Spectral and Condition Number Estimates of the Acoustic Single-Layer Operator for Low-Frequency Multiple Scattering in Dense Media*

Bertrand Thierry† Xavier Antoine‡

Abstract

The aim of this paper is to derive spectral and condition number estimates of the single-layer operator for low-frequency multiple scattering problems. This work extends to dense media the analysis initiated in [7]. Estimates are obtained first in the case of circular cylinders by Fourier analysis and are next formally adapted to disks, ellipses and rectangles in the framework of boundary element methods. Numerical simulations validating the approach are also given.

1 Introduction

Integral equation techniques [13, 28] are an attractive and widely used tool to numerically solve acoustic, electromagnetic and elastic scattering problems. In particular, they can be used for multiple scattering configurations [26, 29] which have many applications in physics and mechanics [9, 15, 16, 17, 18, 21, 22, 25, 27]. A classical way to solve an integral equation is to use an iterative Krylov subspace solver (like e.g. the GMRES [30]) coupled to a Matrix-Vector acceleration scheme like the Fast Multipole Method [14, 19, 29]. A well-known property is that the convergence rate of the iterative solver is related to the condition number and spectral distribution of the integral equation. For this reason, understanding the spectral properties helps in building suitable preconditioners for integral equations. Spectral estimates have already been obtained for single scattering configurations. We refer to [5, 6, 7, 23, 24] for complete studies involving circular cylinders, convex and non convex structures [8, 11] as well as open surfaces or waveguides [2, 3, 12]. Multiple scattering problems are more complex in the sense that interactions between obstacles must be considered in the analysis. In a first part [7], we derived low-frequency spectral and condition number estimates of the single-layer potential for many distant obstacles (disks, ellipses and rectangles). The aim of this second part is to complete the previous results when the scatterers are close (dense media).

The plan of the paper is the following. In Section 2, we briefly introduce the single-layer operator and its spectral formulation. We also recall the analytical formula of the single-layer operator when the obstacle is a collection of circular cylinders. Section 3 explains how to obtain eigenvalues

*The authors are partially supported by the French ANR fundings under the project MicroWave NT09_460489.
†Institut Montefiore, Université de Liège, ACE Team, Sart-Tilman Bldg. B28, B-4000 Liège, Belgique (Bertrand.Thierry@ulg.ac.be).
‡Institut Elie Cartan Nancy, Université de Lorraine, CNRS UMR 7502, INRIA CORIDA Team, Boulevard des Aiguillettes B.P. 239, F-54506 Vandœuvre-lès-Nancy, France (Xavier.Antoine@iecn.u-nancy.fr).
and condition number estimates of the single-layer operator for disks. Section 4 provides some extensions of the results to circular, elliptical and rectangular cylinders in the framework of linear boundary element methods. Finally, Section 5 gives a conclusion and some perspectives.

2 The single-layer operator for multiple scattering by disks

We consider a homogeneous acoustic medium filling the whole space $\mathbb{R}^2$ and containing a bounded open set $\Omega$ with boundary $\Gamma := \partial \Omega$ such that the propagation domain $\mathbb{R}^2 \setminus \overline{\Omega}$ is connected. Throughout Sections 2 and 3, the set $\Omega$ is assumed to be a collection of $M$ strictly disjoint (no sticky case) disks $\Omega_p$ with associated radii $a_p$ and centers $O_p$, for $p = 1, \ldots, M$. For two scatterers $\Omega_p$ and $\Omega_q$, $1 \leq p \neq q \leq M$, we set $b_{pq} = O_q O_p$, $b_{pq} = ||b_{pq}||$ and $\alpha_{pq} = \text{Angle}(\overrightarrow{O_1 O}, b_{pq})$. All along our paper, we try to use the notations from [26]. For any real positive wave number $k$ and smooth density $\rho \in H^{-1/2}(\Gamma)$, the single-layer integral operator is defined by

$$\forall \mathbf{x} \in \Gamma, \quad L_{\rho}(\mathbf{x}) = \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \rho(\mathbf{y}) d\Gamma(\mathbf{y}),$$

where the Green’s function $G(\cdot, \cdot)$ is given by: $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \mathbf{x} \neq \mathbf{y}, G(\mathbf{x}, \mathbf{y}) = \frac{1}{4} H_0^{(1)}(k ||\mathbf{x} - \mathbf{y}||)$ with $H_0^{(1)}$ is the zeroth order Hankel function of the first kind and $||\mathbf{x}|| = (x_1^2 + x_2^2)^{1/2}$. The operator $L$ is known [28] to be invertible from the Sobolev space $H^{-1/2}(\Gamma)$ to the Sobolev space $H^{1/2}(\Gamma)$ except for the set $F_D(\Omega)$ of Dirichlet irregular frequencies, that is the wave numbers $k$ for which the interior homogeneous Dirichlet boundary value problem

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma, \end{cases}$$

admits a non trivial solution. Throughout this paper, we assume that $k \notin F_D(\Omega)$.

By introducing one Fourier basis per circular obstacle, we obtain [31] the following expression of the single-layer integral operator $L$ as an infinite block matrix $\tilde{L}$ defined by

$$\tilde{L} = \begin{pmatrix} \tilde{L}^{1,1} & \tilde{L}^{1,2} & \cdots & \tilde{L}^{1,M} \\ \tilde{L}^{2,1} & \tilde{L}^{2,2} & \cdots & \tilde{L}^{2,1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{L}^{M,1} & \tilde{L}^{M,2} & \cdots & \tilde{L}^{M,M} \end{pmatrix},$$

where each infinite submatrix $\tilde{L}^{p,q} = (\tilde{L}_{m,n}^{p,q})_{m,n \in \mathbb{Z}}$ has for coefficients, for $m, n \in \mathbb{Z}$,

$$\tilde{L}_{m,n}^{p,q} = \begin{cases} \delta_{mn} \frac{i \pi a_p}{2} J_m(k a_p) H_m^{(1)}(k a_p) & \text{if } p = q, \\ \frac{i \pi \sqrt{a_p a_q}}{2} J_m(k a_p) S_{nm}(b_{pq}) J_n(k a_q) & \text{otherwise}. \end{cases} \quad (1)$$

Symbol $\delta_{mn}$ denotes the Kronecker’s delta function, equal to 1 if $m = n$ and 0 otherwise. Functions $J_m$ and $H_m^{(1)}$ are respectively the Bessel and first kind Hankel function of order $m$. The quantity $S_{nm}(b_{pq})$ is given by: $S_{nm}(b_{pq}) = H_{n-m}^{(1)}(b_{pq}) e^{i(n-m) \alpha_{pq}}$, for $p, q = 1, \ldots, M, p \neq q$ and $m, n \in \mathbb{Z}$. Let us remark that the infinite block $\tilde{L}^{p,p}$ located on the diagonal is a diagonal matrix whereas the
off-diagonal block $\tilde{L}^{p,q}$, $p \neq q$, is a full matrix. Moreover, in the particular case of circular scatterers, we have $F_D := \{ k > 0 / \exists (p,m) \in \{1, \ldots, M\} \times \mathbb{N} / J_m(ka_p) = 0\}$ and thus, following (1), $k \in F_D(\Omega)$ if and only if $\tilde{L}_{m,m}^{p,p} = 0$ for some $p = 1, \ldots, M$ and $m \in \mathbb{N}$ (recall that $H_m^{(1)}(x) \neq 0 \forall x \in \mathbb{R}$). For the finite dimensional approximation, we keep only $2N_p + 1$ modes per obstacle $\Omega_p$, $p = 1, \ldots, M$, such that $-N_p \leq m \leq N_p$. The resulting truncated matrix $L \in \mathbb{C}^{N \times N}$ with $N = \sum_{p=1}^{M}(2N_p + 1)$ is then

$$L = \begin{pmatrix}
L_{1,1}^{1,1} & L_{1,1}^{1,2} & \cdots & L_{1,1}^{1,M} \\
L_{2,1}^{1,1} & L_{2,1}^{1,2} & \cdots & L_{2,1}^{1,M} \\
\vdots & \vdots & \ddots & \vdots \\
L_{M,1}^{1,1} & L_{M,1}^{1,2} & \cdots & L_{M,1}^{1,M}
\end{pmatrix},$$

where each $(2N_p + 1) \times (2N_q + 1)$ block $L_{p,q}$ has for coefficient: $L_{m,n}^{p,q} := \tilde{L}_{m,n}^{p,q}$, for $-N_p \leq m \leq N_p$ and $-N_q \leq n \leq N_q$.

