Semantic Data

Practice 1 : First order logic

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Types of exercices

1. Syntax of FOL.
2. Finding the natural language meaning of FOL formulas.
3. Formulating sentences in FOL.
4. Manipulating formulas.
5. Interpreting and reasoning about FOL statements.

1. Syntax of FOL

Exercise 1 : which sentences are well-formed FOL formulas or terms ?

Non logical symbols : constants a, b, functions f^1, g^2, predicates P^1, R^2, Q^3 (with indicated arity).

1. $R(a, g(a, a))$
2. $g(a, g(a, a))$
3. $\forall x \neg P(x)$
4. $\neg R(P(a), x)$
5. $\exists a\ R(a, a)$
6. $\exists x\ Q(x, f(x), b) \rightarrow \forall x\ R(a, x)$
7. $\exists x\ P(R(a, x))$
8. $\forall R(x, a)$

1. well formed formula
2. well formed term
3. well formed formula
4. not well formed
5. not well formed
6. well formed formula
7. not well formed
8. not well formed

(the exercises in this section are inspired from exercises of the course of Mathematical Logic, Ghidini and Serafini, FBK Trento)

1. Syntax of FOL

Suggested exercise 1 : which sentences are well formed FOL formulas or terms ?

Non logical symbols: constants a, b, functions f^1, g^2, predicates P^1, R^2, Q^3 (with indicated arity).

1. Q(a)
2. P(y)
3. P(g(b))
4. ¬R(x, a)
5. Q(x, P(a), b)
6. P(g(f(a), g(x, f(x))))
7. Q(f(a), f(f(x)), f(g(f(z), g(a, b))))
8. R(a, R(a, a))

1. Syntax of FOL

Exercise 2 : For each of the following formulas indicate :

a) the scope of the quantifiers;
b) the free variables.

1. $\exists x\,(A(x, y) \wedge B(x))$
2. $\exists x\,(\exists y\, A(x, y) \rightarrow B(x))$
3. $\neg\exists x\, \exists y\, A(x, y) \rightarrow B(x)$
4. $\forall x\, \neg\exists y\, A(x, y)$
5. $\exists x\, A(x, x) \wedge B(x)$
6. $\exists x\, A(x, x) \wedge \exists y\, B(y)$

1. Scope for $\exists x$: $(A(x, y) \wedge B(x))$. Free variable : y.
2. Scope for $\exists x$: $(\exists y\, A(x, y) \rightarrow B(x))$.
   Scope for $\exists y$: $A(x, y)$. Free variable : none.
3. Scope for $\exists x$: $\exists y\, A(x, y)$. Scope for $\exists y$: $A(x, y)$.
   Free variable : x in $B(x)$.
4. Scope for $\forall x$: $\neg\exists y\, A(x, y)$. Scope for $\exists y$: $A(x, y)$.
   Free variable : none.
5. Scope for $\exists x$: $A(x, x)$. Free variables : x in $B(x)$.
6. Scope for $\exists x$: $A(x, x)$. Scope for $\exists y$: $B(y)$.
   Free variable : none.

1. Syntax of FOL

Suggested exercise 2 : find the free variables in the following formulas :

1. $P(x) \land \neg R(y, a)$
2. $\exists x\ R(x, y)$
3. $\forall x\ P(x) \rightarrow \exists y\ \neg Q(f(x), y, f(y))$
4. $\forall x\ \exists y\ R(x, f(y))$
5. $\forall x\ \exists y\ R(x, f(y)) \rightarrow R(x, y)$

2. Finding the meaning of FOL formulas

Exercise 1 : what is the meaning of the following FOL formulas?

1. $Bought(Frank, dvd)$
2. $\exists x\ Bought(Frank, x)$
3. $\forall x(Bought(Frank, x) \rightarrow Bought(Susan, x))$
4. $\forall x\ Bought(Frank, x) \rightarrow \forall x\ Bought(Susan, x)$
5. $\forall x\ \exists y\ Bought(x, y)$
6. $\exists x\ \forall y\ Bought(x, y)$

1. Frank bought a dvd.
2. Frank bought something.
3. Susan bought everything that Frank bought.
4. If Frank bought everything, so did Susan.
5. Everyone bought something.
6. Someone bought everything.

