \( \Phi_{l2} \).
- Voltages \( v_1(t) \) and \( v_2(t) \).
- Currents \( i_1(t) \) and \( i_2(t) \).
Transformer model, including leakage inductance

Terminal equations can be written in the form

\[
\begin{bmatrix}
  v_1(t) \\
  v_2(t)
\end{bmatrix} =
\begin{bmatrix}
  L_{11} & L_{12} \\
  L_{12} & L_{22}
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix}
  i_1(t) \\
  i_2(t)
\end{bmatrix}
\]

mutual inductance:

\[
L_{12} = \frac{n_1 n_2}{R} = \frac{n_2}{n_1} L_M
\]

primary and secondary self-inductances:

\[
L_{11} = L_{\ell 1} + \frac{n_1}{n_2} L_{12}
\]

\[
L_{22} = L_{\ell 2} + \frac{n_2}{n_1} L_{12}
\]

effective turns ratio

\[
n_e = \sqrt{\frac{L_{22}}{L_{11}}}
\]

coupling coefficient

\[
k = \frac{L_{12}}{\sqrt{L_{11}L_{22}}}
\]
13.3 Loss mechanisms in magnetic devices

Low-frequency losses:
- Dc copper loss
- Core loss: hysteresis loss

High-frequency losses: the skin effect
- Core loss: classical eddy current losses
- Eddy current losses in ferrite cores

High frequency copper loss: the proximity effect
- Proximity effect: high frequency limit
- MMF diagrams, losses in a layer, and losses in basic multilayer windings
- Effect of PWM waveform harmonics
13.3.1 Core loss

Energy per cycle $W$ flowing into $n$-turn winding of an inductor, excited by periodic waveforms of frequency $f$:

$$W = \int_{\text{one cycle}} v(t)i(t)\,dt$$

Relate winding voltage and current to core $B$ and $H$ via Faraday’s law and Ampere’s law:

$$v(t) = nA_c \frac{dB(t)}{dt}$$
$$H(t)l_m = ni(t)$$

Substitute into integral:

$$W = \int_{\text{one cycle}} \left( nA_c \frac{dB(t)}{dt} \right) \left( \frac{H(t)l_m}{n} \right)\,dt$$

$$= (A_c l_m) \int_{\text{one cycle}} H\,dB$$
Core loss: Hysteresis loss

\[ W = (A_c \ell_m) \int_{\text{one cycle}} H \, dB \]

The term \( A_c \ell_m \) is the volume of the core, while the integral is the area of the \( B-H \) loop.

\[ \text{(energy lost per cycle)} = \text{(core volume)} \times \text{(area of } B-H \text{ loop)} \]

\[ P_H = (f) (A_c \ell_m) \int_{\text{one cycle}} H \, dB \]

Hysteresis loss is directly proportional to applied frequency

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Chapter 13: Basic Magnetics Theory
Modeling hysteresis loss

- Hysteresis loss varies directly with applied frequency
- Dependence on maximum flux density: how does area of $B-H$ loop depend on maximum flux density (and on applied waveforms)?

Empirical equation (Steinmetz equation):

$$P_H = K_H f B_{\text{max}}^\alpha (\text{core volume})$$

The parameters $K_H$ and $\alpha$ are determined experimentally.

Dependence of $P_H$ on $B_{\text{max}}$ is predicted by the theory of magnetic domains.
Core loss: eddy current loss

Magnetic core materials are reasonably good conductors of electric current. Hence, according to Lenz’s law, magnetic fields within the core induce currents (“eddy currents”) to flow within the core. The eddy currents flow such that they tend to generate a flux which opposes changes in the core flux $\Phi(t)$. The eddy currents tend to prevent flux from penetrating the core.

$$i(t)^2R$$

[Diagram showing magnetic flux and eddy current]
Modeling eddy current loss

- Ac flux $\Phi(t)$ induces voltage $v(t)$ in core, according to Faraday’s law. Induced voltage is proportional to derivative of $\Phi(t)$. In consequence, magnitude of induced voltage is directly proportional to excitation frequency $f$.

- If core material impedance $Z$ is purely resistive and independent of frequency, $Z = R$, then eddy current magnitude is proportional to voltage: $i(t) = v(t)/R$. Hence magnitude of $i(t)$ is directly proportional to excitation frequency $f$.

- Eddy current power loss $i^2(t)R$ then varies with square of excitation frequency $f$.

- Classical Steinmetz equation for eddy current loss:
  $$P_E = K_E f^2 B_{\text{max}}^2 (\text{core volume})$$

- Ferrite core material impedance is capacitive. This causes eddy current power loss to increase as $f^4$. 
Total core loss: manufacturer’s data

Empirical equation, at a fixed frequency:

$$P_{fe} = K_f e^\text{\(\Delta B\)}^\beta A_c l_m$$
## Core materials

<table>
<thead>
<tr>
<th>Core type</th>
<th>$B_{sat}$</th>
<th>Relative core loss</th>
<th>Applications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laminations iron, silicon steel</td>
<td>1.5 - 2.0 T</td>
<td>high</td>
<td>50-60 Hz transformers, inductors</td>
</tr>
<tr>
<td>Powdered cores</td>
<td>0.6 - 0.8 T</td>
<td>medium</td>
<td>1 kHz transformers, 100 kHz filter inductors</td>
</tr>
<tr>
<td>powdered iron, molypermalloy</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ferrite Manganese-zinc, Nickel-zinc</td>
<td>0.25 - 0.5 T</td>
<td>low</td>
<td>20 kHz - 1 MHz transformers, ac inductors</td>
</tr>
</tbody>
</table>
13.3.2 Low-frequency copper loss

DC resistance of wire

\[ R = \rho \frac{\ell_b}{A_w} \]

where \( A_w \) is the wire bare cross-sectional area, and \( \ell_b \) is the length of the wire. The resistivity \( \rho \) is equal to \( 1.724 \cdot 10^{-6} \) \( \Omega \) cm for soft-annealed copper at room temperature. This resistivity increases to \( 2.3 \cdot 10^{-6} \) \( \Omega \) cm at 100°C.