3 Low frequency condition number estimates of the single-layer potential for close obstacles: the case of circular cylinders

In this Section, we assume that we have a dense medium: the obstacles are small and close. In other words, setting $a = \min_{p=1,\ldots,M} a_p$ and $b = \min_{p,q=1,\ldots,M, p \neq q} b_{pq}$, we consider that both $ka$ and $kb$ tend towards 0 at the same speed. To analyze this regime, we potentially have two methods. The first one consists in choosing a fixed wave number $k$ and applying a dilation to the geometrical configuration. The second one (that we follow here) is to fix the geometry and to let $k$ tends towards 0. As a result, the radii $a_p$, the centers of the obstacles $O_p$ as well as the distances between the centers $b_{pq}$ are supposed to be constant. Following [7], we consider the limit matrix approach when $k$ tends towards zero to derive some estimates of the eigenvalues $\mu_{\min}$ and $\mu_{\max}$ of the matrix $L$, respectively with smallest and largest modulus. In particular, we show that $\mu_{\min}$ can still be approximated by the eigenvalue with smallest modulus related to single scattering. As a by-product, these approximations allow us to derive condition number estimates.

3.1 The limit matrix approach

We extend the approach introduced in [7] to distant obstacles. We develop an asymptotic analysis of the coefficients of $L$ when $k$ tends towards 0. By using these approximations, we build a limit matrix $L^0 \in \mathbb{C}^{N \times N}$ such that: $L = L^0 + O(k^2 \ln(k))$ and approximate the spectrum of $L$ by the one of $L^0$.

Let us recall that each diagonal block $L_{p,p}$, for $p = 1, \ldots, M$, is a diagonal matrix with coefficients $L_{m,m}^{p,p}$

$$L_{m,m}^{p,p} = \frac{i \pi a_p}{2} J_m(ka_p)H_m^{(1)}(ka_p), \quad \text{for } m = -N_p, \ldots, N_p.$$ 

To simplify, let $I := \{(p,m) \in \mathbb{Z} \text{ such that } 1 \leq p \leq M \text{ and } -N_p \leq m \leq N_p\}$ be the set of all indices $p$ and $m$. From Lemma 1 in [7], we have the following result.

**Lemma 1.** When $k \to 0$ and for every $(p,m) \in I$, the following result holds

$$L_{m,m}^{p,p} = \begin{cases}
(L^0_{m,m})^{p,p} + O\left((k)^2 \ln(k)\right) & \text{for } m = 0, \\
(L^0_{m,m})^{p,p} + O\left((k)^2\right) & \text{for } m \neq 0,
\end{cases} \quad (2)$$

3
\( (L^0)_{m,m} = \begin{cases} -a_p \left[ \ln \left( \frac{k a_p}{2} \right) + \gamma \right] + \frac{i \pi a_p}{2} & \text{for } m = 0, \\ \frac{a_p}{2|m|} & \text{for } m \neq 0, \end{cases} \)

where \( \gamma = 0.5772 \ldots \) is the Euler constant.

Following [7], for \( p = 1, \ldots, M \), we build the diagonal matrix \( (L^0)_{p,p} \) of size \( (2N_p+1) \times (2N_p+1) \), defined by

\[
(L^0)_{p,p} = \begin{pmatrix}
(L^0)_{-N_p,-N_p} & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \cdots & 0 \\
\vdots & \ddots & (L^0)_{0,0} & \ddots & \vdots \\
0 & \ddots & \ddots & 0 & 0 \\
0 & \cdots & 0 & 0 & (L^0)_{N_p,N_p}
\end{pmatrix},
\]

where the coefficients \( (L^0)_{m,m} \) are given by Eq. (3). When \( k \to 0 \), relation (2) implies that \( L_{p,p} = (L^0)_{p,p} + O\left(k^2 \ln(k)\right) \), for \( p = 1, \ldots, M \). Let us remark that the limit matrix \( (L^0)_{p,p} \) is the same as for distant obstacles. This is expected since the difference lies in the parameters \( k b_{pq} \) (\( b_{pq} \) is the distance between \( \Omega_p \) and \( \Omega_q \)) which only appears in the off-diagonal blocks \( L_{p,q} \), for \( p \neq q \).

The asymptotic behaviors of the off-diagonal blocks coefficients \( L_{m,n} \) differ according to the indices \( m \) and \( n \). We split each block \( L_{p,q} \), for \( p \neq q \), into 5 zones, labeled from 0 to 4, as shown on Figure 1. We next develop an asymptotic analysis of the coefficients \( L_{m,n} \) when \( k \to 0 \) with respect to each zone. The results are summarized in the following Lemma.

`Figure 1: Decomposition of \( L_{p,q} \) into five different zones, for \( p \neq q \) in the case where \( N_p = N_q \) (in the general case, the matrix is rectangular and not square).`
Lemma 2. Let \((p, m) \in I\) and \((q, n) \in I\), with \(p \neq q\). When \(k\) tends towards 0, the coefficients \(L_{p,q}^{m,n}\) of the matrix \(L\) have the following asymptotic behavior

- **Zone 0**: \((m = n = 0)\)
  \[
  L_{0,0}^{p,q} = -\sqrt{a_p a_q} \left[ \ln \left( \frac{kb_{pq}}{2} \right) + \gamma \right] + i \pi \sqrt{a_p a_q} + O \left( k^2 \ln(k) \right) .
  \]

- **Zone 1**: \((mn \leq 0\) and \((m, n) \neq (0, 0))\)
  \[
  L_{m,n}^{p,q} = \begin{cases} 
  i(-1)^m \sqrt{a_p a_q} \frac{1}{2} \frac{|m|}{|m||n|!} \left( \frac{a_p}{b_{pq}} \right)^{|m|} e^{i(n-m)\alpha_{pq}} \times O \left( k^2 \ln(k) \right) & \text{if } |m| + |n| = 1, \\
  i(-1)^m \sqrt{a_p a_q} \left( \frac{|m| + |n|-1}{|m||n|!} \right) \left( \frac{a_p}{b_{pq}} \right)^{|m|} \left( \frac{a_q}{b_{pq}} \right)^{|n|} e^{i(n-m)\alpha_{pq}} + O \left( k^2 \right) & \text{otherwise}. 
  \end{cases}
  \]

- **Zone 2**: \((n = m, m \neq 0\) and \(n \neq 0))\)
  \[
  L_{m,m}^{p,q} = O \left( k^{2|m|} \ln(k) \right) .
  \]

- **Zone 3**: \((-1 > m > n\) or \((0 < m < n))\)
  \[
  L_{m,n}^{p,q} = O \left( k^{2|m|} \right) .
  \]

- **Zone 4**: \((0 > n > m\) or \((0 < n < m))\)
  \[
  L_{m,n}^{p,q} = O \left( k^{2|n|} \right) .
  \]

**Proof.** We prove the results zone by zone.

- **Zone 0, \(m = n = 0\)**. In this zone, the coefficients \(L_{m,n}^{p,q}\) are
  \[
  L_{0,0}^{p,q} = \frac{\pi}{2} \sqrt{a_p a_q} J_0(ka_p) H_0^{(1)}(kb_{pq}) J_0(ka_q) .
  \]
  Let us recall that, when \(k \to 0\), the first kind Hankel’s function of order zero has the following expansion (see relations (9.1.8) and (9.1.13) in [1])
  \[
  H_0^{(1)}(kb_{pq}) = 1 + \frac{2}{\pi} \left[ \ln \left( \frac{kb_{pq}}{2} \right) + \gamma \right] + O \left( k^2 \ln(k) \right) ,
  \]
  and the Bessel’s function of order \(m \in \mathbb{N}\) is such that (Eq. (9.1.10) in [1])
  \[
  J_m(ka_q) = \frac{1}{m!} \left( \frac{ka_p}{2} \right)^m + O \left( k^{m+2} \right) .
  \]

- **Zone 1 \((mn \leq 0\) and \((m, n) \neq (0, 0))\)**. Let us introduce the sign function, denoted by \(\text{sgn}\), such that for any \(n \in \mathbb{Z}\), \(\text{sgn}(n) = 1\) if \(n \geq 0\) and \(\text{sgn}(n) = -1\) if \(n < 0\). Using some properties of the special functions, we can write the coefficients \(L_{m,n}^{p,q}\) under the following form
  \[
  L_{m,n}^{p,q} = (\text{sgn}(m))^m (\text{sgn}(n))^n (\text{sgn}(n - m))^{n-m} \frac{\pi}{2} \sqrt{a_p a_q} e^{i(n-m)\alpha_{pq}} J_m(ka_p) H_{|m|-m}^{(1)}(kb_{pq}) J_{|n|}(ka_q) .
  \]
Let us note that the indices of the Bessel’s and Hankel’s functions are now positive. For the coefficients in this zone, the indices \( m \) and \( n \) satisfy \( mn \leq 0 \). Since \( |n - m| = |n| + |m| \), sgn\((m) = -\text{sgn}(n)\) and sgn\((n - m) = \text{sgn}(n)\), we have

\[
\left(\text{sgn}(m)\right)^m \left(\text{sgn}(n)\right)^n \left(\text{sgn}(n - m)\right)^{n-m} = -\left(\text{sgn}(n)\right)^m \left(\text{sgn}(n)\right)^n \left(\text{sgn}(n)\right)^{n+m} = (-1)^m.
\]