(after an exercise from Ghidini and Serafini, FBK Trento)

2. Finding the meaning of FOL formulas

Suggested exercise 1 : what is the meaning of the following FOL formulas ?

1. $\forall x\ [(StrongEngine(x) \land Car(x) \land Wheels(x, 4)) \rightarrow Fast(x)]$
2. $\forall x\ \forall y\ [(Parent(x, y) \land Ancestor(y)) \rightarrow Ancestor(x)]$
3. $\forall x\ \forall y\ [(Car(x) \land OnRoad(x, y) \land Highway(y) \land NormalConditions(y)) \rightarrow FastSpeedAllowed(x)]$
4. $\exists t\ \forall p\ (\neg Travel(t, p) \lor FarFrom(p, Mycity))$
   where travel(t, p) represents my travel to p at time t.
5. $\exists t\ \forall p\ (Travel(t, p) \rightarrow FarFrom(p, Mycity))$
6. Are sentences 4 and 5 equivalent ?

(after a booklet of exercise from A. Szalas)

3. Formulating sentences in FOL

Exercise 1 : The function *mapColor* and predicates *In(x, y)*, *Borders(x, y)*, and *Country(x)* are given.
For each of the following sentences and corresponding candidate FOL expressions, indicate if the FOL expression
a) correctly expresses the English sentence;
b) is syntactically invalid and therefore meaningless; or
c) is syntactically valid but incorrect : does not express the meaning of the English sentence.

a) **Paris and Marseilles are both in France.**

i.	$In(Paris \land Marseilles, France)$	Invalid (conjunction in term)
ii.	$In(Paris, France) \land In(Marseilles, France)$	Correct.
iii.	$In(Paris, France) \lor In(Marseilles, France)$	Incorrect.

b) **There is a country that borders both Iraq and Pakistan.**

i.	$\exists c\ (Country(c) \rightarrow [Borders(c, Iraq) \land Borders(c, Pakistan)])$	Incorrect (implication in existential).
ii.	$\exists c\ (Country(c) \land Borders(c, Iraq) \land Borders(c, Pakistan))$	Correct.
iii.	$\exists c\ Country(c) \rightarrow [Borders(c, Iraq) \land Borders(c, Pakistan)]$	Incorrect (variable out of scope of quantifier).
iv.	$\exists c\ Borders(Country(c), Iraq \land Pakistan)$	Invalid (predicate as argument, conjunction in term).

(after Russel and Norvig 2010)

3. Formulating sentences in FOL

c) **All countries that border Ecuador are in South America.**

i.	$\forall c\ (Country(c) \land Borders(c, Ecuador) \rightarrow In(c, SouthAmerica))$	Correct.
ii.	$\forall c\ (Country(c) \rightarrow [Borders(c, Ecuador) \rightarrow In(c, SouthAmerica)])$	Correct (equivalent to *i.* (*)).
iii.	$\forall c\ [Country(c) \rightarrow Borders(c, Ecuador)] \rightarrow In(c, SouthAmerica)$	Incorrect (variable out of scope).
iv.	$\forall c\ (Country(c) \land Borders(c, Ecuador) \land In(c, SouthAmerica))$	Incorrect (conjunction in universal).

$$* : a \rightarrow b \rightarrow c \equiv a \rightarrow (\neg b \lor c) \equiv \neg a \lor \neg b \lor c \equiv \neg(a \land b) \lor c \equiv (a \land b) \rightarrow c$$

3. Formulating sentences in FOL

Suggested exercise 1 : answer question from exercise 1 for the following sentences :

d) **No region in South America borders any region in Europe.**

   i. $\neg[\exists c\ \exists d\ (In(c, SouthAmerica) \land In(d, Europe) \land Borders(c, d))]$

   ii. $\forall c\ \forall d\ [In(c, SouthAmerica) \land In(d, Europe)] \rightarrow \neg Borders(c, d)]$

   iii. $\neg \forall c\ (In(c, SouthAmerica) \rightarrow \exists d\ (In(d, Europe) \land \neg Borders(c, d)))$

   iv. $\forall c\ (In(c, SouthAmerica) \rightarrow \forall d\ (In(d, Europe) \rightarrow \neg Borders(c, d)))$

e) **No two adjacent countries have the same map color (this sentence requires using equality).**

   i. $\forall x\ \forall y\ (\neg Country(x) \lor \neg Country(y) \lor \neg Borders(x, y) \lor \neg(mapColor(x) = mapColor(y)))$

   ii. $\forall x\ \forall y\ ((Country(x) \land Country(y) \land Borders(x, y) \land \neg(x = y)) \rightarrow \neg (mapColor(x) = mapColor(y)))$

   iii. $\forall x\ \forall y\ (Country(x) \land Country(y) \land Borders(x, y) \land \neg(mapColor(x) = mapColor(y)))$