The wire resistance leads to a power loss of

\[ P_{cu} = I_{rms}^2 R \]
13.4 Eddy currents in winding conductors

13.4.1 Intro to the skin and proximity effects
Penetration depth $\delta$

For sinusoidal currents: current density is an exponentially decaying function of distance into the conductor, with characteristic length $\delta$ known as the \textit{penetration depth} or \textit{skin depth}.

$\delta = \sqrt{\frac{\rho}{\pi \mu f}}$

For copper at room temperature:

$\delta = \frac{7.5}{\sqrt{f}} \text{ cm}$
The proximity effect

Ac current in a conductor induces eddy currents in adjacent conductors by a process called the proximity effect. This causes significant power loss in the windings of high-frequency transformers and ac inductors.

A multi-layer foil winding, with $h \gg \delta$. Each layer carries net current $i(t)$. 

![Diagram showing the proximity effect]
Example: a two-winding transformer

Cross-sectional view of two-winding transformer example. Primary turns are wound in three layers. For this example, let’s assume that each layer is one turn of a flat foil conductor. The secondary is a similar three-layer winding. Each layer carries net current $i(t)$. Portions of the windings that lie outside of the core window are not illustrated. Each layer has thickness $h \gg \delta$. 

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Distribution of currents on surfaces of conductors: two-winding example

Skin effect causes currents to concentrate on surfaces of conductors.

Surface current induces equal and opposite current on adjacent conductor.

This induced current returns on opposite side of conductor.

Net conductor current is equal to $i(t)$ for each layer, since layers are connected in series.

Circulating currents within layers increase with the numbers of layers.
Estimating proximity loss: high-frequency limit

The current $i(t)$ having rms value $I$ is confined to thickness $d$ on the surface of layer 1. Hence the effective “ac” resistance of layer 1 is:

$$R_{ac} = \frac{h}{d} R_{dc}$$

This induces copper loss $P_1$ in layer 1:

$$P_1 = I^2 R_{ac}$$

Power loss $P_2$ in layer 2 is:

$$P_2 = P_1 + 4P_1 = 5P_1$$

Power loss $P_3$ in layer 3 is:

$$P_3 = (2^2 + 3^2)P_1 = 13P_1$$

Power loss $P_m$ in layer $m$ is:

$$P_m = I^2 \left( (m-1)^2 + m^2 \right) \left( \frac{h}{d} R_{dc} \right)$$
Total loss in \( M \)-layer winding: high-frequency limit

Add up losses in each layer:

\[
P = I^2 \left( \frac{h}{\delta} R_{dc} \right) \sum_{m=1}^{M} \left[ (m - 1)^2 + m^2 \right]
\]

\[
= I^2 \left( \frac{h}{\delta} R_{dc} \right) \frac{M}{3} \left( 2M^2 + 1 \right)
\]

**Compare with dc copper loss:**

If foil thickness were \( H = \delta \), then at dc each layer would produce copper loss \( P_1 \). The copper loss of the \( M \)-layer winding would be

\[
P_{dc} = P_1 MR_{dc}
\]

So the proximity effect increases the copper loss by a factor of

\[
F_R = \frac{P}{P_{dc}} = \frac{1}{3} \left( \frac{h}{\delta} \right) \left( 2M^2 + 1 \right)
\]
Discussion: design of winding geometry to minimize proximity loss

- Interleaving windings can significantly reduce the proximity loss when the winding currents are in phase, such as in the transformers of buck-derived converters or other converters.
- In some converters (such as flyback or SEPIC) the winding currents are out of phase. Interleaving then does little to reduce the peak MMF and proximity loss. See Vandelac and Ziogas [10].
- For sinusoidal winding currents, there is an optimal conductor thickness near $\varphi = 1$ that minimizes copper loss.
- Minimize the number of layers. Use a core geometry that maximizes the width $l_w$ of windings.
- Minimize the amount of copper in vicinity of high MMF portions of the windings.
Litz wire

- A way to increase conductor area while maintaining low proximity losses
- Many strands of small-gauge wire are bundled together and are externally connected in parallel
- Strands are twisted, or transposed, so that each strand passes equally through each position on inside and outside of bundle. This prevents circulation of currents between strands.
- Strand diameter should be sufficiently smaller than skin depth
- The Litz wire bundle itself is composed of multiple layers
- Advantage: when properly sized, can significantly reduce proximity loss
- Disadvantage: increased cost and decreased amount of copper within core window
Practical realisations and simulations

- Transformer (50 Hz)
- Transformer (20 kHz)
- Inductors
- Simulations with ONELAB
Chapter 4: Switch Realization

▶ Diode
▶ MOSFET
▶ Other power semi-conductors
Forward-biased power diode

conductivity modulation

minority carrier injection
Charge-controlled behavior of the diode

The diode equation:

\[ q(t) = Q_0 \left( e^{\lambda v(t)} - 1 \right) \]

Charge control equation:

\[ \frac{dq(t)}{dt} = i(t) - \frac{q(t)}{\tau_L} \]

With:

\[ \lambda = \frac{1}{26 \text{ mV}} \text{ at } 300 \text{ K} \]

\[ \tau_L = \text{minority carrier lifetime} \]

