Logic - Tutorial 8

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Reminder

Semantic tableaux - Decomposition rules

Prolongation rules (α-rules) and ramification rules (β-rules)

α	α_1	α_2
$A_1 \wedge A_2$	A_1	A_2
$\neg(A_1 \vee A_2)$	$\neg A_1$	$\neg A_2$
$\neg(A_1 \Rightarrow A_2)$	A_1	$\neg A_2$
$\neg(A_1 \Leftarrow A_2)$	$\neg A_1$	A_2

β	β_1	β_2
$B_1 \vee B_2$	B_1	B_2
$\neg(B_1 \wedge B_2)$	$\neg B_1$	$\neg B_2$
$B_1 \Rightarrow B_2$	B_1	$\neg B_2$
$B_1 \Leftarrow B_2$	$\neg B_1$	B_2

Generative rule (γ-rule) and exemplification rule ($\delta - rule$).

γ	$\gamma(a)$
$\forall x\, A(x)$	$\forall x\, A(x), A(c)$
$\neg\exists x\, A(x)$	$\neg\exists x\, A(x), \neg A(c)$

δ	$\delta(a)$
$\exists x\, A(x)$	$A(a)$
$\neg\forall x\, A(x)$	$\neg A(a)$

The choice of constant *c* is free but *a* must be a fresh constant.

Reminder

<u>Construction of a semantic tableaux</u>

Init: root is labelled $\{A\}$; it is an unmarked leaf.

Induction step: select an unmarked leaf l labelled $U(l)$.

- If $U(l)$ is a literal set :
  - if $U(l)$ contains a complementary pair, then mark l as closed 'X' ;

Reminder

- If $U(l)$ is not a literal set, select a non-literal formula in $U(l)$:
  - If it is an α-formula A, generate a child node l' and label it with
    $$U(l') = (U(l) - \{A\}) \cup \{\alpha_1, \alpha_2\};$$
  - if it is a β-formula B, generate two child nodes l' and l'' ; their labels respectively are
    $$U(l') = (U(l) - \{B\}) \cup \{\beta_1\}$$
    $$U(l'') = (U(l) - \{B\}) \cup \{\beta_2\}.$$
  - If it is a γ-formula C, create a single child l' and label it with
    $$U(l') = U(l) \cup \gamma(c)$$
    where c is a constant occurring in U(l) (if any).
  - If it is a δ-formula D, create a single child l' and label it with
    $$U(l') = (U(l) - D) \cup \delta(a)$$
    where a is a fresh constant, not occurring in $U(l)$.

Reminder

Termination : occurs when each leaf is either closed, or contains only literals and fully instantiated γ-formulas; such a leaf can be marked open. The γ-rule may prevent termination.

Reminder

Construction strategy

Two conditions must be fulfilled.

- Every non literal formula on an open branch is decomposed on this branch.
- For each γ-formula A and each constant a occurring on an open-branch, an a-instantiation of A occurs on the branch.

Exercise 1

Exercise 1

What is the link (in terms of logical consequences) between the following formulas?

1. $p(x)$ and $\forall x\, p(x)$
2. $p(x)$ and $\exists x\, p(x)$
3. $\forall x\, p(x) \vee \forall x\, q(x)$ and $\forall x\, (p(x) \vee q(x))$
4. $\forall x\, p(x) \wedge \forall x\, q(x)$ and $\forall x\, [p(x) \wedge q(x)]$
5. $\forall x\, \forall y\, p(x, y)$ and $\forall x\, \forall y\, p(y, x)$

Exercise 1

1) $A \triangleq p(x)$ and $B \triangleq \forall x\, p(x)$

$\underline{A \not\models B}$

Indeed if we have $\mathcal{I}$ such that:

- $D = \{x, b\}$
- $I_c[p(x)] = T$
- $I_c[p(b)] = F$

then $\mathcal{I}[A] = T$ and $\mathcal{I}[B] = F$

$\underline{B \models A}$

To clarify the development, let's change x by a in A.
Then, if $\mathcal{I}[\forall x\, p(x)] = T$, $\mathcal{I}[p(x)] = T$ for all x and in particular for a.

Exercise 1

2) $A \triangleq p(x)$ and $B \triangleq \exists p(x)$

A $\models B$

To clarify the development, let's change x by a in A.
Then, if $\mathcal{I}[p(a)] = T$, as $\mathcal{I}[\exists x\, p(x)] = T$ if there exists one x such that $\mathcal{I}[p(x)] = T$, $\mathcal{I}[B] = T$.

