Logic - Tutorial 7

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Reminder

The syntax

We define

- $A = \{a, a_1, a_2, ..., b, c, ...\}$: a set of *(individual) constants*
- $X = \{x, x_1, x_2, x', ..., y, z, ...\}$: a set of *(individual) variables*
- $P = \{p, q, r, ...\}$: a set of *predicate symbols* (with arity)
- $F = f, g, h, ...$: a set of *function symbols* (each with its arity)

Examples:

- $p(x, y) = T$ if $x < y$
- $f(x, y) = x + y$

Note:
Propositions (i.e. atoms in propositional logic) are 0-ary predicate symbols.
Constants are 0-ary function symbols.

Reminder

How to build formulas:

- *Terms* are recursively defined:
  - A *variable* is a term
  - A *constant* is a term
  - If f is an m-ary *function symbol* and if $t_1, ..., t_m$ are *terms*, then $f(t_1, ..., t_m)$ is a term.
- An *atomic formula (atom)* is an expression $p(t_1, ..., t_n)$ where $p \in P$ is a n-ary predicate symbol and $t_1, ..., t_n$ are *terms*.
- *Formulas* are recursively defined:
  - An *atomic formula* is a formula.
  - *true, false* are formulas.
  - If A is a formula, then $\neg A$ is a formula.
  - If A_1 and A_2 are formulas then $(A_1 \vee A_2), (A_1 \wedge A_2), (A_1 \Rightarrow A_2), (A_1 \equiv A_2), ...$ are formulas
  - If A is a formula and x is a variable, then $\forall x\, A$ and $\exists x\, A$ are formulas.

Reminder

Quantified, bound and free variable occurence

In $\forall x\, A$ and $\exists x\, A$, the *scope* of x is A.

- The occurence of variable x in quantification $\forall x$ or $\exists x$ is *quantified*
- Any occurence of x in the scope of a quantification is *bound*.
- A variable occurence that is neither quantified nor bound is *free*.

Reminder

The semantic

An *interpretation (or valuation)* $\mathcal{I}$ is a triple (D, I_c, I_v) s.t.

- D is a non-empty set, the domain;
- I_c is a function that maps:
  - an object $I_c[a] \in D$ to each constant a,
  - a function $I_c[f]$ of type $D^m \mapsto D$ to each m-ary function symbol f
  - an n-ary relation on D, i.e. a function $I_c[p]$ of type $D^n \mapsto \{T, F\}$ to each n-ary predicate symbol p;
- I_v is a function that maps an object $I_v[x] \in D$ to each variable x

An interpretation $\mathcal{I} = (D, I_c, I_v)$ assigns an element of D to every term and a truth value to every formula.

Reminder

Terms interpretation

- If x is a variable $\mathcal{I}[x] = I_v[x]$
- If a is a constant $\mathcal{I}[a] = I_c[a]$
- If f is a m-ary function symbol and if $t_1, \ldots, t_m$ are terms, then $\mathcal{I}[f(t_1, \ldots, t_m)] = I_c[f](\mathcal{I}[t_1], \ldots, \mathcal{I}[t_m])$

Formulas interpretation

- If p is a n-ary predicate symbol and if $t_1, \ldots, t_n$ are terms, then $\mathcal{I}[p(t_1, \ldots, t_n)] = (I_c[p])(\mathcal{I}[t_1], \ldots, \mathcal{I}[t_n])$
- $\mathcal{I}[true] = T$ and $\mathcal{I}[false] = F$
- If A is a formula, then
  - $\mathcal{I}[\neg A] = T$ if $\mathcal{I}[A] = F$
  - $\mathcal{I}[\neg A] = F$ if $\mathcal{I}[A] = T$
- If A_1 and A_2 are formulas, then $(A_1 \vee A_2)$, $(A_1 \wedge A_2)$, $(A_1 \Rightarrow A_2)$, $(A_1 \equiv A_2)$ are interpreted as in propositional logic.