(above equations don’t include current that charges depletion region capacitance)

In equilibrium: \( dq/dt = 0 \), and hence

\[ i(t) = \frac{q(t)}{\tau_L} = \frac{Q_0}{\tau_L} \left( e^{\lambda v(t)} - 1 \right) = I_0 \left( e^{\lambda v(t)} - 1 \right) \]
Charge-control in the diode: Discussion

- The familiar $i-v$ curve of the diode is an equilibrium relationship that can be violated during transient conditions.
- During the turn-on and turn-off switching transients, the current deviates substantially from the equilibrium $i-v$ curve, because of change in the stored charge and change in the charge within the reverse-bias depletion region.
- Under forward-biased conditions, the stored minority charge causes “conductivity modulation” of the resistance of the lightly-doped $n^-$ region, reducing the device on-resistance.
Diode in OFF state:
reversed-biased, blocking voltage

- Diode is reverse-biased
- No stored minority charge: $q = 0$
- Depletion region blocks applied reverse voltage; charge is stored in capacitance of depletion region

$$v(t)$$

$$i(t)$$

Depletion region, reverse-biased
Turn-on transient

The current $i(t)$ is determined by the converter circuit. This current supplies:

- charge to increase voltage across depletion region
- charge needed to support the on-state current
- charge to reduce on-resistance of $n^-$ region

The diagram shows:
- Diode is forward-biased. Supply minority charge to $n^-$ region to reduce on-resistance
- Diode conducts with low on-resistance
- Charge depletion region
- On-state current determined by converter circuit
Turn-off transient

Removal of stored minority charge $q_{pn-n^+}$

$i (< 0)$

$v$

$p$ $n^-$ $n$

Removal of stored minority charge $q$
Diode turn-off transient
continued

(1) (2) (3) (4) (5) (6)

(4) Diode remains forward-biased. Remove stored charge in n− region
(5) Diode is reverse-biased. Charge depletion region capacitance.

\[ v(t) \]

\[ i(t) \]

\[ \frac{di}{dt} \]

Area \[ -Q_r \]

Fundamentals of Power Electronics

Chapter 4: Switch realization
The diode switching transients induce switching loss in the transistor

- Diode recovered stored charge $Q_r$ flows through transistor during transistor turn-on transition, inducing switching loss

- $Q_r$ depends on diode on-state forward current, and on the rate-of-change of diode current during diode turn-off transition

Fundamentals of Power Electronics
Switching loss calculation

Energy lost in transistor:

\[ W_D = \int_{t_0}^{t_1} v_A(t) i_A(t) \, dt \]

With abrupt-recovery diode:

\[ W_D \approx \int_{t_0}^{t_1} V_g (i_L - i_B(t)) \, dt \]

\[ = V_g \, i_L \, t_r + V_g \, Q_r \]

- Often, this is the largest component of switching loss

Soft-recovery diode:

\[ (t_2 - t_1) >> (t_1 - t_0) \]

Abrupt-recovery diode:

\[ (t_2 - t_1) << (t_1 - t_0) \]

Often, this is the largest component of switching loss
Ringing induced by diode stored charge

- Diode is forward-biased while \( i_L(t) > 0 \)
- Negative inductor current removes diode stored charge \( Q_r \)
- When diode becomes reverse-biased, negative inductor current flows through capacitor \( C \).
- Ringing of \( L-C \) network is damped by parasitic losses. Ringing energy is lost.

see Section 4.3.3
Energy associated with ringing

Recovered charge is

\[ Q_r = - \int_{t_2}^{t_3} i_L(t) \, dt \]

Energy stored in inductor during interval \( t_2 \leq t \leq t_3 \):

\[ W_L = \int_{t_2}^{t_3} v_L(t) \, i_L(t) \, dt \]

Applied inductor voltage during interval \( t_2 \leq t \leq t_3 \):

\[ v_L(t) = L \frac{di_L(t)}{dt} = -V_2 \]

Hence,

\[ W_L = \int_{t_2}^{t_3} L \frac{di_L(t)}{dt} \, i_L(t) \, dt = \int_{t_2}^{t_3} (-V_2) \, i_L(t) \, dt \]

\[ W_L = \frac{1}{2} L \, i_L^2(t_3) = V_2 \, Q_r \]
Excerpt of IXYS DSEP29-12A diode data-sheet:

Fig. 1 Forward current $I_F$ vs. $V_F$
Excerpt of IXYS DSEP29-12A diode data-sheet:

Fig. 2  Typ. reverse recovery charge $Q_r$ versus $-\text{di}_F/\text{dt}$

$T_{VJ} = 100^\circ\text{C}$
$V_R = 600\ \text{V}$
$I_F = 60\ \text{A}$
$30\ \text{A}$
$15\ \text{A}$

Fig. 3  Typ. peak reverse current $I_{RM}$ versus $-\text{di}_F/\text{dt}$

$T_{VJ} = 100^\circ\text{C}$
$V_R = 600\ \text{V}$
$I_F = 60\ \text{A}$
$30\ \text{A}$
$15\ \text{A}$
Types of power diodes

**Standard recovery**
Reverse recovery time not specified, intended for 50/60Hz

**Fast recovery and ultra-fast recovery**
Reverse recovery time and recovered charge specified
Intended for converter applications

**Schottky diode**
A majority carrier device
Essentially no recovered charge
Model with equilibrium $i-v$ characteristic, in parallel with depletion region capacitance
Restricted to low voltage (few devices can block 100V or more)
### Characteristics of several commercial power rectifier diodes