B $\not\models A$

Let $\mathcal{I} = (D, I_c, I_v)$ be an interpretation such that:

- $D = \mathbb{N}$
- $I_c[p] \rightarrow$ even predicate
- $I_v[x] = 3$

then $I[B] = T$ but $I[A] = F$

Exercise 1

3) $A \triangleq \forall x\, p(x) \land \forall x\, q(x)$ and $B \triangleq \forall x\, [p(x) \land q(x)]$

$\underline{A \models B}$

If $\mathcal{I}[\forall x\, p(x) \land \forall x\, q(x)] = T$, it means that for any $d' \in D$, $\mathcal{I}_{x/d'}[p(x)] = T$ and for all $d'' \in D$, $\mathcal{I}_{x/d''}[q(x)] = T$.
d' and d'' both covering the whole of D, we thus have than for any $d \in D$, $\mathcal{I}_{x/d}[p(x) \land q(x)] = T$ and thus $\mathcal{I}[B] = T$

$\underline{B \models A}$

If $\mathcal{I}[\forall x\, [p(x) \land q(x)]] = T$, it means that for any $d \in D$, $\mathcal{I}_{x/d}[p(x) \land q(x)] = T$ and therefore $\mathcal{I}_{x/d}[p(x)] = T$ and $\mathcal{I}_{x/d}[q(x)] = T$.
If $\mathcal{I}_{x/d}[p(x)] = T$ for all $d \in D$, then $\mathcal{I}[\forall x\, p(x)] = T$. Similarly we have that $\mathcal{I}[\forall x\, q(x)] = T$ and therefore $\mathcal{I}[A] = T$.

We therefore have that $A \leftrightarrow B$.

Exercise 1

4) $A \triangleq \forall x\, p(x) \vee \forall x\, q(x)$ and $B \triangleq \forall x\, (p(x) \vee q(x))$

$B \not\models A$

1st counter-example

$D = \{a, b\}$
$I_c[p(a)] = T$
$I_c[p(b)] = F$
$I_c[q(a)] = F$
$I_c[q(b)] = T$

2nd counter-example

$D = \mathbb{N}$
$I_c[p] \rightarrow$ even predicate
$I_c[q] \rightarrow$ odd predicate

Exercise 1

$A \models B$

For any $\mathcal{I}$, if $\mathcal{I}[\forall x\, p(x) \vee \forall x\, q(x)] = T$, we have:

- $\mathcal{I}[\forall x\, p(x)] = T$ or $\mathcal{I}[\forall x\, q(x)] = T$

Each interpretation makes $\forall x\, p(x)$ or $\forall x\, q(x)$ true.

Case 1: $\mathcal{I}[\forall x\, p(x)] = T$

This means that for any d, $\mathcal{I}_{x/d}[p(x)] = T$.
Therefore $\mathcal{I}_{x/d}[\forall x\, p(x) \vee q(x)] = T$ for any d and $\mathcal{I}[B] = T$.

Case 2: $\mathcal{I}[\forall x\, p(x)] = T$

Idem then case 1.

Exercise 1

5) $A \triangleq \forall x \, \forall y \, p(x, y)$ and $B \triangleq \forall x \, \forall y \, p(y, x)$

We can just rename x to y and y to x in B and then we directly have that $A \leftrightarrow B$ as $\forall x \, \forall y \, \phi \leftrightarrow \forall y \, \forall x \, \phi$.

Exercise 2

Exercise 2

What is the link between the following formulas?

1. $A \triangleq \forall x\, P(x) \Rightarrow \forall x\, Q(x)$
2. $B \triangleq \exists x\, P(x) \Rightarrow \forall x\, Q(x)$
3. $C \triangleq \forall x\, P(x) \Rightarrow \exists x\, Q(x)$
4. $D \triangleq \forall x\, [P(x) \Rightarrow Q(x)]$

Need reminder on formula changes and also maybe on simplification procedure

Exercise 2

$$
\begin{aligned}
A &\triangleq \forall x\, P(x) \Rightarrow \forall x\, Q(x) && (1)\\
&\leftrightarrow \forall x\, P(x) \Rightarrow \forall y\, Q(y) && (2)\\
&\leftrightarrow \neg(\forall x\, P(x)) \vee \forall y\, Q(y) && (3)\\
&\leftrightarrow \exists x\, \neg P(x) \vee \forall y\, Q(y) && (4)\\
&\leftrightarrow \exists x\, \forall y\, [\neg P(x) \vee Q(y)] && (5)\\
&\leftrightarrow \exists x\, \forall y\, [P(x) \Rightarrow Q(y)] && (6)
\end{aligned}
$$