By using (8), the coefficients \( \mathbb{L}_{m,n}^{p,q} \) in this zone can be written

\[
\mathbb{L}_{m,n}^{p,q} = (-1)^m \frac{\sqrt{a_p a_q}}{2} e^{i(n-m)\alpha_{pq}} J_{|n|}(ka_p) H_{m+|n|}^{(1)}(kb_{pq}) J_{|n|}(ka_q).
\]  

Let us recall the asymptotic expansions of the first kind Hankel’s functions of order \( m > 0 \) when \( k \to 0 \) (see relations (9.1.9) and (9.1.11) in [1])

\[
H_{m}^{(1)}(kb_{pq}) = \frac{(m-1)!}{\pi} \left(\frac{kb_{pq}}{2}\right)^{-m} + O \left( f_m(k) \right),
\]  

where the functions \( f_m \) are defined by: \( f_1(k) = k \ln(k) \) and \( f_m(k) = k^{2-m} \) for \( m \geq 2 \). Next, we use the expansions (7) and (10) of the Bessel’s and Hankel’s functions into the expression (9) to get

\[
\mathbb{L}_{m,n}^{p,q} = i(-1)^m \frac{\sqrt{a_p a_q}}{2} e^{i(n-m)\alpha_{pq}} \left[ \frac{1}{|m|!} \left(\frac{ka_p}{2}\right)^{|m|} + O \left( k^{|m|+2} \right) \right] \left[ \frac{1}{|n|!} \left(\frac{ka_q}{2}\right)^{|n|} + O \left( k^{|n|+2} \right) \right].
\]

Let us note that we can use the asymptotic relation (10) of the Hankel’s functions of order \( |m| + |n| \). Indeed, for the coefficients in this zone, the indices \((m, n)\) satisfy \( mn \leq 0 \) and \((m, n) \neq (0, 0)\), which in particular implies that \( |m| + |n| \neq 0 \). Hence we study the Hankel’s functions with non null index. We then develop the previous relation to obtain

\[
\mathbb{L}_{m,n}^{p,q} = i(-1)^m \frac{\sqrt{a_p a_q}}{2} e^{i(n-m)\alpha_{pq}} \left[ \frac{|m| + |n| - 1}{|m|!|n|!} \left(\frac{ka_p}{2}\right)^{|m|} \left(\frac{ka_q}{2}\right)^{|n|} \left(\frac{2}{kb_{pq}}\right)^{|m|+|n|} + O \left( k^{|m|+|n|+2-|m|-|n|} \right) + O \left( k^{|m|+|n|} f_{m+|n|}(k) \right) \right].
\]

After some simplifications, we have

\[
\mathbb{L}_{m,n}^{p,q} = i(-1)^m \frac{\sqrt{a_p a_q}}{2} \left(\frac{a_p}{b_{pq}}\right)^{|m|} \left(\frac{a_q}{b_{pq}}\right)^{|n|} e^{i(n-m)\alpha_{pq}} + O \left( k^2 \right) + O \left( k^{|m|+|n|} f_{m+|n|}(k) \right). \]  

From the definition of the functions \( f_{|m|+|n|} \), we have

\[
k^{|m|+|n|} f_{|m|+|n|}(k) = \begin{cases} k^2 \ln(k) & \text{if } |m| + |n| = 1, \\ k^2 & \text{otherwise}. \end{cases}
\]

By injecting these relations into (11), we obtain the expected relation.
• Zone 2, \((n = m, m \neq 0 \text{ and } n \neq 0)\). To prove the relations in the zones 2, 3 and 4, we only need to analyze the modulus of the coefficient \(L_{m,n}^{p,q}\)

\[
\|L_{m,n}^{p,q}\| = \frac{\sqrt{a_p a_q}}{2} |J_{|m|}(k a_p)| \left| H_{|n-m|}^{(1)}(k b_{pq}) \right| |J_{|n|}(k a_q)|. \tag{12}
\]

For zone 2, we have \(|n - m| = 0\), with \(m \neq 0\) and \(n \neq 0\) and then

\[
\|L_{m,n}^{p,q}\| = \frac{\sqrt{a_p a_q}}{2} |J_{|m|}(k a_p)| \left| H_{|0|}^{(1)}(k b_{pq}) \right| |J_{|n|}(k a_q)|.
\]

By using the asymptotics (7) and (6) we obtain

\[
\|L_{m,n}^{p,q}\| = \left| \frac{\pi \sqrt{a_p a_q}}{2} \left( k a_p \right)^{|m|} + O \left( k^{|m|+2} \right) \right| \left[ 1 + i \frac{2}{\pi} \ln \left( \frac{k b_{pq}}{2} \right) + O \left( k^2 \ln(k) \right) \right]
\]

\[
+ \left| \frac{\left( k a_q \right)^{|n|}}{2} + O \left( k^{|n|+2} \right) \right|.
\]

This directly proves that \(L_{m,n}^{p,q} = O \left( k^{2|m|} \ln(k) \right)\).

• Zone 3: since \((0 > m > n)\) or \((0 < m < n)\), we have \(|n - m| = |n| - |m|\) and

\[
\|L_{m,n}^{p,q}\| = \frac{\sqrt{a_p a_q}}{2} |J_{|m|}(k a_p)| \left| H_{|n|-|m|}^{(1)}(k b_{pq}) \right| |J_{|n|}(k a_q)|.
\]

By using (7) and (10) one gets

\[
\|L_{m,n}^{p,q}\| = \left| \frac{\pi \sqrt{a_p a_q}}{2} \left( \frac{1}{|m|!} \frac{k a_p}{2} \right)^{|m|} + O \left( k^{|m|+2} \right) \right|
\]

\[
\left( \frac{|n| - |m| - 1)!}{\pi} \left( \frac{k b_{pq}}{2} \right)^{|n|+|m|} + O \left( f_{|n|-|m|}(k) \right) \right]
\]

and obtain: \(L_{m,n}^{p,q} = O \left( k^{2|m|} \right)\).

• Zone 4: since \((0 > n > m)\) or \((0 < n < m)\), we have: \(|n - m| = |m| - |n|\). Let us point out that, by changing the role of \(m\) and \(n\), we recover the results of zone 3. Thus, the proof developed above can be adapted here to directly obtain \(L_{m,n}^{p,q} = O \left( k^{2|m|} \right)\).

\[
\square
\]

For any \(1 \leq p \neq q \leq M\), let us introduce the limit matrix \((L^0)^{p,q}\) with coefficients

\[
(L^0)^{p,q}_{m,n} = \begin{cases} 
- \sqrt{a_p a_q} \left( \ln \left( \frac{k b_{pq}}{2} \right) + \gamma \right) + i \frac{\pi \sqrt{a_p a_q}}{2} & \text{if } (m, n) = (0, 0), \\
(-1)^m \frac{\sqrt{a_p a_q} \left( |m| + |n| - 1 \right)!}{|m|!|n|!} \left( \frac{a_p}{b_{pq}} \right)^{|m|} \left( \frac{a_q}{b_{pq}} \right)^{|n|} e^{i(n-m)\alpha_{pq}} & \text{if } (m, n) \neq (0, 0) \text{ and } mn \leq 0, \\
0 & \text{otherwise},
\end{cases}
\tag{13}
\]

for \(-N_p \leq m \leq N_p\) and \(-N_q \leq n \leq N_q\). Moreover, from Lemma 2, we have the following relation

\[
L^{p,q} = (L^0)^{p,q} + O \left( k^2 \ln(k) \right), \quad \text{for } p \neq q.
\tag{14}
\]
Let us now introduce the block matrix $\mathbb{L}^0$ containing each submatrix $(\mathbb{L}^0)^{p,q}$ and defined by

$$
\mathbb{L}^0 = \begin{pmatrix}
(\mathbb{L}^0)^{1,1} & (\mathbb{L}^0)^{1,2} & \cdots & (\mathbb{L}^0)^{1,M} \\
(\mathbb{L}^0)^{2,1} & (\mathbb{L}^0)^{2,2} & \cdots & (\mathbb{L}^0)^{2,1} \\
\vdots & \vdots & \ddots & \vdots \\
(\mathbb{L}^0)^{M,1} & (\mathbb{L}^0)^{M,2} & \cdots & (\mathbb{L}^0)^{M,M}
\end{pmatrix}.
$$

By using (4) and (14), the following proposition holds.

**Proposition 3.** When $k$ tends towards zero, the truncated matrix $\mathbb{L} \in \mathbb{C}^{N \times N}$ of the single-layer operator $L$ satisfies the relation

$$
\mathbb{L} = \mathbb{L}^0 + O \left( k^2 \ln(k) \right).
$$

(15)

To visualize the relatively sparse structure of $\mathbb{L}^0$, we represent on Figure 2 in grey color the non zero coefficients (skeleton of $\mathbb{L}^0$) in the case of two circular scatterers.

![Figure 2: Non null coefficients of the matrix $\mathbb{L}^0$ for a configuration with two obstacles and in the square case $N_1 = N_2$. The grey zones correspond to the non zero coefficients.](image)

Figure 2: Non null coefficients of the matrix $\mathbb{L}^0$ for a configuration with two obstacles and in the square case $N_1 = N_2$. The grey zones correspond to the non zero coefficients.