Reminder

If $\mathcal{I} = (D_{\mathcal{I}}, I_c, I_v)$ is an interpretation, if x is a variable and d is an element of $D_{\mathcal{I}}$, then $\mathcal{J} = \mathcal{I}_{x/d}(D_{\mathcal{J}}, J_c, J_v)$ such that

- $D_{\mathcal{J}} = D_{\mathcal{I}}$
- $J_c = I_c$
- $J_v[x] = d$ and $J_v[y] = I_v[y]$ for each variable y other than x

How to interpret $\forall x\, A$ and $\exists x\, A$? If A is a formula and x is a variable,

$$\mathcal{I}[\forall x\, A] = T \text{ if } \mathcal{I}_{x/d}[A] = T \text{ for each element } d \in D, \\ = F \text{ else.}$$

$$\mathcal{I}[\exists x\, A] = T \text{ if } \mathcal{I}_{x/d}[A] = T \text{ for at least one element } d \in D, \\ = F \text{ else.}$$

Reminder

$\mathcal{I}$ is a *model* of formula A if $\mathcal{I}[A] = T$.

As with propositional logic, based on models, we can define consistency (or satisfiability), validity and inconsistency (or unsatisfiability).

Note: A formula that is simply consistent or *contingent* means it is consistent but not valid.

As with propositional logic, we can also define the concepts of logical consequence and logical equivalance.

Reminder

Some formulas relationships

- $(\forall x\, A \wedge \forall x\, B) \leftrightarrow \forall x\, (A \wedge B)$
- $(\forall x\, A \vee \forall x\, B) \models \forall x\, (A \vee B)$
- $\forall x (A \Rightarrow B) \models (\forall x\, A \Rightarrow \forall x\, B)$
- $\forall x (A \equiv B) \models (\forall x\, A \equiv \forall x\, B)$

- $\exists x (A \vee B) \leftrightarrow (\exists x\, A \vee \exists x\, B)$
- $\exists x (A \wedge B) \models (\exists x\, A \wedge \exists x\, B)$
- $\exists x (A \Rightarrow B) \leftrightarrow (\forall x\, A \Rightarrow \exists x\, B)$

- $\neg \forall x\, A \leftrightarrow \exists x\, \neg A$
- $\neg \exists x\, A \leftrightarrow \forall x\, \neg A$

- $\forall x \forall y\, A \leftrightarrow \forall y \forall x\, A$
- $\exists x \exists y\, A \leftrightarrow \exists y \exists x\, A$
- $\exists x \forall y\, A \models \forall y \exists x\, A$

Exercise 1

In the definition of the interpretation of a predicate formula, what hypothesis is absolutely necessary in order for $\exists x \, (p(x) \Rightarrow p(x))$ to be valid.

Exercise 1

The domain must be non empty.

Exercise 2

Exercise 2

Is the following reasoning correct?

Some students do not work
All students want to pass

Some people want to pass without working

Exercise 2

Let define D to be the ensemble of students and let $x \in D$. Then we define the predicates:

- $w(x)$ is true if x works
- $p(x)$ is true if x wants to pass

The reasoning the becomes:

$$H_1 \triangleq \exists x \, \neg w(x)$$
$$H_2 \triangleq \forall x \, p(x)$$

$$C \triangleq \exists x (p(x) \wedge \neg w(x))$$

Is this correct?

Let $\mathcal{I} = (D, I_c, I_v)$ be an interpretation that makes H_1 and H_2 true.
Then $\exists a \in D$, s.t. $\mathcal{I}[\neg w(a)] = T$.
Moreover as, $\mathcal{I}[\forall x \, p(x)] = T$, in particular $\mathcal{I}[p(a)] = T$.
Hence: $\mathcal{I}[p(a) \wedge \neg w(a)] = T$.
And so: $\mathcal{I}[C] = \mathcal{I}[\exists x(p(x) \wedge \neg w(x)] = T$

<u>Exercise 3</u>

Tony, Mike and John are members of an alpine club. Each member is a skier or an alpinist or both. No alpinist likes rain but all skiers like snow. Mike likes nothing that Tony likes and likes everything that Tony doesn't like. Tony likes rain and snow. Is there a member of the alpine club that is an alpinist but not a skier?