(2) Renaming quantified x
(4) $\neg\forall x\, p(x) \leftrightarrow \exists x \neg p(x)$
(5) $\exists x\, p(x) \vee \forall y\, q(y) \leftrightarrow \exists x\, \forall y\, [p(x) \vee q(y)]$

Exercise 2

$$
\begin{aligned}
B &\triangleq \exists x\, P(x) \Rightarrow \forall x\, Q(x) && (1)\\
&\leftrightarrow \exists x\, P(x) \Rightarrow \forall y\, Q(y) && (2)\\
&\leftrightarrow \neg(\exists x\, P(x)) \vee \forall y\, Q(y) && (3)\\
&\leftrightarrow \forall x\, \neg P(x) \vee \forall y\, Q(y) && (4)\\
&\leftrightarrow \forall x\, \forall y\, [\neg P(x) \vee Q(y)] && (5)\\
&\leftrightarrow \forall x\, \forall y\, [P(x) \Rightarrow Q(y)] && (6)
\end{aligned}
$$

(5) $\forall x\, p(x) \vee \forall y\, q(y) \leftrightarrow \forall x\, \forall y\, [p(x) \vee q(y)]$

Let's denote $R(x) = \forall y\, [P(x)) \Rightarrow Q(y)]$, we then have $A \leftrightarrow \exists x\, R(x)$ and $B \leftrightarrow \forall x\, R(x)$. As we know that $\forall x\, p(x) \models \exists x\, p(x)$ and $\exists x\, p(x) \not\models \forall x\, p(x)$, we have $B \models A$ and $A \not\models B$.

Exercise 2

$$\begin{aligned} C &\triangleq \forall x\, P(x) \Rightarrow \exists x\, Q(x) && (1)\\ &\leftrightarrow \forall x\, P(x) \Rightarrow \exists y\, Q(y) && (2)\\ &\leftrightarrow \neg(\forall x\, P(x)) \vee \exists y\, Q(y) && (3)\\ &\leftrightarrow \exists x\, \neg P(x) \vee \exists y\, Q(y) && (4)\\ &\leftrightarrow \exists x\, \neg P(x) \vee \exists y\, Q(y) && (5)\\ &\leftrightarrow \exists x\, \exists y\, [\neg P(x) \vee Q(y)] && (6)\\ &\leftrightarrow \exists x\, \exists y\, [P(x) \Rightarrow Q(y)] && (7) \end{aligned}$$

(6) $\exists x\, p(x) \vee \exists y\, q(y) \leftrightarrow \exists x\, \exists y\, [p(x) \vee q(y)]$

Looking at A and C, we can see that $A \models C$ and $C \not\models A$.
We therefore also have $B \models C$ (as $B \models A$ and $A \models C$).
And finally, $C \not\models B$ because if it was not the case, $A \models C \models B$ which is not the case.

Exercise 2

$$
\begin{align}
B &\triangleq \exists x\, P(x) \Rightarrow \forall x\, Q(x) \tag{1}\\
&\leftrightarrow \neg(\exists x\, P(x)) \vee \forall x\, Q(x) \tag{2}\\
&\leftrightarrow \forall x\, \neg P(x) \vee \forall x\, Q(x) \tag{3}\\
&\models \forall x\, [\neg P(x) \vee Q(x)] \tag{4}\\
&\leftrightarrow \forall x\, [P(x) \Rightarrow Q(x)] \tag{5}\\
&\leftrightarrow D \tag{6}
\end{align}
$$

(4) $\forall x\, p(x) \vee \forall x\, q(x) \models \forall x\, [p(x)) \vee q(x)]$

Therefore $B \models D$ and $D \not\models B$.

Exercise 2

$\underline{A \not\models D}$

Counter-example: $\mathcal{I}$ with:

- $\mathcal{D} = \{a, b\}$
- $I_c[P(a)] = T$
- $I_c[P(b)] = F$
- $I_c[Q(a)] = F$
- $I_c[Q(b)] = T$

$\underline{D \models A}$

If $\mathcal{I}[C] = F$ for any interpretation $\mathcal{I}$, then $\mathcal{I}_{x/d}[P(x)] = T$ and $\mathcal{I}_{x/d}[Q(x)] = F$ for each $d \in \mathcal{D}$.
Thus for each $d \in \mathcal{D}$, $\mathcal{I}_{x/d}[P(x) \Rightarrow Q(x)] = F$ and thus $\mathcal{I}[D] = F$

Exercise 2

We therefore also have $D \models C$ (as $D \models A$ and $A \models C$).