From now on, let us respectively denote by $(\mu^p_m)_{(p,m) \in I}$ and $((\mu^0)^p_m)_{(p,m) \in I}$ the eigenvalues of $\mathbb{L}$ and $\mathbb{L}^0$. For $(p,m) \in I$, we moreover assume that

$$
\mu^p_m \simeq (\mu^0)^p_m,
$$

(16)

which is coherent with (15). To motivate our approach, we compare the eigenvalues of $\mathbb{L}$ and $\mathbb{L}^0$ on Figures 3(b)-3(d), for $k = 0.1$ and 30 randomly distributed disks of radius $a_p = 0.1$ in $[0,4]^2$ (see the example of Figure 3(a)). For $p, q = 1, \ldots, M$, with $p \neq q$, $N_p = 5$ and the intercenter distance $b_{pq}$ is such that: $b_{pq} \geq 0.3(= 3a_p)$. The eigenvalues are computed by the Matlab function `eig`. We observe that the approximation of the eigenvalues $(\mu^p_m)_{(p,m) \in I}$ of $\mathbb{L}$ by the eigenvalues $((\mu^0)^p_m)_{(p,m) \in I}$ of $\mathbb{L}^0$ is satisfactory. Let us denote by $\mu_{\text{max}}$ and $\mu^0_{\text{max}}$ the eigenvalues with largest

8
modulus of respectively \( L \) and \( L^0 \). On Figure 3(b), we observe that they are very close which is numerically confirmed since: \( \mu_{\text{max}} \simeq 5.393 + 4.623i \) and \( \mu_{\text{max}}^0 \simeq 5.494 + 4.695i \). This implies that: \( |\mu_{\text{max}}| \simeq 7.103 \) and \( |\mu_{\text{max}}^0| \simeq 7.227 \). Furthermore, the eigenvalues with smallest modulus of \( L \) and \( L^0 \), respectively denoted by \( \mu_{\text{min}} \) and \( \mu_{\text{min}}^0 \), are also very close (see Figure 3(d)). The numerical values confirm this result: \( \mu_{\text{min}} \simeq 0.01 + 10^{-13}i \), and \( \mu_{\text{min}}^0 \simeq 0.01 + 10^{-13}i \).

![Figure 3: Comparison of the eigenvalues of \( L \) and \( L^0 \) for 30 randomly and identical distributed disks of radius \( a_p = 0.1 \) in \([0, 4]^2\), \( N_p = 5 \) for each obstacle \((k = 0.1 \) and \( b \geq 0.3(= 3a_p)\)].

This example confirms that our approach seems reasonable. Further simulations have been performed and show that the eigenvalues of \( L^0 \) are close to the ones associated with \( L \). To simplify the computations, we assume that, for any \( p = 1, \ldots, M \), we have \( a_p = a \). Following [4], a good choice of the parameters \( N_p \) only involves the dimensionless quantities \( ka_p \). Thus, as we set \( a = a_p \) for any \( p \), we also assume that \( N = N_p \) for \( p = 1, \ldots, M \). With these assumptions, matrices \( L \) and \( L^0 \) are now of size \( M(2N + 1)^2 \). For \( p = 1, \ldots, M \), the diagonal blocks \( (L^0)^{p,p} \in \mathbb{C}^{(2N+1)\times(2N+1)} \) are square and independent of \( p \). Moreover, we have \((L^0)_{m,m}^{p,q} = (L^0)_{m,m}^{q,q} = (L^0)_{m,m}^{q,-q} = (L^0)_{m,m}^{q,-q} = (L^0)_{m,m}^{q,-q} = (L)_{m,m}^{q,-q} = (L^0)_{m,m}^{q,q}, \) for \((p,m) \in I \) and \( q = 1, \ldots, M \). These terms are now denoted
by $\hat{\mu}_m$

$$\forall p = 1, \ldots, M, \forall m = 0, \ldots, N, \quad \hat{\mu}_m = (L^0)_{\hat{p}, \hat{m}} = \begin{cases} -a \left[ \ln \left( \frac{ka}{2} \right) + \gamma \right] + \frac{i\pi a}{2M}, & \text{if } m = 0, \\ \frac{a^2}{2m}, & \text{otherwise.} \end{cases}$$

### 3.2 Estimates of the eigenvalue with smallest modulus

Let us now estimate the eigenvalue with smallest modulus $\mu_{\text{min}}^0$ of $L^0$. Like for the case of distant obstacles [7], an approach based on the Gershgorin-Hadamard discs has been developed but provides inaccurate estimates of the eigenvalue with smallest modulus. In addition, the obtained approximation for the largest eigenvalue is not precise, most particularly for many obstacles.

Like in [7], we show that the eigenvalue of $L^0$ (and next $L^1$) with smallest modulus can be approximated correctly by the associated single scattering eigenvalue: $\hat{\mu}_N = a/(2N)$, with multiplicity $2M$. To prove this result, we build an approximate eigenvector of $L^0$ for the approximate eigenvalue $\hat{\mu}_{|m|}$, for large $|m|$. Let us introduce the vector $X^{(p,m)} \in \mathbb{C}^{N \times 1}$, for $(p, m) \in I$, given by

$$X^{(p,m)} = \begin{pmatrix} X^{(p,m)}_1 \\ X^{(p,m)}_2 \\ \vdots \\ X^{(p,m)}_M \end{pmatrix}.$$ 

Each block $X^{(p,m)}_q = (X^{(p,m)}_{q,n})_{n=-N, \ldots, N}$, for $q = 1, \ldots, M$, has size $2N + 1$ and its components are defined by the following relations

$$X^{(p,m)}_{p,n} = \delta_{pq}\delta_{mn}. \tag{17}$$

Hence, the blocks $X^{(p,m)}_q$ are zero for $q \neq p$, and $X^{(p,m)}_p = (X^{(p,m)}_{p,n})_{n=-N, \ldots, N}$ has only one nonzero component localized at $n = m$ (and equal to 1). Let us remark that $X^{(p,m)}$ is also normalized for both the infinity and euclidian norms. We introduce now $Y^{(p,m)}$ as the vector resulting from the matrix-vector product between $L^0$ and $X^{(p,m)}$

$$Y^{(p,m)} = L^0 X^{(p,m)} = \begin{pmatrix} Y^{(p,m)}_1 \\ Y^{(p,m)}_2 \\ \vdots \\ Y^{(p,m)}_M \end{pmatrix} = \begin{pmatrix} (L^0)_{1,p} X^{(p,m)}_p \\ (L^0)_{2,p} X^{(p,m)}_p \\ \vdots \\ (L^0)_{M,p} X^{(p,m)}_p \end{pmatrix}.$$

Let us focus on the vectorial block $Y^{(p,m)}_p$. From (4), we have

$$Y^{(p,m)}_{p,n} = \begin{cases} (L^0)_{\hat{p}, \hat{m}} = \hat{\mu}_{|m|} & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, relation (17) implies that $Y^{(p,m)}_p = \hat{\mu}_{|m|} X^{(p,m)}_p$. Let us now analyze the vectors $Y^{(p,m)}_q = (L^0)_{\hat{q}, \hat{p}} X^{(p,m)}_p$, for $q = 1, \ldots, M$ and $q \neq p$. By using the particular structure of $X^{(p,m)}_p$ (see (17)), the coefficients $Y^{(p,m)}_{q,n}$ satisfy

$$Y^{(p,m)}_{q,n} = (L^0)_{\hat{q}, \hat{n}} X^{(p,m)}_p, \quad \text{for } -N \leq n \leq N.$$

10
If \( \| \cdot \|_2 \) denotes the usual 2-norm, we then have the following equality

\[
\| L_0 X^{(p,m)} - \hat{\mu}_{|m|} X^{(p,m)} \|_2^2 = \sum_{q=1}^{M} \sum_{n=-N}^{N} |(L_0)_{pq}^{p,q} |^2.
\] (18)

Our goal is to prove that \( X^{(p,m)} \) is an approximate eigenvector of \( L_0 \), with approximate eigenvalue \( \hat{\mu}_{|m|} \), for \( |m| \) sufficiently large. To this end, we need to find an upper bound for

\[
\| L_0 X^{(p,m)} - \hat{\mu}_{|m|} X^{(p,m)} \|_2^2.
\]

Indeed, this term measures the error related to the approximation of an eigenvector of \( L_0 \) by \( X^{(p,m)} \) with approximate eigenvalue \( \hat{\mu}_{|m|} \). We first have

\[
\| L_0 X^{(p,m)} - \hat{\mu}_{|m|} X^{(p,m)} \|_2^2 \leq \left( \sum_{q=1}^{M} \sum_{n=-N}^{N} |(L_0)_{pq}^{p,q} |^2 \right)^2. \tag{19}
\]

Let us now state the following Lemma.