<table>
<thead>
<tr>
<th>Part number</th>
<th>Rated max voltage</th>
<th>Rated avg current</th>
<th>$V_F$ (typical)</th>
<th>$t_r$ (max)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fast recovery rectifiers</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1N3913</td>
<td>400V</td>
<td>30A</td>
<td>1.1V</td>
<td>400ns</td>
</tr>
<tr>
<td>SD453N25S20PC</td>
<td>2500V</td>
<td>400A</td>
<td>2.2V</td>
<td>2µs</td>
</tr>
<tr>
<td><strong>Ultra-fast recovery rectifiers</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MUR815</td>
<td>150V</td>
<td>8A</td>
<td>0.975V</td>
<td>35ns</td>
</tr>
<tr>
<td>MUR1560</td>
<td>600V</td>
<td>15A</td>
<td>1.2V</td>
<td>60ns</td>
</tr>
<tr>
<td>RHRU100120</td>
<td>1200V</td>
<td>100A</td>
<td>2.6V</td>
<td>60ns</td>
</tr>
<tr>
<td><strong>Schottky rectifiers</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MBR6030L</td>
<td>30V</td>
<td>60A</td>
<td>0.48V</td>
<td></td>
</tr>
<tr>
<td>444CNQ045</td>
<td>45V</td>
<td>440A</td>
<td>0.69V</td>
<td></td>
</tr>
<tr>
<td>30CPQ150</td>
<td>150V</td>
<td>30A</td>
<td>1.19V</td>
<td></td>
</tr>
</tbody>
</table>
4.2.2. The Power MOSFET

- Gate lengths approaching one micron
- Consists of many small enhancement-mode parallel-connected MOSFET cells, covering the surface of the silicon wafer
- Vertical current flow
- n-channel device is shown
MOSFET: Off state

- $p$-$n^-$ junction is reverse-biased
- off-state voltage appears across $n^-$ region

source $-$

\begin{center}
\begin{tikzpicture}
  \draw[thick, dashed] (0,0) -- (8,0);
  \draw[thick, dashed] (0,1) -- (8,1);
  \draw[thick, dashed] (0,2) -- (8,2);

  \draw[fill, gray!20] (0,0) -- (4,0) -- (4,1) -- (0,1) -- cycle;
  \draw[fill, gray!20] (4,0) -- (8,0) -- (8,1) -- (4,1) -- cycle;

  \draw[fill, gray!40] (0,2) -- (4,2) -- (4,3) -- (0,3) -- cycle;
  \draw[fill, gray!40] (4,2) -- (8,2) -- (8,3) -- (4,3) -- cycle;

  \node at (2,0) {$n$}; \node at (2,1) {$n$}; \node at (2,2) {$p$}; \node at (2,3) {$n$};
  \node at (6,0) {$n$}; \node at (6,1) {$n$}; \node at (6,2) {$p$}; \node at (6,3) {$n$};

  \node at (0,4) {source}; \node at (8,4) {drain};
  \node at (4,0) {$-$}; \node at (4,4) {$+$};
  \node at (4,0.5) {depletion region}; \node at (4,2.5) {$n^-$};
\end{tikzpicture}
\end{center}
MOSFET: on state

- $p$-$n$ junction is slightly reverse-biased
- Positive gate voltage induces conducting channel
- Drain current flows through $n^-$ region and conducting channel
- On resistance = total resistances of $n^-$ region, conducting channel, source and drain contacts, etc.
MOSFET body diode

- $p$-$n$ junction forms an effective diode, in parallel with the channel
- negative drain-to-source voltage can forward-bias the body diode
- diode can conduct the full MOSFET rated current
- diode switching speed not optimized — body diode is slow, $Q_r$ is large
Typical MOSFET characteristics

- Off state: $V_{GS} < V_{th}$
- On state: $V_{GS} >> V_{th}$
- MOSFET can conduct peak currents well in excess of average current rating —characteristics are unchanged
- on-resistance has positive temperature coefficient, hence easy to parallel
A simple MOSFET equivalent circuit

- $C_{gs}$: large, essentially constant
- $C_{gd}$: small, highly nonlinear
- $C_{ds}$: intermediate in value, highly nonlinear
- switching times determined by rate at which gate driver charges/discharges $C_{gs}$ and $C_{gd}$

$$C_{ds}(v_{ds}) = \frac{C_0}{\sqrt{1 + \frac{v_{ds}}{V_0}}}$$

$$C_{ds}(v_{ds}) \approx C_0 \sqrt{\frac{V_0}{v_{ds}}} = \frac{C_0}{\sqrt{v_{ds}}}$$
Switching loss caused by semiconductor output capacitances

Buck converter example

Energy lost during MOSFET turn-on transition (assuming linear capacitances):

\[ W_C = \frac{1}{2} (C_{ds} + C_j) V_g^2 \]
MOSFET nonlinear $C_{ds}$

Approximate dependence of incremental $C_{ds}$ on $v_{ds}$:

$$C_{ds}(v_{ds}) \approx C_0 \sqrt{\frac{V_0}{v_{ds}}} = \frac{C'_0}{\sqrt{v_{ds}}}$$

Energy stored in $C_{ds}$ at $v_{ds} = V_{DS}$:

$$W_{Cds} = \int_{0}^{V_{DS}} v_{ds} i_C \, dt = \int_{0}^{V_{DS}} v_{ds} C_{ds}(v_{ds}) \, dv_{ds}$$

$$W_{Cds} = \int_{0}^{V_{DS}} C'_0(v_{ds}) \sqrt{v_{ds}} \, dv_{ds} = \frac{2}{3} C_{ds}(V_{DS}) \frac{V_{DS}^2}{2}$$