Exercise 3

Let define D to be the set of alpine club members. From the first sentence, we define three constants: $t, m, j \in D$.

For $x \in D$ we have the following sets of predicates:

- $k(x)$ is true if x is a skier
- $a(x)$ is true if x is an alpinist
- $r(x)$ is true if x likes rain
- $s(x)$ is true if x likes snow

Let's transform the sentences in a mathematical reasoning:

$$H_1 \triangleq \forall x\,(k(x) \vee a(x))$$
$$H_2 \triangleq \forall x\,(r(x) \Rightarrow \neg a(x))$$
$$H_3 \triangleq \forall x\,(k(x) \Rightarrow s(x))$$
$$H_4 \triangleq (r(t) \equiv \neg r(m)) \wedge (s(t) \equiv \neg s(m))$$
$$H_5 \triangleq r(t) \wedge s(t)$$

$$C \triangleq \exists x\,(a(x) \wedge \neg k(x))$$

Exercise 3

Let $\mathcal{I}$ be an interpretation making the hypothesis true. Is the conclusion true?

- $\mathcal{I}[H_5] = T$ implies $\mathcal{I}[r(t)] = \mathcal{I}[s(t)] = T$
- $\mathcal{I}[H_4] = T$ implies $\mathcal{I}[r(m)] = \mathcal{I}[\neg r(t)] = F$ and $\mathcal{I}[s(m)] = \mathcal{I}[\neg s(t)] = F$
- $\mathcal{I}[H_3] = T$ and $\mathcal{I}[s(m)] = F$ implies $\mathcal{I}[k(m)] = F$
- $\mathcal{I}[H_1] = T$ and $\mathcal{I}[k(m)] = F$ implies $\mathcal{I}[a(m)] = T$

Therefore $\mathcal{I}[a(m) \wedge \neg k(m)] = T$ and $\mathcal{I}[\exists x\,(a(x) \wedge \neg k(x))] = T$.

Exercise 3

Subsidiary question (asked by students): Does John satisfies the conclusion as he is not constrained by the hypothesis?

This would be the case if for **any** interpretation $\mathcal{I}$ making the hypothesis true, $\mathcal{I}[a(j) \wedge \neg k(j)] = T$.
However, the following interpretation:

- $\mathcal{I}[a(j)] = T$
- $\mathcal{I}[k(j)] = T$
- $\mathcal{I}[s(j)] = T$
- $\mathcal{I}[r(j)] = F$

respects the hypothesis but makes the conclusion false. Therefore, we have no guarantee that *John* is just an alpinist and not a skier.

Exercise 4

If $A \models \forall x\, p(x)$ and $\exists x\, p(x) \models B$, then $A \Rightarrow \exists x\, p(x) \models \forall x\, p(x) \Rightarrow B$.
Is this statement correct for all formulas A and B?

Exercise 4

Hypothesis:

- $A \models \forall x\, p(x)$
- $\exists x\, p(x) \models B$

Show: $A \Rightarrow \exists x\, p(x) \models \forall x\, p(x) \Rightarrow B$

Let $\mathcal{I}$ be an interpretation s.t. $\mathcal{I}[A \Rightarrow \exists x\, p(x)] = T$.
Thus either $\mathcal{I}[A] = F$ or $\mathcal{I}[\exists x\, p(x)] = T$

Exercise 4

Case 1: $\mathcal{I}[\exists x\, p(x)] = T$
Then using the second hypothesis, $\mathcal{I}[B] = T$ and thus
$\mathcal{I}[\forall x\, p(x) \Rightarrow B] = T$

Case 2: $\mathcal{I}[A] = F$
Then using hypothesis one, $\mathcal{I}[\forall x\, p(x)] = T$ or $\mathcal{I}[\forall x\, p(x)] = F$.