**Lemma 4.** Let \( (p,m) \in I \) and \( q = 1, \ldots, M \), such that \( p \neq q \) and \( m \neq 0 \). The following inequality holds

\[
\sum_{n=-N}^{N} |(L_0)_{pq}^{q,p} | \leq \frac{a}{2|m|} \left( \frac{a}{b_{pq}} \right)^{|m|} \left( \frac{a}{b_{pq}} \right)^{|n|} . \tag{20}
\]

**Proof.** Let us consider three integers \( p, q = 1, \ldots, M \) and \( m = -N, \ldots, N \) such that \( p \neq q \) and \( m \neq 0 \). From definition (13) of the coefficients \( (L_0)_{pq}^{p,q} \), we have

\[
\sum_{n=-N}^{N} |(L_0)_{pq}^{q,p} | = \frac{a}{2|m|} \left( \frac{a}{b_{pq}} \right)^{|m|} \sum_{n=0}^{N} \frac{|m| + |n| - 1)!}{|m|!|n|!} \left( \frac{a}{b_{pq}} \right)^{|n|} .
\]

Since \( |m|! = |m|(|m| - 1)! \), we can write

\[
\sum_{n=-N}^{N} |(L_0)_{pq}^{q,p} | = \frac{a}{2|m|} \left( \frac{a}{b_{pq}} \right)^{|m|} \sum_{n=0}^{N} \frac{|m| + |n| - 1)!}{|m|!|n|!} \left( \frac{a}{b_{pq}} \right)^{|n|} .
\]

Then one gets

\[
\sum_{n=-N}^{N} |(L_0)_{pq}^{q,p} | = \frac{a}{2|m|} \left( \frac{a}{b_{pq}} \right)^{|m|} \sum_{n=0}^{N} \left( \frac{|m| + |n| - 1)!}{|m|!|n|!} \right) \left( \frac{a}{b_{pq}} \right)^{|n|} , \tag{21}
\]

where

\[
\left( \frac{|m| + |n| - 1)!}{|m|!|n|!} \right) = \left( \frac{|m| + |n| - 1)!}{(|m| - 1)!|n|!} .
\]
Since $a < b_{pq}$, the series indexed by $n$ appearing in the right-hand side of the equality (21) is converging when $N$ tends towards infinity. More precisely, we have

$$
\lim_{N \to \infty} \sum_{n=0}^{N} \left( \left| m \right| + \left| n \right| - 1 \right) \left( \frac{a}{b_{pq}} \right)^{\left| n \right|} = \left( 1 - \frac{a}{b_{pq}} \right)^{-\left| m \right|} = \left( \frac{b_{pq}}{b_{pq} - a} \right)^{\left| m \right|}.
$$

Furthermore, since each term of the series is positive, the following inequality is fulfilled for any $N \in \mathbb{N}$

$$
\sum_{n=0}^{N} \left( \left| m \right| + \left| n \right| - 1 \right) \left( \frac{a}{b_{pq}} \right)^{\left| n \right|} \leq \left( \frac{b_{pq}}{b_{pq} - a} \right)^{\left| m \right|}.
$$

By using the upper bound in (21), we finally obtain

$$
\sum_{n=-N}^{N} |(L^0)_{n,m}| \leq \frac{a}{2|m|} \left( \frac{a}{b_{pq} - a} \right)^{|m|}.
$$

This Lemma leads to a fine upper bound of the euclidian norm

$$
\left\| L^0 X^{(p,m)} - \hat{\mu}_{|m|} X^{(p,m)} \right\|_2^2 \leq \frac{a}{2|m|} \left( \frac{a}{b_{pq} - a} \right)^{|m|},
$$

and shows that $X^{(p,m)}$ can be used as an approximate eigenvector of $L^0$. We summarize these results in the following proposition.

**Proposition 5.** Let $(p,m) \in I$ with $m \neq 0$. The vector $X^{(p,m)}$ defined by relation (17) is an approximate eigenvector of $L^0$ with approximate eigenvalue $\hat{\mu}_{|m|}$, in the sense that the relative error between $L^0 X^{(p,m)}$ and $\hat{\mu}_{|m|} X^{(p,m)}$ satisfies the following upper bound

$$
\left\| L^0 X^{(p,m)} - \hat{\mu}_{|m|} X^{(p,m)} \right\|_2 \leq \frac{M}{\sum_{q=1 \atop q \neq p}^{M} \left( \frac{a}{b_{pq} - a} \right)^{|m|}}.
$$

**Proof.** By applying Lemma 4 to the first inequality (19) for $\left\| L^0 X^{(p,m)} - \hat{\mu}_{|m|} X^{(p,m)} \right\|_2$ we obtain

$$
\left\| L^0 X^{(p,m)} - \hat{\mu}_{|m|} X^{(p,m)} \right\|_2 \leq \left( \sum_{q=1 \atop q \neq p}^{M} \frac{a}{2|m|} \left( \frac{a}{b_{pq} - a} \right)^{|m|} \right)^2.
$$

Since all the terms with index $q$ appearing in the sum are positive, we can take the square-root of the inequality to get

$$
\left\| L^0 X^{(p,m)} - \hat{\mu}_{|m|} X^{(p,m)} \right\|_2 \leq \frac{a}{2|m|} \sum_{q=1 \atop q \neq p}^{M} \left( \frac{a}{b_{pq} - a} \right)^{|m|}.
$$
Finally, we use \( \hat{\mu}[m] = \frac{a}{2|m|} \) and obtain

\[
\frac{\|L^0X^{(p,m)} - \hat{\mu}[m]X^{(p,m)}\|_2}{\hat{\mu}[m]} \leq \sum_{q=1}^{M} \left( \frac{a}{b_{pq} - a} \right)^{|m|}.
\]

For \((p,m) \in I\) with \(m \neq 0\) and \(|m|\) sufficiently large, this proposition shows that \(X^{(p,m)}\) is an approximate eigenvector of \(L^0\) with approximate eigenvalue \(\hat{\mu}[m]\). Moreover, the \(2M\) vectors \((X^{(p,m)})_{p=1,...,M}\) and \((X^{(p,-m)})_{p=1,...,M}\) are approximate eigenvectors associated with the same approximate eigenvalue \(\hat{\mu}[m]\). Hence, for \(|m|\) large enough, the quantity \(\hat{\mu}[m]\) is an approximate eigenvalue of \(L^0\) with multiplicity \(2M\). The sequence \((\hat{\mu}[m])_{m \geq 1} = (\frac{a}{2M})_{m \geq 1}\) decays and tends towards 0 when \(m\) tends to infinity. The term \(\hat{\mu}_N\) is then the smallest approximate eigenvalue \((\hat{\mu}[m])_{m \geq 1}\). Furthermore, \(\hat{\mu}_N\) tends to 0 when \(N\) tends to infinity. This is the reason why we estimate \(\mu_{min}^0\), and next \(\mu_{min}\), by \(\hat{\mu}_N\) with a multiplicity equal to \(2M\), that is

\[
\mu_{min} \simeq \mu_{min}^0 \simeq \hat{\mu}_N. \tag{23}
\]

Let us remark that \(\hat{\mu}_N\) is also the approximation of the smallest eigenvalue in the framework of single scattering as well as multiple scattering for distant scatterers [7]. This approximation is more accurate as \(N\) is large and that the coupling between the obstacles is weak, that is when the obstacles are not too close. Indeed, the larger the distance \(b_{pq}\) is, the smaller the left hand side of the inequality (22) is. Similarly, the right hand side term of (22) (with \(|m| = N\)) is smaller as \(N\) is larger. Let us now come back to the example in Figure 3. The parameters were: 30 randomly distributed disks of radius 0.1 in \([0,4]^2\), \(N = 5\), \(k = 0.1\) and \(b_{pq} \geq 0.3(=3a)\). The numerical computation of the smallest eigenvalues of \(L\) and \(L^0\) provide \(\mu_{min} \approx 0.01 + 10^{-13}i\) and \(\mu_{min}^0 \approx 0.01 + 10^{-13}i\). Our estimate gives \(\hat{\mu}_N = 0.01\), with a small relative error on \(\mu_{min}\) equal to

\[
100\frac{|\mu_{min} - \hat{\mu}_N|}{|\mu_{min}|} \approx 0.08%.
\]

**Remark 6.** For distances \(b_{pq}\) closer to \(2a\), the parameter \(N\) must be chosen large enough, otherwise the approximation \(\hat{\mu}_N\) of the smallest eigenvalue \(\mu_{min}\) would be inaccurate. When \(N\) is set to 5, as is the case in our simulations, satisfactory results have been observed for \(b_{pq} \geq 0.3(=3a)\).

### 3.3 Estimates of the eigenvalue with largest modulus

Unlike the previous case, we cannot construct an approximate eigenvector of \(L^0\) to provide an estimate of the largest eigenvalue \(\mu_{max}^0\) of \(L^0\). By analyzing the expression (13) of the coefficients of the limit matrix \(L^1\), none of them depends on \(k\), except the coefficients associated with the indices \(m = n = 0\) which have a logarithmic growth with respect to \(k\) (zone 0 in Lemma 2). When the wavenumber \(k\) is small enough, the information related to the largest eigenvalue is *a priori* contained in these coefficients.