And finally, $C \not\models D$ because if it was not the case, $A \models C \models D$ which is not the case.

Exercise 3

Exercise 3

What is the link between the following formulas?

1. $A \triangleq \forall x \exists y \, [P(x) \Rightarrow Q(x,y)]$
2. $B \triangleq \forall x \, [P(x) \Rightarrow \exists y \, Q(x,y)]$
3. $C \triangleq \forall x P(x) \Rightarrow \exists y \, Q(x,y)$
4. $D \triangleq \forall x \, [P(x) \Rightarrow \forall x \exists y \, Q(x,y)]$

Exercise 3

$$\begin{aligned} A &\triangleq \forall x \exists y [P(x) \Rightarrow Q(x,y)] \quad (1) \\ &\leftrightarrow \forall x [\forall y\, P(x) \Rightarrow \exists y\, Q(x,y)] \quad (2) \\ &\leftrightarrow \forall x [P(x) \Rightarrow \exists y\, Q(x,y)] \quad (3) \\ &\leftrightarrow B \quad (4) \end{aligned}$$

(2): $\exists x (A(x) \Rightarrow B(x)) \leftrightarrow \forall x\, A(x) \Rightarrow \exists x\, B(x)$

Exercise

$B \models C$ and $C \not\models B$
$B \not\models D$, $D \models B$

Exercise 4

Exercise 4

What is the link between the following formulas?

1. $\alpha \triangleq \exists x \, \exists y \, \exists z \, [P(x, y) \Rightarrow [Q(x, z) \Rightarrow R(y, z)]]$
2. $\beta \triangleq \exists x \, \exists y \, [P(x, y) \Rightarrow [\forall z \, Q(x, z) \Rightarrow \exists z \, R(y, z)]]$
3. $\gamma \triangleq \forall x \, \forall y \, P(x, y) \Rightarrow [\forall x \, \forall z \, Q(x, z) \Rightarrow \exists y \, \exists z \, R(y, z)]$

Exercise 4

We are going to use the relation:
$\exists x (A(x) \Rightarrow B(x)) \leftrightarrow \forall x\, A(x) \Rightarrow \exists x\, B(x)$.

$$
\begin{align}
\alpha &\triangleq \exists x\, \exists y\, \exists z\, [P(x,y) \Rightarrow [Q(x,z) \Rightarrow R(y,z)]] \tag{1}\\
&\leftrightarrow \exists x\, \exists y\ \ [\forall z\, P(x,y) \Rightarrow \exists z\, [Q(x,z) \Rightarrow R(y,z)]] \tag{2}\\
&\leftrightarrow \exists x\, \exists y\ \ [P(x,y) \Rightarrow [\forall z\, Q(x,z) \Rightarrow \exists z\, R(y,z)]] \tag{3}\\
&\leftrightarrow \beta \tag{4}\\
&\leftrightarrow \exists x\ [\forall y\, P(x,y) \Rightarrow \exists y\, [\forall z\, Q(x,z) \Rightarrow \exists z\, R(y,z)]] \tag{5}\\
&\leftrightarrow \exists x\ [\forall y\, P(x,y) \Rightarrow [\forall z\, Q(x,z) \Rightarrow \exists y \exists z\, R(y,z)]] \tag{6}\\
&\leftrightarrow \forall x \forall y\, P(x,y) \Rightarrow \exists x\, [\forall z\, Q(x,z) \Rightarrow \exists y \exists z\, R(y,z) \tag{7}\\
&\leftrightarrow \forall x \forall y\, P(x,y) \Rightarrow [\forall x \forall z\, Q(x,z) \Rightarrow \exists y \exists z\, R(y,z) \tag{8}\\
&\leftrightarrow \gamma \tag{9}
\end{align}
$$

Exercise 5

Exercise 5

What can you say about the following inference rule?

$$\frac{H \Rightarrow \forall x A(x),\, H \Rightarrow \exists x \left[A(x) \Rightarrow \forall y B(x,y)\right]}{H \Rightarrow \exists x \forall y B(x,y)}$$

Exercise 5

We have 2 possibilities for H:

- If $\mathcal{I}[H] = F$, then the rule is correct.
- If $\mathcal{I}[H] = T$, then the rule becomes:

$$\frac{\forall x A(x), \exists x \left[A(x) \Rightarrow \forall y B(x,y)\right]}{\exists x \forall y B(x,y)}$$

We have that $\exists x \left[A(x) \Rightarrow \forall y B(x,y)\right] \leftrightarrow \forall x\, A(x) \Rightarrow \exists x \forall y\, B(x,y)$.