— same energy loss as linear capacitor having value $\frac{4}{3} C_{ds}(V_{DS})$
\[ W_{C_{ds}} = \int_{0}^{V_{DS}} C'_{0} \sqrt{v_{ds}} dv_{ds} = \frac{2}{3} C_{ds}(V_{ds}) V_{ds}^{2} \]
MOSFET: hard switching losses (waveforms)

Excerpt of [?]:

The area $P_{on}$ and $P_{off}$ represent the switching losses.
MOSFET: hard switching losses (explanations)

- (a) Turn-On:
  1. The gate voltage $V_{GS}$ rises from 0 to $V_{th}$.
  2. During $t_{CR}$, the drain current $I_{DS}$ rises according to $V_{GS}$ change (linked by the transconductance).
  3. Once $I_{DS}$ reaches $I_L$, the transistor carries the full load current.
  4. $V_{GS}$ stays at the "plateau" voltage $V_{pl}$ due to the Miller effect and the drain voltage $V_{DS}$ falls linearly during $t_{VF}$.
  5. Once $V_{DS}$ reaches 0V, $V_{GS}$ continues to rise up to $V_{DR}$.

- (b) Turn-Off:
  1. $V_{GS}$ falls from $V_{DR}$ to $V_{pl}$.
  2. When $V_{GS}$ reaches $V_{pl}$, $V_{DS}$ starts rising linearly during $t_{VR}$. $V_{GS}$ stays at $V_{pl}$ due to the Miller effect.
  3. Once $V_{DS}$ reaches $V_{BUS}$, $V_{GS}$ starts falling and $I_{DS}$ also starts falling accordingly (linked by the transconductance) during $t_{CF}$.
  4. Once $V_{GS}$ reaches $V_{th}$, $I_{DS}$ reaches 0V.
  5. Finally, $V_{GS}$ goes to 0V.
MOSFET: static characteristics example

Excerpt of IR IRFP4668PbF MOSFET data-sheet:

- **Graph 1:**
  - **Y-axis:** Drain-to-Source Current (A)
  - **X-axis:** Drain-to-Source Voltage (V)
  - Curves for different gate voltages (VGS = 4.5V, 6.8V, 8.0V, 7.0V, 6.0V, 5.0V, 10V, 15V)
  - Note: <60μs PULSE WIDTH, TJ = 175°C

- **Graph 2:**
  - **Y-axis:** Drain-to-Source On Resistance (Normalized)
  - **X-axis:** Junction Temperature (°C)
  - Curve for ID = 8A, VGS = 10V
Excerpt of IR IRFP4668PbF MOSFET data-sheet:
## Characteristics of several commercial power MOSFETs

<table>
<thead>
<tr>
<th>Part number</th>
<th>Rated max voltage</th>
<th>Rated avg current</th>
<th>$R_{on}$</th>
<th>$Q_g$ (typical)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IRFZ48</td>
<td>60V</td>
<td>50A</td>
<td>0.018Ω</td>
<td>110nC</td>
</tr>
<tr>
<td>IRF510</td>
<td>100V</td>
<td>5.6A</td>
<td>0.54Ω</td>
<td>8.3nC</td>
</tr>
<tr>
<td>IRF540</td>
<td>100V</td>
<td>28A</td>
<td>0.077Ω</td>
<td>72nC</td>
</tr>
<tr>
<td>APT10M25BNR</td>
<td>100V</td>
<td>75A</td>
<td>0.025Ω</td>
<td>171nC</td>
</tr>
<tr>
<td>IRF740</td>
<td>400V</td>
<td>10A</td>
<td>0.55Ω</td>
<td>63nC</td>
</tr>
<tr>
<td>MTM15N40E</td>
<td>400V</td>
<td>15A</td>
<td>0.3Ω</td>
<td>110nC</td>
</tr>
<tr>
<td>APT5025BN</td>
<td>500V</td>
<td>23A</td>
<td>0.25Ω</td>
<td>83nC</td>
</tr>
<tr>
<td>APT1001RBNR</td>
<td>1000V</td>
<td>11A</td>
<td>1.0Ω</td>
<td>150nC</td>
</tr>
</tbody>
</table>
MOSFET: conclusions

- A majority-carrier device: fast switching speed
- Typical switching frequencies: tens and hundreds of kHz
- On-resistance increases rapidly with rated blocking voltage
- Easy to drive
- The device of choice for blocking voltages less than 500V
- 1000V devices are available, but are useful only at low power levels (100W)
- Part number is selected on the basis of on-resistance rather than current rating
Other power semi-conductors (brief overview)

Thyristor: high voltage, high current, switches off at zero current,

GTO (gate turn off Thyristor): similar to Thyristor but can be switched off with the gate signal,

IGBT (Isolated Gate Bipolar Transistor): high voltage, high current, controlled like a MOSFET,

BJT transistor: not often used, replaced by MOSFET,

Schottky diode: diode with higher conduction and switching performances but lower breakdown voltage,

SiC diode: emerging component that could/will replace diodes,

SiC transistor: emerging component that could/will replace IGBT,

GaN transistor: emerging component that could/will replace MOSFET.
Other power semi-conductors

Excerpt of [?]:

![Graph showing power semi-conductors](image-url)
Chapter 5. The Discontinuous Conduction Mode