As in [7], we propose to extract the elements in \(L^0 \in \mathbb{C}^{(2N+1)\times(2N+1)}\) referring to the zero order modes and collect them in the matrix \(L^1 \in \mathbb{C}^{N \times N}\). From relation (13), this \(M \times M\) matrix \(L^1\) is
defined, for \( p, q = 1, \ldots, M \), by

\[
(L^1)^{p,q} = (L^0)_{p,q}^{0,0} = \begin{cases} 
-a \left[ \ln \left( \frac{ka}{2} \right) + \gamma \right] + \frac{i \pi a}{2}, & \text{if } p = q, \\
-a \left[ \ln \left( \frac{k_{pq}}{2} \right) + \gamma \right] + \frac{i \pi a}{2}, & \text{if } p \neq q.
\end{cases}
\]  

(24)

Let us denote by \( \mu_{max}^0 \), respectively \( \mu_{max}^1 \), the eigenvalue with largest modulus of \( L^0 \), respectively \( L^1 \). We then estimate the largest eigenvalue of \( L^0 \) by the one of \( L^1 \), that is:

\[
\mu_{max}^0 \simeq \mu_{max}^1.
\]

Next, from (16), we also estimate \( \mu_{max} \) by \( \mu_{max}^1 \). We compared three approaches to estimate \( \mu_{max}^1 \).

Two consist in bounding the spectral radius of \( L^1 \), and so the modulus of \( \mu_{max}^1 \), by computing the Frobenius norm of \( L^1 \) or by applying the Gershgorin’s discs theorem to \( L^1 \). In both cases, the result was less accurate than for the third approach. Moreover, the expression of the estimate obtained by this last method is simpler. This is the reason why we only present this approach here.

The principle of our approach is to obtain a mean distance \( d \) related to the inter center distances \( b_{pq} \). Suppose that this “equivalent” distance \( d \) satisfies the inequality

\[
d \geq \min_{1 \leq p \neq q \leq M} b_{pq},
\]

and define the matrix \( L^1_{eqv} \) of size \( M \times M \) through two parameters \( l_0 \) and \( l_1 \) by

\[
L^1_{eqv} = \begin{pmatrix}
  l_0 & l_1 & l_1 & \ldots & l_1 \\
l_1 & l_0 & l_1 & \ldots & l_1 \\
l_1 & l_1 & l_0 & \ldots & l_1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
l_1 & \ldots & l_1 & l_1 & l_0
\end{pmatrix},
\]

with

\[
\begin{align*}
  l_0 &= (L^1_{eqv})_{0,0}^0 = -a \left[ \ln \left( \frac{ka}{2} \right) + \gamma \right] + \frac{i \pi a}{2}, \\
  l_1 &= (L^1_{eqv})_{1,1}^1 = -a \left[ \ln \left( \frac{kd}{2} \right) + \gamma \right] + \frac{i \pi a}{2},
\end{align*}
\]

(26)

(the coefficients of the matrix \( L^1 \) are obtained by replacing \( b_{pq} \) by \( d \) in (24)). The main property of this matrix is that we explicitly know its eigenvalues, and most particularly its largest one, accordingly to the next Lemma.

**Lemma 7.** The eigenvalues of \( L^1_{eqv} \) are given by

- \( \mu_1^1 = l_0 + (M - 1)l_1 = -a \left[ \ln \left( \frac{ka}{2} \right) + (M - 1) \ln \left( \frac{kd}{2} \right) \right] - aM\gamma + i\frac{M\pi a}{2} \), with multiplicity 1,

- \( \mu_2^1 = l_0 - l_1 = a \ln \left( \frac{d}{a} \right) \), with multiplicity \( (M - 1) \).

Moreover, for \( kd < 1 \), we have the following inequality

\[
|\mu_1| \geq |\mu_2|.
\]

(27)
Proof. The characteristic polynomial of $\mathbb{I}_{eqv}^1$ is given by: $P(\mu) = \text{det}(\mathbb{I}_{eqv}^1 - \mu \mathbb{I})$, where $\mathbb{I}$ is the $M \times M$ identity matrix,

$$
\text{det}(\mathbb{I}_{eqv}^1 - \mu \mathbb{I}) = \text{det} \left( \begin{array}{cccc}
  l_0 - \mu & l_1 & l_1 & \cdots & l_1 \\
  l_1 & l_0 - \mu & l_1 & \cdots & l_1 \\
  l_1 & l_1 & l_0 - \mu & \cdots & l_1 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  l_1 & \cdots & l_1 & l_0 - \mu \\
\end{array} \right).
$$

By subtracting from the first row to rows 2, 3, $\ldots$, $M$, and next adding to the first column the other $M - 1$ columns, we easily obtain the expression

$$
\text{det}(\mathbb{I}_{eqv}^1 - \mu \mathbb{I}) = (-1)^M [\mu - (l_0 + (M - 1)l_1)] [\mu - (l_0 - l_1)]^{M-1}.
$$

We then deduce the eigenvalues of $\mathbb{I}_{eqv}^1$

$$
\mu_1^1 = l_0 + (M - 1)l_1 = -a \left[ \ln \left( \frac{ka}{2} \right) + (M - 1) \ln \left( \frac{kd}{2} \right) \right] - aM\gamma + i\frac{M\pi a}{2},
$$

with multiplicity 1, and $\mu_2^1 = l_0 - l_1 = a \ln \left( \frac{d}{2} \right)$, with multiplicity $M - 1$.

Let us now state the inequality (27). We first remark that the eigenvalue $\mu_2^1$ is real. From relation (26) and since $d > a$, $\mu_2^1$ is positive. It remains to prove that the real part of $\mu_1^1$ is larger than $\mu_2^1$. Let first remark that $\mu_2^1 = \Re(l_0) - \Re(l_1)$ since $\mu_2^1$ is real. We next compare the real part of $\mu_1^1$ with $\mu_2^1$:

$$
\Re(\mu_1^1) - \mu_2^1 = (M - 2)\Re(l_1). \tag{28}
$$

Since $M - 2 \geq 0$, $\Re(\mu_1^1) - \mu_2^1$ and $\Re(l_1)$ have the same sign. Let us prove that $\Re(l_1)$ is positive. The real part of $l_1$ is: $\Re(l_1) = -a \left[ \ln \left( \frac{kd}{2} \right) - \gamma \right]$. Let us recall that: $-\ln(0.5) \simeq 0.69 > \gamma \simeq 0.58$. We assume that we have $kd < 1$ (since $k$ tends towards 0). Since $\ln$ is an increasing function, we get: $-\ln \left( \frac{kd}{2} \right) > \gamma > 0$, that is $\Re(l_1) = a \left[ -\ln \left( \frac{kd}{2} \right) - \gamma \right] \geq 0$. By using (28), we have $\Re(\mu_1^1) - \mu_2^1 \geq 0$, and next $|\mu_1^1| \geq |\mu_2^1|$.

This Lemma shows that the eigenvalue $\mu_{\max}^{1, eqv}$ of $\mathbb{I}_{eqv}^1$ with largest modulus has a multiplicity equal to 1 and is given by

$$
\mu_{\max}^{1, eqv} = l_0 + (M - 1)l_1 = -a \left[ \ln \left( \frac{ka}{2} \right) + (M - 1) \ln \left( \frac{kd}{2} \right) \right] - aM\gamma + i\frac{M\pi a}{2}. \tag{29}
$$

We now estimate $\mu_{\max}$ by $\mu_{\max}^{1, eqv}$, that is

$$
\mu_{\max} \simeq -a \left[ \ln \left( \frac{ka}{2} \right) + (M - 1) \ln \left( \frac{kd}{2} \right) \right] - aM\gamma + i\frac{M\pi a}{2}.
$$

We propose to choose $d$ as: $d = \frac{b_{\min} + b_{\max}}{2}$. The term $b_{\min}$, respectively $b_{\max}$, represents the smallest, respectively largest, possible distance $b_{pq}$ between the centers of two obstacles. When the obstacles are contained in a rectangular box of sides $\ell$ and $L$, we fix $b_{\min}$ and $b_{\max}$ as: $b_{\min} = 2a$ and $b_{\max} = \sqrt{\ell^2 + L^2} - 2a$. Finally, $d$ is defined as the average of $b_{\min}$ and $b_{\max}$

$$
d = \frac{b_{\min} + b_{\max}}{2} = \frac{\sqrt{\ell^2 + L^2}}{2}. \tag{30}
$$
Let us remark that $d$ can also be seen as the half diagonal of the box with sidelengths $\ell$ and $L$. In particular, parameters $b_{\text{min}}$ and $b_{\text{max}}$ (and thus $d$) are independent of the placement of the scatterers and are fictitious ($b_{pq} = 2a$ implies that obstacle $\Omega_p$ and $\Omega_q$ have an intersection point). However, this choice gives satisfactory results and avoids the computation of the inter-center distances of the obstacles. Let us also remark that, in our simulations, the “practical” minimal and maximal distances between the centers of two scatterers will not be related to $b_{\text{min}}$ and $b_{\text{max}}$ (see also Remark 6).