If we set $A' = \forall x\, A(x)$ and $B' = \exists x \forall y\, B(x,y)$, then the rule is equivalent to:

$$\frac{A', A' \Rightarrow B'}{B'}$$

which is the Modus Ponens rule (see slide 72 of course) and therefore the rule is true.

Exercise 6

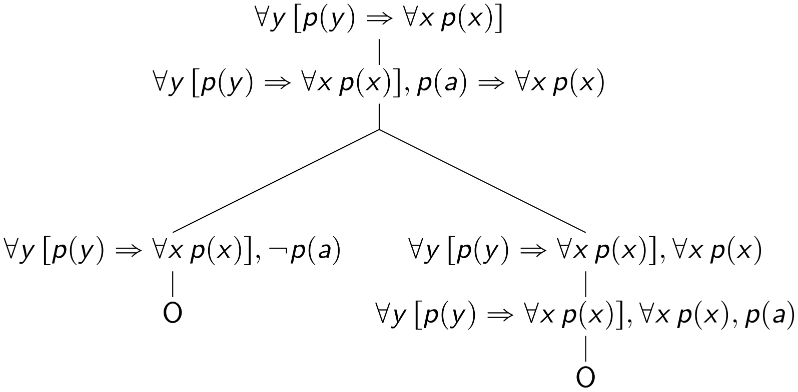
Exercise 6

Using the semantic tableaux method, determine whether the following formulas are valid, consistent or inconsistent.

1. $\forall y\,[p(y) \Rightarrow \forall x\, p(x)]$
2. $\forall x\,[p(x) \Rightarrow q(x)] \Rightarrow [\forall x\, p(x) \Rightarrow \forall x\, q(x)]$
3. $[\forall x\, p(x) \wedge \neg\forall y\, q(y)] \vee \forall z\,[p(z) \Rightarrow q(z)]$
4. $\forall x \exists y\, p\,(x, y) \wedge \forall x\, \neg p(x, x) \wedge \forall x \forall y \forall z\,[(p(x, y) \wedge p(y, z)) \Rightarrow p(x, z)]$

Exercise 6

1)



$\forall y\,[p(y) \Rightarrow \forall x\,p(x)]$

$\forall y\,[p(y) \Rightarrow \forall x\,p(x)], p(a) \Rightarrow \forall x\,p(x)$

$\forall y\,[p(y) \Rightarrow \forall x\,p(x)], \neg p(a)$

O

$\forall y\,[p(y) \Rightarrow \forall x\,p(x)], \forall x\,p(x)$

$\forall y\,[p(y) \Rightarrow \forall x\,p(x)], \forall x\,p(x), p(a)$

O

The formula is consistent.

The first constant a is chosen freely as there is no constant in the set initially.

The second time, we instantiate in a, we do so to confirm the condition that each γ-formula on the branch has been instantiated for a.

Exercise 6

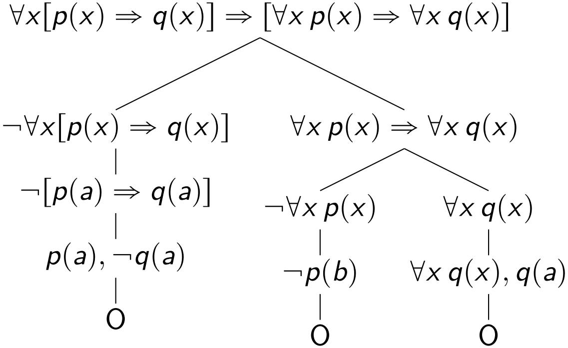
Let's analyze the negated formula:

$$\begin{array}{c}
\neg\forall y\,[p(y) \Rightarrow \forall x\, p(x)] \\
| \\
\exists y\, \neg[p(y) \Rightarrow \forall x\, p(x)] \\
| \\
\neg[p(a) \Rightarrow \forall x\, p(x)] \\
| \\
p(a), \neg\forall x\, p(x) \\
| \\
p(a), \exists\neg x\, p(x) \\
| \\
p(a), \neg p(b) \\
| \\
\mathrm{O}
\end{array}$$

The formula is not valid.

