1 Objectives

At the end of this exercise session you should be able to:

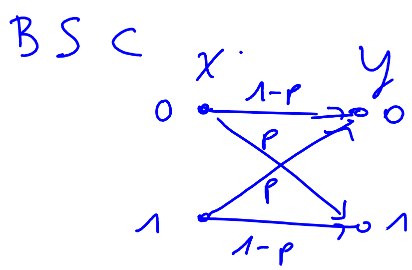
- Compute the capacity of a channel

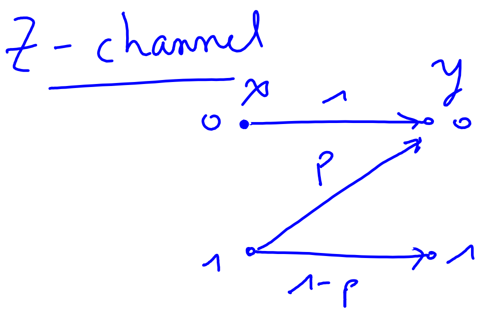
2 Exercises

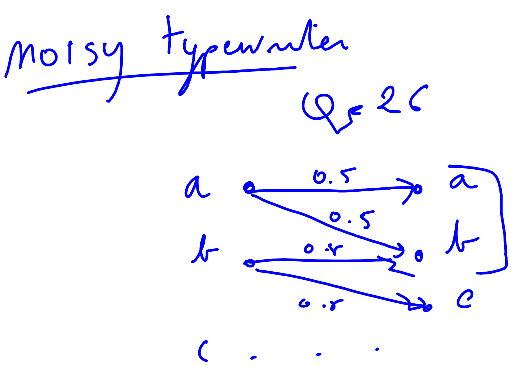
Channel coding

By definition

$$C = \max_{P(x)} I(x;y)$$







$$I(x;y) = H(y) - H(y|x)$$

If $P(x)$ is uniform, then $P(y)$ is uniform

$$H(y) = \log_2 26 = \cancel{26} \cdot \frac{-1}{\cancel{26}} \log_2 \frac{1}{26}$$

$$H(y|x) = 1$$

$$C = \log_2 26 - 1 = \log_2 13$$

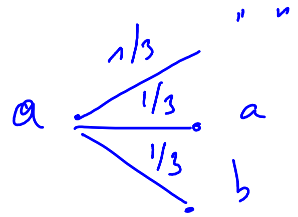
$a \longrightarrow a$ or $b \longrightarrow a$

$\cancel{b}$

$a \longrightarrow a$ or b ?

b

$Q = 27$



$$P(x) = [\,1/27,\ 1/27,\ \dots\,]$$

$$= \underline{[\,0\ \ 1/9\ \ 0\ \ \dots\ 0\ \ 1/9\ \ 0\,]}$$

$$C = \max_{P(x)} I(x;y)$$

$$= \max_{P(x)} H(y) - H(y|x)$$

$$= \log_2 27 - \log_2 3$$

$$= \log_2 9$$

Exercise 1. [9.7] Compute the mutual information between X and Y for the binary symmetric channel with $p = 0.15$ when the input distribution is $P(\mathcal{X})$ is uniform.

Exercise 2. [9.8] Compute the mutual information between X and Y for the Z-channel with $p = 0.15$ when the input distribution $P(\mathcal{X})$ is uniform.

Ex 1

$P(X)$ is uniform : $[0.5 \quad 0.5]$

$P(y)$ is uniform : $[0.5 \quad 0.5]$

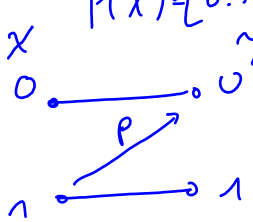
$P(y \mid X=0) = [0.85 \quad 0.15]$

$P(y \mid X=1) = [0.15 \quad 0.85]$

$$I(X;y) = H(y) - H(y \mid X)$$
$$= 1 - [p(X=0)\, H_2(0.85) + p(X=1)\, H_2(0.15)]$$
$$= 1 - H_2(0.85)\, \underbrace{(P(X=0) + P(X=1))}_{=1} =$$
$$= 1 - H_2(0.15)$$
$$= 1 - H_2(p)$$
$$= 0.39 \text{ Shannon}$$

Ex 2

$P(X) = [0.5, 0.5]$



$P(y \mid x)$ (y rows, x columns)	0	1
0	1	p
1	0	$1-p$

$$P(Y=0) = P(X=0)\,P(Y=0 \mid X=0) + P(X=1)\,P(Y=0 \mid X=1)$$
$$= 0.575$$
$$P(Y=1) = 0.425$$

$$I(X;Y) = H(Y) - H(Y|X)$$

$$= H_2(0.575) - \Big[P(X=0)\, \underbrace{H(Y|X=0)}_{\substack{[1\ \ 0] \\ =0}} + P(X=1)\, \underbrace{H(Y|X=1)}_{[p\ \ 1-p]} \Big]$$

$$= H_2(0.575) - 0.5\, H_2(p)$$

$$= 0.68 \text{ Shannon}$$

Exercise 3. [9.12] What is the capacity of the binary symmetric channel for general p?

By symmetry, $P(x) = $ uniform $= [0.5 \quad 0.5]$

Then $C = I(x;y) = H(y) - H(y|x)$
$= 1 - H_2(p)$

Without invoking symmetry. $P(x) = [p_0 \quad p_1]$

$$I(x;y) = H(y) - H(y|x)$$
$$= \underbrace{H_2\left(p_0(1-p) + p_1 p\right)} - H_2(p)$$

$$\max_{p_0, p_1} H_2\left(p_0(1-p) + p_1 p\right)$$

$$\Rightarrow \begin{cases} p_0(1-p) + p_1 p = 0.5 \\ p_1(1-p) + p_0 p = 0.5 \end{cases}$$

$$\Rightarrow \begin{cases} p_0 = 0.5 \\ p_1 = 0.5 \end{cases}$$

Exercise 4. [9.17] What is the capacity of the five-input, ten-output channel whose transition probability matrix is

$$
\begin{array}{cc}
 & \begin{array}{ccccc} & & X & & \\ 0 & 1 & 2 & 3 & 4 \end{array} \\
y \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \\ \\ \\ \\ \\ 9 \end{array} &
\begin{bmatrix}
\boxed{0.25} & 0 & 0 & 0 & 0.25 \\
0.25 & 0 & 0 & 0 & 0.25 \\
0.25 & 0.25 & 0 & 0 & 0 \\
0.25 & 0.25 & 0 & 0 & 0 \\
0 & 0.25 & 0.25 & 0 & 0 \\
0 & 0.25 & 0.25 & 0 & 0 \\
0 & 0 & 0.25 & 0.25 & 0 \\
0 & 0 & 0.25 & 0.25 & 0 \\
0 & 0 & 0 & 0.25 & 0.25 \\
0 & 0 & 0 & 0.25 & 0.25
\end{bmatrix}
\end{array}?
$$

$P(y \mid X=0)$

$\Rightarrow P(Y=0 \mid X=0) = 1/4$

$P(Y=1 \mid X=0) = 1/4$

$= 1/4$

$Y=2 \quad = 1/4$

$Y=3 \quad = 0$

$\vdots \quad = 0$

$$H(y) = \log_2(10)$$

$$H(y \mid X) = \log_2 4 = 2$$

$$C = \max_{P(X)} I(X;y) = \log_2(10) - \log_2 4$$

$$= \log_2\left(\frac{5}{2}\right) \quad \text{Shannon}$$