Let us come back again to the example presented in Figure 3. The numerical computation of the largest eigenvalue $\mu_{\text{max}}$ of $L$ was $\mu_{\text{max}} \simeq 5.39 + 4.62i$, with $|\mu_{\text{max}}| \simeq 7.1$. By using relation (30) for $d$, one gets $d = 2\sqrt{2}$. The proposed formula (29) then gives $\mu_{\text{max}}^{\text{equiv}} \simeq 4.47 + 4.71i$, with $|\mu_{\text{max}}^{\text{equiv}}| \simeq 6.5$. The relative error when estimating $\mu_{\text{max}}$ by $\mu_{\text{max}}^{\text{equiv}}$ is then equal to

$$100 \frac{|\mu_{\text{max}} - \mu_{\text{max}}^{\text{equiv}}|}{|\mu_{\text{max}}|} = 13.04\%,$$

and, in term of the modulus, to

$$100 \frac{|\mu_{\text{max}}| - |\mu_{\text{max}}^{\text{equiv}}|}{|\mu_{\text{max}}|} = 8.4\%,$$

which means that our approach is consistent. Moreover, let us point out that, unlike the dilute medium case [7], we propose here an estimate of the eigenvalue $\mu_{\text{max}}$ and not only of its modulus. We launched 100 tests respecting the same parameters set ($k = 0.1$, $N = 5$, 30 randomly distributed disks of radius 0.1 in $[0, 4]^2$ with $b_{pq} \geq 0.3$). We observe on Figures 4(a) and 4(b) that the error essentially affects the real part of $\mu_{\text{max}}$ while the estimate of the imaginary part is acceptable. The average relative error on the modulus of $\mu_{\text{max}}$ for these 100 realizations is about 15.6%.

### 3.4 Condition number estimate

Like in the dilute medium case [7], the condition number $\text{cond}_2(L)$ of the matrix $L$ is estimated by the quantity

$$\frac{|\mu_{\text{max}}|}{|\mu_{\text{min}}|}.$$  

Thus, an approximate condition number $\text{cond}_{\text{app}}(L)$ is obtained by replacing $\mu_{\text{min}}$ and $\mu_{\text{max}}$ by their respective estimates (23) and (29) in Equation (33)

$$\text{cond}_2(L) \simeq \text{cond}_{\text{app}}(L) = 2N \left[ \ln \left( \frac{ka}{2} \right) + \ln \left( \frac{kd}{2} \right) \right] - M \gamma + i \frac{M \pi}{2}.$$  

Let us consider again the example of Figure 3. Then, the numerical condition number of $L$ is: $\text{cond}_2(L) \simeq 713$ while the estimate (34) yields $\text{cond}_{\text{app}}(L) \simeq 650$, leading to a relative error

$$100 \frac{|\text{cond}_2(L) - \text{cond}_{\text{app}}(L)|}{|\text{cond}_2(L)|} \simeq 9\%.$$  

Essentially, this error is related to the estimate of the largest eigenvalue. We report on Figure 5 the results for 100 launches. This gives a mean relative error equal to 11%, which is satisfactory.
Figure 4: Comparisons between $\mu_{\text{max}}$ and its estimate $\mu^1_{\text{max}}$ for 100 configurations of $M = 30$ obstacles with radius $a = 0.1$ randomly placed in $[0, 4]^2$, with $k = 0.1$, $N = 5$ and $b \geq 0.3 (= 3a)$. 
Figure 5: Comparison between the exact and approximate condition numbers of $L$, for 100 configurations of $M = 30$ disks with radius $a = 0.1$ randomly distributed in $[0, 4]^2$ ($k = 0.1$, $N = 5$, $b \geq 0.3 (= 3a)$).

4 Connections with the boundary element approximation and extension to other geometries

We approximate the single-layer potential by a linear boundary element method. For more details we refer to [7] where a similar approach is developed for distant obstacles. The boundary $\Gamma$ (which is the union of the $M$ boundaries $\Gamma_p$) is approximated by polygonal curves $\Gamma_h$, where $h$ designates the smallest element size of the $N_{\text{tot},h}$ segments composing $\Gamma_h$. The boundary element space $V_h$ is the space of piecewise linear elements on each segment of $\Gamma_h$. Let us introduce $[L_h] \in \mathcal{M}_{N_{\text{tot},h},N_{\text{tot},h}}(\mathbb{C})$ as the single-layer matrix and $[M_h] \in \mathcal{M}_{N_{\text{tot},h},N_{\text{tot},h}}(\mathbb{C})$ as the global mass matrix for linear finite elements. Finally, we denote by $\mu_{\text{min}}^h$, respectively $\mu_{\text{max}}^h$, the eigenvalue of $[M_h]^{-1} [L_h]$ with the smallest, respectively largest, modulus.

4.1 The circular case

As in [7], we begin by considering the case of circular scatterer and then formally extend the results to rectangular and elliptical shaped objects. We consider $M$ disks $\Omega_p$ with the same radius $a$ and we uniformly mesh each circle $\Gamma_p$ by fixing the meshsize to $h$. The number of degrees of freedom $N_E$ associated to one disk is then: $N_E = N_h \simeq \frac{2\pi a}{h}$ for a uniform mesh. On the other hand, the spectral method requires $2N + 1$ modes. Thus and as in [7], we propose to formally substitute $N$ by $\frac{\pi ah^{-1}}{2} - 1/2$ in the estimates (23), respectively (29), of $\mu_{\text{min}}$, respectively $\mu_{\text{max}}$. When $k$ tends towards 0, we therefore obtain $\mu_{\text{min}} \simeq \mu_{\text{min}}^{\text{app}}(a,h)$ and $\mu_{\text{max}} \simeq \mu_{\text{max}}^{\text{app}}(a,k)$, with

$$
\begin{cases}
\mu_{\text{min}}^{\text{app}}(a,h) = \frac{a}{2\pi ah^{-1} - 1}, \\
\mu_{\text{max}}^{\text{app}}(a,k) = -a \left[ \ln \left( \frac{ka}{2} \right) + (M - 1) \ln \left( \frac{kd}{2} \right) \right] - aM\gamma + i\frac{M\pi a}{2},
\end{cases}
$$

(36)
where $d$ represents a mean distance between obstacles. When the obstacles are contained in the rectangular box $[0, \ell] \times [0, L]$, we use the previous expression (30): $d = \sqrt{\pi a^2 + \ell^2}$. In addition, when $k$ tends towards zero, the condition number of the matrix $[M_h]^{-1}[L_h]$ is approximated by

$$\text{cond}(k, a, \Gamma_h) := \text{cond}_2([M_h]^{-1}[L_h]) \simeq \text{cond}_{\text{app}}(k, a, h),$$

with

$$\text{cond}_{\text{app}}(k, a, h) = a(2\pi ah^{-1} - 1)\left| -\ln \left(\frac{ka}{2}\right) - (M - 1)\ln \left(\frac{kd}{2}\right) - M\gamma + i\frac{M\pi}{2}\right|. \quad (37)$$

We propose to numerically validate the above approximations for the preceding example: $M = 30$ small disks of radius $a = 0.1$ are placed inside the box $[0, 4]^2$. The smallest inter centers distance $b$ is equal to 0.3$(= 3a)$ and $k = 0.1$. Moreover, each obstacle is meshed with $N_h = 50$ elements. We numerically compute the eigenvalues and the condition number of $[M_h]^{-1}[L_h]$ for 100 configurations as well as their corresponding estimates. For each quantity, $\mu^h_{\text{min}}, \mu^h_{\text{max}}$ and the condition number, we compute the relative error $100\left|\frac{\mu^h_{\text{app}} - \mu^h_{\text{nu}}}{\mu^h_{\text{nu}}}\right|$ (in %), where $\nu$ designates one of the listed quantities and $\mu^h_{\text{app}}$ its estimate. The average and standard deviation of the relative errors are reported in the following table:

|                  | $\mu^h_{\text{min}}$ | $\mu^h_{\text{max}}$ | $\Re(\mu^h_{\text{max}})$ | $\Im(\mu^h_{\text{max}})$ | $|\mu^h_{\text{max}}|$ | $\text{cond}_2$ |
|------------------|----------------------|-----------------------|-----------------------------|-----------------------------|------------------------|----------------|
| Average rel. error on... | 2.4% | 20% | 1.8% | 15% | 10% | 13% |
| Std. dev. of the rel. error on... | $< 10^{-12}$% | 1.1% | 0.1% | 0.9% | 0.8% | 0.7% |

We observe that the relative error on the smallest eigenvalue $\mu^h_{\text{min}}$ is about 2.4%, which is very satisfactory and similar to the distant obstacles case [7]. Concerning the largest eigenvalue, the relative error is about 15%, the main deterioration being on the real part of $\mu^h_{\text{max}}$. However, the mean relative error on the modulus of $\mu^h_{\text{max}}$ is lower and is about 10%. By comparison, we get similar errors with the spectral method. About the condition number of $[M_h]^{-1}[L_h]$, we obtain a mean relative error of 13% which is about the same as for the spectral method (11%). Let us finally point out that the standard deviation remains reasonably good for each approximation.