Exercise 6

2)



$\forall x[p(x) \Rightarrow q(x)] \Rightarrow [\forall x\, p(x) \Rightarrow \forall x\, q(x)]$

$\neg\forall x[p(x) \Rightarrow q(x)]$

$\neg[p(a) \Rightarrow q(a)]$

$p(a), \neg q(a)$

O

$\forall x\, p(x) \Rightarrow \forall x\, q(x)$

$\neg\forall x\, p(x)$

$\neg p(b)$

O

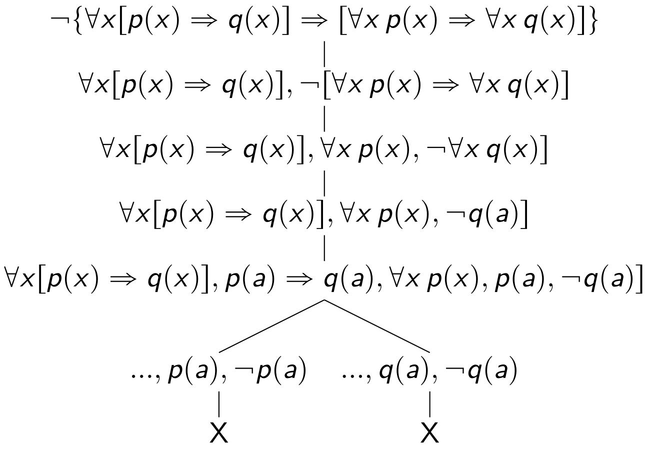
$\forall x\, q(x)$

$\forall x\, q(x), q(a)$

O

The formula is consistent.

Exercise 6



$$\neg\{\forall x[p(x) \Rightarrow q(x)] \Rightarrow [\forall x\, p(x) \Rightarrow \forall x\, q(x)]\}$$

$$\forall x[p(x) \Rightarrow q(x)], \neg[\forall x\, p(x) \Rightarrow \forall x\, q(x)]$$

$$\forall x[p(x) \Rightarrow q(x)], \forall x\, p(x), \neg\forall x\, q(x)]$$

$$\forall x[p(x) \Rightarrow q(x)], \forall x\, p(x), \neg q(a)]$$

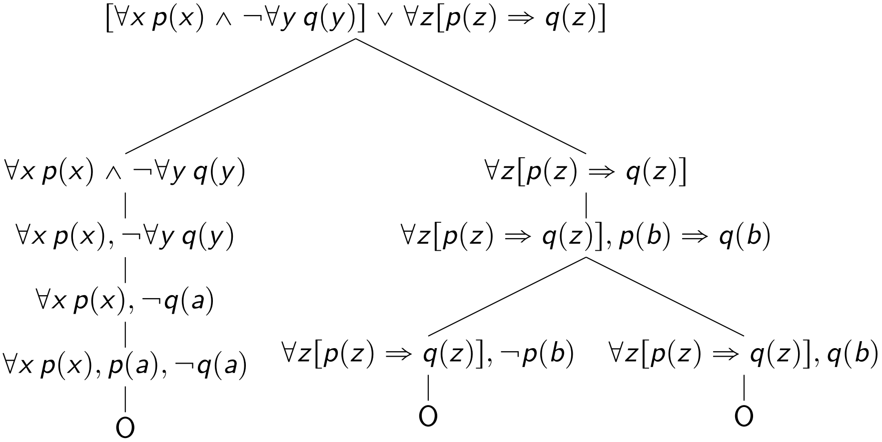
$$\forall x[p(x) \Rightarrow q(x)], p(a) \Rightarrow q(a), \forall x\, p(x), p(a), \neg q(a)]$$

$$\ldots, p(a), \neg p(a) \qquad \ldots, q(a), \neg q(a)$$

The formula is valid

Exercise 6

3)



$[\forall x\, p(x) \wedge \neg\forall y\, q(y)] \vee \forall z[p(z) \Rightarrow q(z)]$