3. Formulating sentences in FOL

- **Exercise 2** : which of the following formulas is a formalization of the sentence :

"There is a computer which is not used by any student".

i. $\exists x \, (Computer(x) \land \forall y \, (\neg Student(y) \land \neg Uses(y, x)))$

ii. $\exists x \, (Computer(x) \rightarrow \forall y \, (Student(y) \rightarrow \neg Uses(y, x)))$

iii. $\exists x \, (Computer(x) \land \forall y \, (Student(y) \rightarrow \neg Uses(y, x)))$

Answer: iii

(after an exercise from Ghidini and Serafini, FBK Trento, Italy)

3. Formulating sentences in FOL

Suggested exercise 2 : translate into FOL :

1. Everyone is mad.
2. There is at least one doctor.
3. Doctors are not lawyers.
4. Lawyers sue everyone.
5. Doctors sue back if they are sued.
6. There is an individual who does not sue.

3. Formulating sentences in FOL

Suggested exercise 3 : define an appropriate language and translate into FOL :

1. Bill has at least one sister.
2. Bill has no sister.
3. Every student takes at least one course.
4. No student failed Geometry but at least one student failed Analysis.
5. Every student who takes Analysis also takes Geometry.

(after an exercise from Ghidini and Serafini, FBK Trento)

3. Formulating sentences in FOL

Suggested exercise 4 : In a world of labeled colored blocks, translate the following sentences in FOL :

1. A is above C, D is on E and above F.
2. A is green while C is not.
3. Everything is on something.
4. Everything that is free has nothing on it.
5. Everything that is green is free.
6. There is something that is red and is not free.
7. Everything that is not green and is above B, is red.

(after an exercise from Ghidini and Serafini, FBK Trento)

4. Manipulating formulas

Background information on normal forms in propositional logic (PL) :

- Negation normal form (NNF)

  - A formula is in negation normal form if it uses only literals and the connectives ∧ and ∨. Literals are positive or negative propositional variables or atoms : p or ¬p.

    In NNF, negation occurs only directly in front of atoms.

  - Negation normal form is important for reasoning algorithms seen in chapter 8.

¬a ∧ ¬b is in NNF.

¬ (a ∨ b) is not in NNF.

- Conjunctive normal form (CNF)

  - A formula is in conjunctive normal form if it is a conjunction of disjunctions of literals.

(a ∨ ¬b) ∧ (b ∨ ¬c ∨ ¬d) is in CNF.

Reminder : standard equivalences in propositional logic

- $(a \wedge b) \equiv (b \wedge a)$ — commutativity of $\wedge$
- $(a \vee b) \equiv (b \vee a)$ — commutativity of $\vee$
- $((a \wedge b) \wedge c) \equiv (a \wedge (b \wedge c))$ — associativity of $\wedge$
- $((a \vee b) \vee c) \equiv (a \vee (b \vee c))$ — associativity of $\vee$
- $\neg\neg a \equiv a$ — double-negation elimination
- $(a \rightarrow b) \equiv (\neg b \rightarrow \neg a)$ — contraposition
- $(a \rightarrow b) \equiv (\neg a \vee b)$ — implication elimination
- $(a \leftrightarrow b) \equiv ((a \rightarrow b) \wedge (b \rightarrow a))$ — biconditional elimination
- $\neg(a \wedge b) \equiv (\neg a \vee \neg b)$ — De Morgan's laws
- $\neg(a \vee b) \equiv (\neg a \wedge \neg b)$ — De Morgan's laws
- $(a \wedge (b \vee c)) \equiv ((a \wedge b) \vee (a \wedge c))$ — distributivity of $\wedge$ over $\vee$
- $(a \vee (b \wedge c)) \equiv ((a \vee b) \wedge (a \vee c))$ — distributivity of $\vee$ over $\wedge$

These formulas should be known !