- Case 2.1.: $\mathcal{I}[\forall x\, p(x)] = T$ implies $\mathcal{I}[\exists x\, p(x)] = T$ and we reach the same conclusion as case 1.
- Case 2.2.: If $\mathcal{I}[\forall x\, p(x)] = F$, we directly have $\mathcal{I}[\forall x\, p(x) \Rightarrow B] = T$.

Exercise 5

Determine whether the following formulas are valid, consistent or inconsistent.

1. $\forall x\,[p(x) \Rightarrow p(a)]$
2. $\forall x\,[p(x) \Rightarrow p(x)]$
3. $\forall x\,[p(y) \Rightarrow q(x)] \Rightarrow [p(y) \Rightarrow \forall x\, q(x)]$
4. $\forall x\,[p(x) \Rightarrow q(x)] \Rightarrow [p(x) \Rightarrow \forall x\, q(x)]$

Exercise 5

1) $\forall x\,[p(x) \Rightarrow p(a)]$: consistent but not valid

Model $\mathcal{I} = (D, I_c, I_v)$ with:

- D = even numbers
- $I_c[a] = 2$
- $I_c[p]$ = prime (number) predicate

Anti-model $\mathcal{I} = (D, I'_c, I'_v)$ with:

- $I'_c[a] = 4$
- $I'_c[p] = I_c[p]$

Exercise 5

2) $\forall x\,[p(x) \Rightarrow p(x)]$: valid

Indeed, given an interpretation $\mathcal{I}$, $\mathcal{I}_{x/d}[p(x) \Rightarrow p(x)] = T$ for each $d \in D$.

Thus $\mathcal{I}[\forall x\,(p(x) \Rightarrow p(x))] = T$

Exercise 5

3) $\forall x\,[p(y) \Rightarrow q(x)] \Rightarrow [p(y) \Rightarrow \forall x\, q(x)]$: valid

We have

$$\begin{aligned}\forall x\,[p(y) \Rightarrow q(x)] &\leftrightarrow [\forall x\, p(y) \Rightarrow \forall x\, q(x)] \\ &\leftrightarrow [p(y) \Rightarrow \forall x\, q(x)]\end{aligned}$$

Exercise 5

4) $\forall x\,[p(x) \Rightarrow q(x)] \Rightarrow [p(x) \Rightarrow \forall x\, q(x)]$: consistent but not valid
Model $\mathcal{I} = (D, I_c, I_v)$ with:

- D = we don't care
- $I_c[p] = T$
- $I_c[q] = T$

Anti-model $\mathcal{I} = (D', I'_c, I'_v)$ with:

- $D = \{0, 1\}$
- $I'_c[p(0)] = T$
- $I'_c[p(1)] = F$
- $I'_c[q(0)] = T$
- $I'_c[q(1)] = F$
- $I'_v[x] = 0$ (x being the only free one)

Exercise 6

What can you say about the following inference rule?

$$\frac{p(a), \forall x\,[p(x) \Rightarrow p(f(x))]}{\forall x\, p(x)}$$

Exercise 6

Let $\mathcal{I} = (D, I_c, I_v)$ where:

- $D = \{a, b\}$
- $I_c[f] =$ identity function
- $I_c[p(a)] = T$
- $I_c[p(b)] = F$

Then the two hypothesis are true but the conclusion is false, which means the rule is incorrect.

Exercise 7

What can you say about the following inference rule?

$$\frac{\forall x\, p(x,x),\ \forall x \forall y\, [p(x,y) \Rightarrow p(x,f(x))]}{\forall x \forall y\, p(x,y)}$$

Exercise 7

Let $\mathcal{I} = (D, I_c, I_v)$ where:

- $D = \{a, b\}$
- $I_c[f]$ = identity function
- $I_c[p(a, a)] = I_c[p(b, b)] = T$
- $I_c[p(a, b)] = I_c[p(b, a)] = F$

Then the two hypothesis are true but the conclusion is false, which means the rule is incorrect.

Other proof

Let $\mathcal{I} = (D, I_c, I_v)$ where:

- $D = \mathbb{N}$
- $I_c[f]$ = identity function
- $I_c[p]$ = divide?