$\forall x\, p(x) \wedge \neg\forall y\, q(y)$

$\forall x\, p(x), \neg\forall y\, q(y)$

$\forall x\, p(x), \neg q(a)$

$\forall x\, p(x), p(a), \neg q(a)$

O

$\forall z[p(z) \Rightarrow q(z)]$

$\forall z[p(z) \Rightarrow q(z)], p(b) \Rightarrow q(b)$

$\forall z[p(z) \Rightarrow q(z)], \neg p(b)$

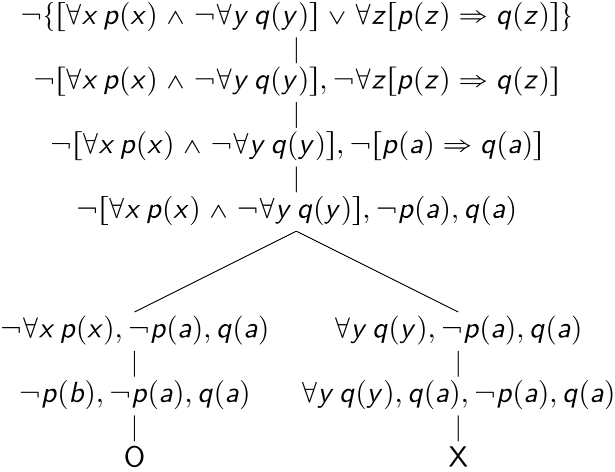
O

$\forall z[p(z) \Rightarrow q(z)], q(b)$

O

The formula is consistent.

Exercise 6



$$\neg\{[\forall x\, p(x) \wedge \neg\forall y\, q(y)] \vee \forall z[p(z) \Rightarrow q(z)]\}$$

$$\neg[\forall x\, p(x) \wedge \neg\forall y\, q(y)], \neg\forall z[p(z) \Rightarrow q(z)]$$

$$\neg[\forall x\, p(x) \wedge \neg\forall y\, q(y)], \neg[p(a) \Rightarrow q(a)]$$

$$\neg[\forall x\, p(x) \wedge \neg\forall y\, q(y)], \neg p(a), q(a)$$

$$\neg\forall x\, p(x), \neg p(a), q(a) \qquad \forall y\, q(y), \neg p(a), q(a)$$

$$\neg p(b), \neg p(a), q(a) \qquad \forall y\, q(y), q(a), \neg p(a), q(a)$$

$$\mathrm{O} \qquad \mathrm{X}$$

The formula is not valid.

Exercise 6

4) Consistent but not valid

Exercise 7

Exercise 7

Consider the following inference rules:

$$1)\ \frac{\forall x\, A \qquad \forall x\, (A \Rightarrow B)}{\forall x\, B} \qquad 2)\ \frac{\exists x\, A \qquad \forall x\, (A \Rightarrow B)}{\exists x\, B}$$

$$3)\ \frac{\exists x\, A \qquad \exists x\, (A \Rightarrow B)}{\exists x\, B} \qquad 4)\ \frac{\forall x\, A \qquad \exists x\, (A \Rightarrow B)}{\exists x\, B}$$

Are they correct?
If not, do they become correct if one adds restrictions on the occurences of the variable x within A and/or B? Motivate your answers.

Exercise 7

1) Let's see if the set $\{\forall x\, A, \forall x(A \Rightarrow B), \neg\forall x\, B\}$ is inconsistent.

$$\begin{array}{c}
\forall x\, A(x), \forall x(A(x) \Rightarrow B(x)), \neg\forall x\, B(x) \\
| \\
\forall x\, A(x), \forall x(A(x) \Rightarrow B(x)), \neg B(a) \\
| \\
\forall x\, A(x), A(a), \forall x(A(x) \Rightarrow B(x)), A(a) \Rightarrow B(a), \neg B(a) \\
| \\
\forall x\, A(x), A(a), \forall x(A(x) \Rightarrow B(x)), \neg A(a), B(a), \neg B(a) \\
| \\
\mathrm{X}
\end{array}$$

The set is inconsistent, therefore the rule is correct.