### 4.2 Extension to other geometries

We now formally adapt the estimates (36) first to elliptical and then to rectangular obstacle. We proceed as [7]: we formally replace the radius $a$ and the meshsize $h$ in the estimates (36) by respectively an equivalent radius $a_{\text{equiv}}$ and an equivalent step $h_{\text{equiv}}$. For an ellipse with semi-axis $a_{x_1}$ along the direction $x_1$ and $a_{x_2}$ along $x_2$, we proposed to choose an equivalent mesh parameter $h_{\text{equiv}}$ equal to the smallest discretization meshsize. Moreover the three equivalent radii were considered [7]

$$a_{\text{equiv}}^1 = \frac{a_{x_1} + a_{x_2}}{2}, \quad a_{\text{equiv}}^2 = \frac{2a_{x_1}a_{x_2}}{a_{x_1} + a_{x_2}} \quad \text{and} \quad a_{\text{equiv}}^3 = \frac{\sqrt{a_{x_1}^2 + a_{x_2}^2}}{\sqrt{2}}.$$ 

We propose to validate these approximations for small ellipses with semi-axes $a_{x_1} = 0.1$ and $a_{x_2} = 0.025$. We keep the same parameters as before (30 obstacles randomly placed in $[0, 4]^2$, with $k = 0.1$ and $b \geq 0.3(= 3a_{x_1})$, where $b$ is the smallest distance between the centers of two ellipses).
We numerically compute the smallest and largest eigenvalues of $[M_h]^{-1} [L_h]$ as well as its condition number for 100 configurations and compare it to their estimates. As in the circular case, we show the average and standard deviation of the relative errors for the 100 configurations and for each quantity:

|                  | $\mu^h_{\text{min}}$ | $\mu^h_{\text{max}}$ | $\Re(\mu^h_{\text{max}})$ | $\Im(\mu^h_{\text{max}})$ | $\mu^h_{\text{max}}$ | $|\mu^h_{\text{max}}|$ | $\text{cond}_2$ |
|------------------|-----------------------|-----------------------|-----------------------------|-----------------------------|-----------------------|------------------------|-----------------|
| Average rel. error for $a_{\text{eqv}}^1$ | 18%                   | 28%                   | 7%                          | 22%                         | 19%                   | 7%                     |                 |
| Average rel. error for $a_{\text{eqv}}^2$ | 18%                   | 53%                   | 40%                         | 49%                         | 48%                   | 40%                    |                 |
| Average rel. error for $a_{\text{eqv}}^3$ | 18%                   | 16%                   | 9%                          | 14%                         | 5.4%                  | 9%                     |                 |
| Std. dev. of the rel. err. for $a_{\text{eqv}}^1$ | $< 10^{-5\%}$         | 1.6%                  | 0.1%                        | 1.4%                        | 1.1%                  | 1.3%                   |                 |
| Std. dev. of the rel. err. for $a_{\text{eqv}}^2$ | $< 10^{-5\%}$         | 1%                    | $< 0.1\%$                  | 0.8%                        | 0.7%                  | 0.8%                   |                 |
| Std. dev. of the rel. err. for $a_{\text{eqv}}^3$ | $< 10^{-5\%}$         | 1.8%                  | 0.1%                        | 1.4%                        | 1.3%                  | 1.5%                   |                 |

First, let us begin by comparing the estimate of $\mu^h_{\text{min}}$. Like for the single scattering situation [7], choosing an equivalent radius has almost no effect on the estimate of $\mu^h_{\text{min}}$. Furthermore, the relative error on $\mu^h_{\text{min}}$ is of the order of 18%, for each radius. This error can be explained by the fact that $\mu^h_{\text{min}}$ is strongly mesh dependent, and most particularly relatively to the smallest mesh size (strong curvature effects). Hence, the estimate of the smallest eigenvalue can clearly degenerate. Concerning the eigenvalue with largest modulus $\mu^h_{\text{max}}$, it is directly impacted by the choice of the equivalent radius. Indeed, the table shows that, among the three equivalent radii that we proposed, $a_{\text{eqv}}^2$ leads to the best approximation of $\mu^h_{\text{max}}$ and $a_{\text{eqv}}^3$ to the worst one. More precisely, the mean relative error on $\mu^h_{\text{max}}$ is about 14% for $a_{\text{eqv}}^1$ compared with 22% for $a_{\text{eqv}}^1$ and 49% for $a_{\text{eqv}}^2$. On the other hand, the estimates of the modulus of $\mu^h_{\text{max}}$ are more precise with a mean relative error of about 5.4% for $a_{\text{eqv}}^3$ compared with 19% for $a_{\text{eqv}}^1$ and 48% for $a_{\text{eqv}}^2$. When the obstacles are distant, we observed a different behavior since $a_{\text{eqv}}^1$ gave better of $\mu^h_{\text{max}}$ than $a_{\text{eqv}}^3$ (the radius $a_{\text{eqv}}^3$ still led to the worst estimate) [7]. Finally, as $a_{\text{eqv}}^3$ leads to the best estimate of $\mu^h_{\text{max}}$, we keep this equivalent radius to estimate the condition number of the matrix $[M_h]^{-1} [L_h]$. The relative error on the condition number is then about 9% which is very satisfactory. As for the circular case, the standard deviation of the relative error is still reasonable with a maximum of 1.5%.

We end this numerical study by considering rectangular obstacles with half side lengths $a_{x_1}$ and $a_{x_2}$. We take the equivalent radius $a_{\text{eqv}}^4$ given by [7]: $a_{\text{eqv}}^4 = \frac{(1+\sqrt{2})}{2} \sqrt{a_{x_1}^2+a_{x_2}^2}$. Moreover, the equivalent step $h_{\text{eqv}}$ is always chosen equal to the smallest discretization meshsize $h_{\text{eqv}} = \min_{1 \leq p \leq M} \min_{1 \leq j \leq N_{h,p}} h_{p,j}$. We provide a numerical example considering the previous parameters: $M = 30$ randomly placed rectangular scatterers with half side lengths $a_{x_1} = 0.1$ and $a_{x_2} = 0.025$ in $[0, 4]^2$. Moreover, we set $k = 0.1$ and $b \geq 0.3 (= 3a_{x_1})$. Numerically, we compute the eigenvalues of the matrix $[M_h]^{-1} [L_h]$, its condition number as well as their respective estimates for 100 configurations. The following table shows the average and standard deviation of the relative errors for each quantity:

|                  | $\mu^h_{\text{min}}$ | $\mu^h_{\text{max}}$ | $\Re(\mu^h_{\text{max}})$ | $\Im(\mu^h_{\text{max}})$ | $\mu^h_{\text{max}}$ | $|\mu^h_{\text{max}}|$ | $\text{cond}_2$ |
|------------------|-----------------------|-----------------------|-----------------------------|-----------------------------|-----------------------|------------------------|-----------------|
| Average rel. error for $a_{\text{eqv}}^4$ | 2.2%                  | 13%                   | 12%                         | 13%                         | 2.4%                  | 14%                    |                 |
| Std. dev. of the rel. err. for $a_{\text{eqv}}^4$ | $< 10^{-4\%}$         | 1.9%                  | 0.2%                        | 1.1%                        | 1.1%                  | 1.2%                   |                 |
We begin by comparing the estimates of the eigenvalues with smallest, respectively largest, modulus $\mu^h_{\text{min}}$, respectively $\mu^h_{\text{max}}$. The average relative error on $\mu^h_{\text{min}}$ is 2%, compared with 18% in the elliptical case. This can be explained by the property that, in the rectangular case, the mesh is non uniform but the mesh size is constant on each of the four rectangle sides (unlike the ellipse). Concerning the eigenvalue with largest modulus $\mu^h_{\text{max}}$, the relative error is about 13%, which is of the same order as for disks and ellipses, and about 2.4% on its modulus, which is very precise. The condition number estimate is satisfactory since the average relative error is about 14% like for disks. Finally and as for the previous cases, the standard deviation remains in a reasonable range ($\sim 1\%$).

5 Conclusion

In this second and last part, we developed and validated low-frequency spectral and condition number estimates of the single-layer integral operator for dense multiple scattering media. They have been formally extended to circular, elliptical and rectangular shaped obstacles when a linear boundary element is considered. These estimates provide explicit dependence of the eigenvalues with respect to the different problem parameters.

These studies open different directions which should complete this work. First, spectral estimates related to the Brakhage-Werner integral equation [5, 6, 10] and Combined Field Integral Equation [5, 6, 20] can be expected in similar situations since they involve the four basic integral operators: the single- and double-layer potentials as well as their normal derivatives. A difficult situation that is not studied here is the case where the distance between the obstacles is of the order of the characteristic size of the scatterers. We did not succeed yet in deriving similar estimates. One of the main difficulties is that an asymptotic regime is not available. Finally, considering the medium/high frequency as well as three-dimensional case are of interest but require further studies.

References