5.1. Origin of the discontinuous conduction mode, and mode boundary

5.2. Analysis of the conversion ratio $M(D,K)$

5.3. Boost converter example

5.4. Summary of results and key points
Introduction to Discontinuous Conduction Mode (DCM)

- Occurs because switching ripple in inductor current or capacitor voltage causes polarity of applied switch current or voltage to reverse, such that the current- or voltage-unidirectional assumptions made in realizing the switch are violated.
- Commonly occurs in dc-dc converters and rectifiers, having single-quadrant switches. May also occur in converters having two-quadrant switches.
- Typical example: dc-dc converter operating at light load (small load current). Sometimes, dc-dc converters and rectifiers are purposely designed to operate in DCM at all loads.
- Properties of converters change radically when DCM is entered:
  \( M \) becomes load-dependent
  Output impedance is increased
  Dynamics are altered
  Control of output voltage may be lost when load is removed
5.1. Origin of the discontinuous conduction mode, and mode boundary

Buck converter example, with single-quadrant switches

Minimum diode current is \((I - \Delta i_L)\)

Dc component \(I = V/R\)

Current ripple is

\[
\Delta i_L = \frac{(V_g - V)}{2L} DT_s = \frac{V_g DD'T_s}{2L}
\]

Note that \(I\) depends on load, but \(\Delta i_L\) does not.

\[Fundamentals of Power Electronics\]
Reduction of load current

Increase $R$, until $I = \Delta i_L$

Minimum diode current is $(I - \Delta i_L)$

Dc component $I = V/R$

Current ripple is

$$\Delta i_L = \frac{(V_g - V)}{2L} DT_s = \frac{V_g DD'T_s}{2L}$$

Note that $I$ depends on load, but $\Delta i_L$ does not.
Further reduce load current

Increase $R$ some more, such that $I < \Delta i_L$

Discontinuous conduction mode

Minimum diode current is $(I - \Delta i_L)$

Dc component $I = \frac{V}{R}$

Current ripple is

$$\Delta i_L = \frac{(V_g - V)}{2L} DT_s = \frac{V_g DD'T_s}{2L}$$

Note that $I$ depends on load, but $\Delta i_L$ does not.

The load current continues to be positive and non-zero.

Fundamentals of Power Electronics

Chapter 5: Discontinuous conduction mode
Mode boundary

\[ I > \Delta i_L \] for CCM
\[ I < \Delta i_L \] for DCM

Insert buck converter expressions for \( I \) and \( \Delta i_L \):
\[
\frac{DV_g}{R} < \frac{DD'T_sV_g}{2L}
\]

Simplify:
\[
\frac{2L}{RT_s} < D'
\]

This expression is of the form
\[
K < K_{crit}(D) \] for DCM
where \( K = \frac{2L}{RT_s} \) and \( K_{crit}(D) = D' \)
$K$ and $K_{crit}$ vs. $D$

for $K < 1$:

$K_{crit}(D) = 1 - D$

for $K > 1$:

$K_{crit}(D) = 1 - D$

$K = \frac{2L}{RT_s}$
Critical load resistance $R_{crit}$

Solve $K_{crit}$ equation for load resistance $R$:

$R < R_{crit}(D)$ for CCM

$R > R_{crit}(D)$ for DCM

where $R_{crit}(D) = \frac{2L}{D'T_s}$
Summary: mode boundary

\[ K > K_{\text{crit}}(D) \quad \text{or} \quad R < R_{\text{crit}}(D) \quad \text{for CCM} \]
\[ K < K_{\text{crit}}(D) \quad \text{or} \quad R > R_{\text{crit}}(D) \quad \text{for DCM} \]

Table 5.1. CCM-DCM mode boundaries for the buck, boost, and buck-boost converters

<table>
<thead>
<tr>
<th>Converter</th>
<th>( K_{\text{crit}}(D) )</th>
<th>( \max_{0 \leq D \leq 1} (K_{\text{crit}}) )</th>
<th>( R_{\text{crit}}(D) )</th>
<th>( \min_{0 \leq D \leq 1} (R_{\text{crit}}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buck</td>
<td>((1 - D))</td>
<td>1</td>
<td>( \frac{2L}{(1 - D)T_s} )</td>
<td>( 2 \frac{L}{T_s} )</td>
</tr>
<tr>
<td>Boost</td>
<td>(D (1 - D)^2)</td>
<td>( \frac{4}{27} )</td>
<td>( \frac{2L}{D (1 - D)^2 T_s} )</td>
<td>( \frac{27 L}{2 T_s} )</td>
</tr>
<tr>
<td>Buck-boost</td>
<td>((1 - D)^2)</td>
<td>1</td>
<td>( \frac{2L}{(1 - D)^2 T_s} )</td>
<td>( 2 \frac{L}{T_s} )</td>
</tr>
</tbody>
</table>
5.2. Analysis of the conversion ratio $M(D,K)$

Analysis techniques for the discontinuous conduction mode:

Inductor volt-second balance

$$\langle v_L \rangle = \frac{1}{T_s} \int_0^{T_s} v_L(t) \, dt = 0$$

Capacitor charge balance

$$\langle i_C \rangle = \frac{1}{T_s} \int_0^{T_s} i_C(t) \, dt = 0$$

Small ripple approximation sometimes applies:

$$v(t) \approx V \quad \text{because} \quad \Delta v \ll V$$

$$i(t) \approx I \quad \text{is a poor approximation when} \quad \Delta i > I$$

Converter steady-state equations obtained via charge balance on each capacitor and volt-second balance on each inductor. Use care in applying small ripple approximation.
Example: Analysis of DCM buck converter $M(D,K)$

Fundamentals of Power Electronics

Chapter 5: Discontinuous conduction mode
Subinterval 1

\[ v_L(t) = V_g - v(t) \]
\[ i_C(t) = i_L(t) - v(t) / R \]