Exercise 7

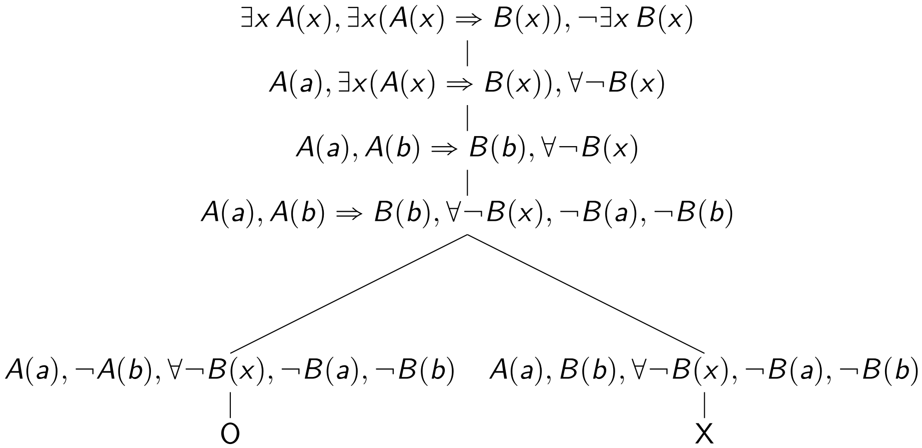
2) Let's see if the set $\{\exists x\, A, \forall x(A \Rightarrow B), \neg\exists x\, B\}$ is inconsistent.

$$\begin{array}{c}
\exists x\, A(x), \forall x(A(x) \Rightarrow B(x)), \neg\exists x\, B(x) \\
| \\
A(a), \forall x(A(x) \Rightarrow B(x)), \forall\neg B(x) \\
| \\
A(a), \forall x(A(x) \Rightarrow B(x)), A(a) \Rightarrow B(a), \forall\neg B(x), \neg B(a) \\
| \\
A(a), \forall x(A(x) \Rightarrow B(x)), \neg A(a), B(a), \forall\neg B(x), \neg B(a) \\
| \\
\mathrm{X}
\end{array}$$

The set is inconsistent, therefore the rule is correct.

Exercise 7

3) Let's see if the set $\{\exists x\, A, \exists x(A \Rightarrow B), \neg\exists x\, B\}$ is inconsistent.



$$\exists x\, A(x), \exists x(A(x) \Rightarrow B(x)), \neg\exists x\, B(x)$$

$$A(a), \exists x(A(x) \Rightarrow B(x)), \forall\neg B(x)$$

$$A(a), A(b) \Rightarrow B(b), \forall\neg B(x)$$

$$A(a), A(b) \Rightarrow B(b), \forall\neg B(x), \neg B(a), \neg B(b)$$

$$A(a), \neg A(b), \forall\neg B(x), \neg B(a), \neg B(b) \qquad A(a), B(b), \forall\neg B(x), \neg B(a), \neg B(b)$$

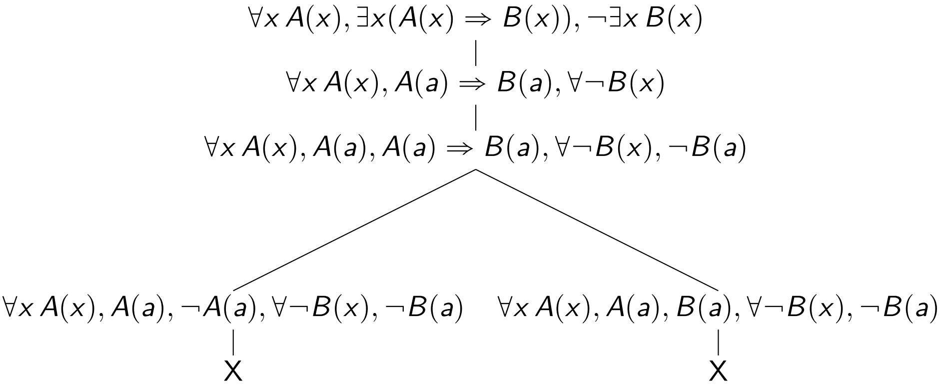
The set is consistent, therefore the rule if false.

Exercise 7

To make it correct, we can consider that A does not depend on x. The rules then becomes $\frac{A, A \Rightarrow \exists B}{\exists B}$ which is correct as it is a Modus Ponens.

Exercise 7

4) Let's see if the set $\{\forall x\, A, \exists x(A \Rightarrow B), \neg\exists x\, B\}$ is inconsistent.



$$\forall x\, A(x), \exists x(A(x) \Rightarrow B(x)), \neg\exists x\, B(x)$$

$$\forall x\, A(x), A(a) \Rightarrow B(a), \forall\neg B(x)$$

$$\forall x\, A(x), A(a), A(a) \Rightarrow B(a), \forall\neg B(x), \neg B(a)$$

$$\forall x\, A(x), A(a), \neg A(a), \forall\neg B(x), \neg B(a) \qquad \forall x\, A(x), A(a), B(a), \forall\neg B(x), \neg B(a)$$

The set is inconsistent, therefore the rule is correct.