4. Manipulating formulas

- **Exercise 1 :** convert the following formula in conjunctive normal form :

  $(\neg p \rightarrow q) \rightarrow (q \rightarrow \neg r)$

- Method :
  - Convert to NNF (use implication elimination, de Morgan rules and double negation elimination).
  - Transform the NNF into CNF (use distributivity of disjunction).

- Solution

  $(\neg p \rightarrow q) \rightarrow (q \rightarrow \neg r)$

  $\neg(\neg p \rightarrow q) \lor (\neg q \lor \neg r)$

  $\neg(p \lor q) \lor (\neg q \lor \neg r)$

  $(\neg p \land \neg q) \lor (\neg q \lor \neg r)$ NNF

  $((\neg q \lor \neg r) \lor \neg p) \land ((\neg q \lor \neg r) \lor \neg q)$ applying $(a \lor (b \land c)) \equiv ((a \lor b) \land (a \lor c))$ with $a = (\neg q \lor \neg r)$

  $(\neg p \lor \neg q \lor \neg r) \land (\neg q \lor \neg r)$ CNF

4. Manipulating formulas

- **Suggested exercise 1 :** convert the following formulas in conjunctive normal form :

1. $\neg(\neg p \lor q) \lor (r \rightarrow \neg s)$
2. $p \rightarrow (q \land r)$
3. $p \rightarrow (q \rightarrow r)$
4. $(p \rightarrow q) \rightarrow r$
5. $(\neg p \rightarrow (p \rightarrow q))$
6. $(p \rightarrow (q \rightarrow r)) \rightarrow (p \rightarrow (r \rightarrow q))$

4. Manipulating formulas

Background information on normal forms in FOL :

- Conjunctive normal form (CNF)

  - A FOL formula is in conjunctive normal form if it is a first-order instance of a propositional formula in CNF, obtained by uniform substitution of atomic formulae for propositional variables.

$(\neg P(x) \lor Q(x, y)) \land (P(x) \lor R(y))$ is in CNF.

- Prenex conjunctive normal form (PCNF)

  - A formula is in prenex form if all quantifiers have been moved outwards : it is of the form $Q_1 x_1 \ldots Q_n x_n\, A$ where A is an open formula (without quantifiers).
  - A formula is in prenex conjunctive normal form if in addition A is in CNF.

$\forall x\, \exists x\, ((\neg P(x) \lor Q(x, y)) \land (P(x) \lor R(y)))$ is in PCNF.

4. Manipulating formulas

- **Exercise 2 :** convert the following formula in prenex conjunctive normal form :

$\forall x\ (\forall y\ (Animal(y) \rightarrow Loves(x, y)) \rightarrow \exists y\ Loves(y, x))$

- Method :
  - Convert first to negation normal form.
  - Rename variables where necessary (check quantifier scopes and free variables)
  - Move quantifiers outwards

$\forall x\ (\forall y\ (Animal(y) \rightarrow Loves(x, y)) \rightarrow \exists y\ Loves(y, x))$

$\forall x\ (\forall y\ (\neg Animal(y) \lor Loves(x, y)) \rightarrow \exists y\ Loves(y, x))$

$\forall x\ (\neg \forall y\ (\neg Animal(y) \lor Loves(x, y)) \lor \exists y\ Loves(y, x))$

$\forall x\ (\exists y\ \neg(\neg Animal(y) \lor Loves(x, y)) \lor \exists y\ Loves(y, x))$

$\forall x\ (\exists y\ (\neg\neg Animal(y) \land \neg Loves(x, y)) \lor \exists y\ Loves(y, x))$

$\forall x\ (\exists y\ (Animal(y) \land \neg Loves(x, y)) \lor \exists z\ Loves(z, x))$

$\forall x\ \exists z\ \exists y\ ((Animal(y) \land \neg Loves(x, y)) \lor Loves(z, x))$

$\forall x\ \exists z\ \exists y\ ((Animal(y) \lor Loves(z, x)) \land (\neg Loves(x, y) \lor Loves(z, x)))$

4. Manipulating formulas

- **Suggested exercise 2 :** convert the following formula in prenex conjunctive normal form :

$\exists z\ (\exists x\ Q(x, z) \lor \exists x\ P(x)) \to \neg(\neg\exists x\ P(x) \land \forall x\ \exists z\ Q(z, x))$