Small ripple approximation for \( v(t) \) (but not for \( i(t) \)!

\[ v_L(t) \approx V_g - V \]
\[ i_C(t) \approx i_L(t) - V / R \]
Subinterval 2

\[ v_L(t) = -v(t) \]
\[ i_C(t) = i_L(t) - \frac{v(t)}{R} \]

Small ripple approximation for \( v(t) \) but not for \( i(t) \):

\[ v_L(t) \approx -V \]
\[ i_C(t) \approx i_L(t) - \frac{V}{R} \]
Subinterval 3

\[ v_L = 0, \quad i_L = 0 \]
\[ i_C(t) = i_L(t) - \frac{v(t)}{R} \]

Small ripple approximation:

\[ v_L(t) = 0 \]
\[ i_C(t) = -\frac{V}{R} \]
Inductor volt-second balance

Volt-second balance:
\[
\langle v_L(t) \rangle = D_1(V_g - V) + D_2(-V) + D_3(0) = 0
\]

Solve for \( V \):
\[
V = V_g \frac{D_1}{D_1 + D_2}
\]

note that \( D_2 \) is unknown
Capacitor charge balance

node equation:
\[ i_L(t) = i_c(t) + \frac{V}{R} \]

capacitor charge balance:
\[ \langle i_c \rangle = 0 \]
hence
\[ \langle i_L \rangle = \frac{V}{R} \]

must compute dc component of inductor current and equate to load current (for this buck converter example)
Inductor current waveform

peak current:

\[ i_L(D_1 T_s) = i_{pk} = \frac{V_g - V}{L} D_1 T_s \]

average current:

\[ \langle i_L \rangle = \frac{1}{T_s} \int_0^{T_s} i_L(t) \, dt \]

triangle area formula:

\[ \int_0^{T_s} i_L(t) \, dt = \frac{1}{2} i_{pk} (D_1 + D_2) T_s \]

\[ \langle i_L \rangle = (V_g - V) \frac{D_1 T_s}{2L} (D_1 + D_2) \]

\[ \frac{V}{R} = \frac{D_1 T_s}{2L} (D_1 + D_2) (V_g - V) \]
Solution for $V$

Two equations and two unknowns ($V$ and $D_2$):

$$V = V_g \frac{D_1}{D_1 + D_2}$$  \hspace{1cm} \text{(from inductor volt-second balance)}

$$\frac{V}{R} = \frac{D_1 T_s}{2L} (D_1 + D_2) (V_g - V)$$  \hspace{1cm} \text{(from capacitor charge balance)}

Eliminate $D_2$, solve for $V$:

$$\frac{V}{V_g} = \frac{2}{1 + \sqrt{1 + 4K / D_1^2}}$$

where $K = \frac{2L}{RT_s}$

valid for $K < K_{crit}$
Buck converter $M(D,K)$

$$M = \begin{cases} 
D & \text{for } K > K_{\text{crit}} \\
\frac{2}{1 + \sqrt{1 + 4K/D^2}} & \text{for } K < K_{\text{crit}}
\end{cases}$$

$K = 0.01$

$K = 0.1$

$K = 0.5$

$K \geq 1$
5.3. Boost converter example

Mode boundary:

\[ I > \Delta i_L \text{ for CCM} \]
\[ I < \Delta i_L \text{ for DCM} \]

Previous CCM soln:

\[ I = \frac{V_g}{D^2 R} \]
\[ \Delta i_L = \frac{V_g}{2L} DT_s \]
Mode boundary

\[
\frac{V_g}{D^2 R} > \frac{D T_s V_g}{2L} \quad \text{for CCM}
\]

\[
\frac{2L}{RT_s} > DD'^2 \quad \text{for CCM}
\]

\[
K > K_{\text{crit}}(D) \quad \text{for CCM}
\]

\[
K < K_{\text{crit}}(D) \quad \text{for DCM}
\]

where \( K = \frac{2L}{RT_s} \) and \( K_{\text{crit}}(D) = DD'^2 \)

\[K_{\text{crit}}\left(\frac{1}{3}\right) = \frac{4}{27}\]
Mode boundary

\[ K > K_{crit}(D) \quad \text{for CCM} \]
\[ K < K_{crit}(D) \quad \text{for DCM} \]

where \( K = \frac{2L}{RT_s} \) and \( K_{crit}(D) = DD'^2 \)
Conversion ratio: DCM boost

Fundamentals of Power Electronics

Chapter 5: Discontinuous conduction mode
Subinterval 1

Small ripple approximation for \( v(t) \) (but not for \( i(t) \)):

\[
\begin{align*}
v_L(t) &\approx V_g \\
i_C(t) &\approx -V / R
\end{align*}
\]

\[
v_L(t) = V_g \\
i_C(t) = -v(t) / R
\]

\[
0 < t < D_1T_s
\]
Subinterval 2

\[ v_L(t) = V_g - v(t) \]
\[ i_C(t) = i(t) - v(t) / R \]

Small ripple approximation for \( v(t) \) but not for \( i(t) \):

\[ v_L(t) \approx V_g - V \]
\[ i_C(t) \approx i(t) - V / R \]

\[ D_1 T_s < t < (D_1 + D_2) T_s \]
Subinterval 3

Small ripple approximation:

\[ v_L(t) = 0, \quad i = 0 \]
\[ i_C(t) = -\frac{v(t)}{R} \]

\[ (D_1 + D_2)T_s < t < T_s \]
Inductor volt-second balance

Volt-second balance:

\[ D_1 V_g + D_2 (V_g - V) + D_3(0) = 0 \]