5. Interpretations and reasoning

- **Exercise 1 :** given the model M defined by $\Delta = \{0, 1\}$ and the interpretation function $\mathcal{I}$:

  $P^{\mathcal{I}} = \{<0>, <1>\}$,

  $R^{\mathcal{I}} = \{<0, 0>, <0, 1>\}$,

  verify whether the following formulas are true :

a) $\forall x\, P(x)$
b) $P(0)$
c) $\neg R(0, 0)$
d) $\exists x\, R(x, x)$
e) $\forall x\, R(x, x)$
f) $\forall x\, (R(x, x) \rightarrow P(x))$
g) $\forall x\, (\neg R(x, x) \rightarrow P(x))$
h) $\forall x\, (P(x) \rightarrow \neg R(x, x))$

a) True
b) True
c) False
d) True
e) False ($x = 1$)
f) True
g) True
h) False ($x = 0$)

(This exercise and the following one are inspired from exercised of course 6.825, Techniques in Artificial Intelligence, CSAI lab, MIT).

5. Interpretations and reasoning

- **Suggested exercise 1 :** given the model M defined by Δ = {A, B, C} and the interpretation function $\mathcal{I}$:

$X^{\mathcal{I}}$ = A, $Y^{\mathcal{I}}$ = A, $Z^{\mathcal{I}}$ = B

$f^{\mathcal{I}}$ = {<A, B>, <B, C>, <C, C>}

$P^{\mathcal{I}}$ = {A, B}

$Q^{\mathcal{I}}$ = {C}

$R^{\mathcal{I}}$ = {<B, A>, <C, B>, <C, C>}

verify whether the following formulas are true :

a) Q(f(Z))

b) R(X, Y)

c) $\forall$w R(f(w), w)

d) $\forall$u $\forall$v (R(u, v) $\rightarrow$ $\forall$w R(u, w))

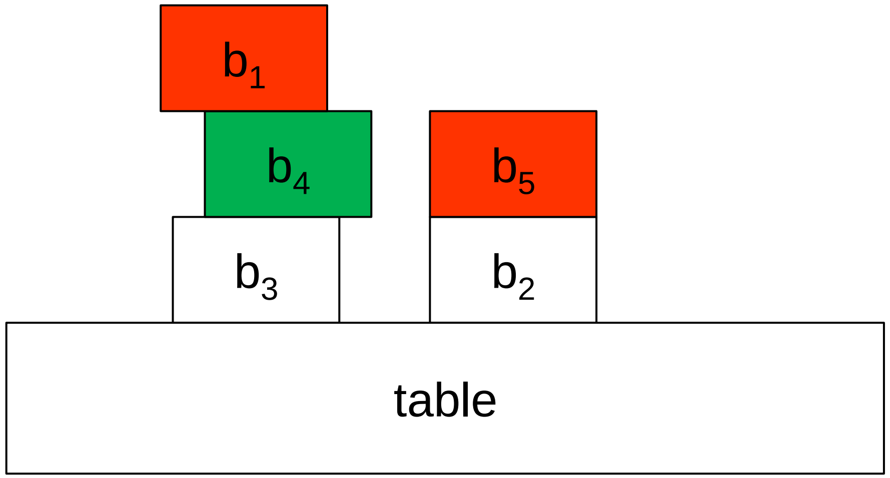
5. Interpretations and reasoning

- **Suggested exercise 2 :** for the world of labeled colored blocks of suggested exercise 4 of section 3,

Consider the interpretation $\mathcal{I}$ defined by $A^{\mathcal{I}} = b_1$, $B^{\mathcal{I}} = b_2$, $C^{\mathcal{I}} = b_3$, $D^{\mathcal{I}} = b_4$, $E^{\mathcal{I}} = b_5$, $F^{\mathcal{I}} =$ table and by the picture below, where *On*, *Above, Green, Red* and *Free* have their normal meaning;

a) Complete the formal definition of that interpretation;
b) For each formula in suggested exercise 4 of section 3 (natural language formulation repeated below), determine whether it is satisfied of not by that interpretation.

1. A is above C, D is on E and above F.
2. A is green while C is not.
3. Everything is on something.
4. Everything that is free has nothing on it.
5. Everything that is green is free.
6. There is something that is red and is not free.
7. Everything that is not green and is above B, is red.



(after an exercise from Ghidini and Serafini, FBK Trento)

THANK YOU