Solve for \( V \):

\[ V = \frac{D_1 + D_2}{D_2} V_g \]

note that \( D_2 \) is unknown
Capacitor charge balance

node equation:

\[ i_D(t) = i_C(t) + \frac{v(t)}{R} \]

capacitor charge balance:

\[ \langle i_C \rangle = 0 \]

hence

\[ \langle i_D \rangle = \frac{V}{R} \]

must compute dc component of diode current and equate to load current
(for this boost converter example)
Inductor and diode current waveforms

peak current:

\[ i_{pk} = \frac{V_g}{L} D_1 T_s \]

average diode current:

\[ \langle i_D \rangle = \frac{1}{T_s} \int_{0}^{T_s} i_D(t) \, dt \]

triangle area formula:

\[ \int_{0}^{T_s} i_D(t) \, dt = \frac{1}{2} i_{pk} D_2 T_s \]
Equate diode current to load current

average diode current:

$$\langle i_D \rangle = \frac{1}{T_s} \left( \frac{1}{2} i_{pk} D_2 T_s \right) = \frac{V_g D_1 D_2 T_s}{2L}$$

equate to dc load current:

$$\frac{V_g D_1 D_2 T_s}{2L} = \frac{V}{R}$$
Solution for $V$

Two equations and two unknowns ($V$ and $D_2$):

\[ V = \frac{D_1 + D_2}{D_2} V_g \]  
(from inductor volt-second balance)

\[ \frac{V_g D_1 D_2 T_s}{2L} = \frac{V}{R} \]  
(from capacitor charge balance)

Eliminate $D_2$, solve for $V$. From volt-sec balance eqn:

\[ D_2 = D_1 \frac{V_g}{V - V_g} \]

Substitute into charge balance eqn, rearrange terms:

\[ V^2 - VV_g - \frac{V_g^2 D_1^2}{K} = 0 \]
Solution for $V$

$$V^2 - VV_g - \frac{V_g^2 D_1^2}{K} = 0$$

Use quadratic formula:

$$\frac{V}{V_g} = \frac{1 \pm \sqrt{1 + 4D_1^2 / K}}{2}$$

Note that one root leads to positive $V$, while other leads to negative $V$. Select positive root:

$$\frac{V}{V_g} = M(D_1,K) = \frac{1 + \sqrt{1 + 4D_1^2 / K}}{2}$$

where $K = \frac{2L}{RT_s}$

valid for $K < K_{crit}(D)$

Transistor duty cycle $D = interval$ 1 duty cycle $D_1$
Boost converter characteristics

Approximate $M$ in DCM:

$$M \approx \frac{1}{2} + \frac{D}{\sqrt{K}}$$

for $K < K_{\text{crit}}$

$$M = \begin{cases} \frac{1}{1 - D} & \text{for } K > K_{\text{crit}} \\ \frac{1 + \sqrt{1 + 4D^2 / K}}{2} & \text{for } K < K_{\text{crit}} \end{cases}$$
## Summary of DCM characteristics

### Table 5.2. Summary of CCM-DCM characteristics for the buck, boost, and buck-boost converters

<table>
<thead>
<tr>
<th>Converter</th>
<th>$K_{\text{crit}}(D)$</th>
<th>$\text{DCM } M(D,K)$</th>
<th>$\text{DCM } D_2(D,K)$</th>
<th>$\text{CCM } M(D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buck</td>
<td>$(1 – D)$</td>
<td>$\frac{2}{1 + \sqrt{1 + 4K/D^2}}$</td>
<td>$\frac{K}{D} M(D,K)$</td>
<td>$D$</td>
</tr>
<tr>
<td>Boost</td>
<td>$D (1 – D)^2$</td>
<td>$\frac{K}{2} \sqrt{1 + 4D^2/K}$</td>
<td>$\frac{K}{D} M(D,K)$</td>
<td>$\frac{1}{1-D}$</td>
</tr>
<tr>
<td>Buck-boost</td>
<td>$(1 – D)^2$</td>
<td>$- \frac{D}{\sqrt{K}}$</td>
<td>$\sqrt{K}$</td>
<td>$- \frac{D}{1-D}$</td>
</tr>
</tbody>
</table>

with $K = 2L/R_\text{ts}$. DCM occurs for $K < K_{\text{crit}}$. 

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**Fundamentals of Power Electronics**  
Chapter 5: Discontinuous conduction mode
Summary of DCM characteristics

- DCM buck and boost characteristics are asymptotic to $M = 1$ and to the DCM buck-boost characteristic.
- DCM buck-boost characteristic is linear.
- CCM and DCM characteristics intersect at mode boundary. Actual $M$ follows characteristic having larger magnitude.
- DCM boost characteristic is nearly linear.
Summary of key points

1. The discontinuous conduction mode occurs in converters containing current- or voltage-unidirectional switches, when the inductor current or capacitor voltage ripple is large enough to cause the switch current or voltage to reverse polarity.

2. Conditions for operation in the discontinuous conduction mode can be found by determining when the inductor current or capacitor voltage ripples and dc components cause the switch on-state current or off-state voltage to reverse polarity.

3. The dc conversion ratio $M$ of converters operating in the discontinuous conduction mode can be found by application of the principles of inductor volt-second and capacitor charge balance.
Summary of key points

4. Extra care is required when applying the small-ripple approximation. Some waveforms, such as the output voltage, should have small ripple which can be neglected. Other waveforms, such as one or more inductor currents, may have large ripple that cannot be ignored.

5. The characteristics of a converter changes significantly when the converter enters DCM. The output voltage becomes load-dependent, resulting in an increase in the converter output impedance.

