Logic - Tutorial 6

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Reminder

Yet a new decision procedure: **Resolution**
$\Rightarrow$ Requires formulas in clausal form, or conjunctive normal form.

Normal form

A *disjunctive normal form* (DNF) is a **disjunction** of *cubes* which are **conjunctions** of literals.

Expl: $(p \wedge q) \vee (p \wedge r)$

A *conjunctive normal form* (CNF) is a **conjunction** of *clauses* which are **disjunctions** of literals.

Expl: $(r \vee s) \wedge (s \vee t) \wedge (t \vee u)$

Reminder

Normalization algorithm

1. Eliminate all connections but $\neg$, $\wedge$, $\vee$
2. Use *De Morgan laws* for propagating $\neg$ occurences downwards
   - $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$
   - $\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$
3. Eliminate double negatives: $\neg\neg A \leftrightarrow A$
4. Use *distributivity laws*:
   - $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$
   - $A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$

Reminder

Resolution procedure

The resolution procedure is divided into three steps:

1. Convert a formula A into its causal form.
2. All the clauses of this form compose a set of clause that we denote S.
3. Prove that A is inconsistent by showing that the set S is inconsistent.

A set of clauses S is inconsistent iff $S \models \square$ ($\square$ is the empty clause also denote false).
Therefore, we can prove that S is inconsistent by "deriving" $\square$ from S.

Reminder

<u>Procedure to find if $\Box$ can be derived from S</u>

$S := S_0$;

While $\Box \notin S$, do:

  select $p \in \Pi_S$ such that

    $C_1 \triangleq (C_1' \vee p) \in S$,

    $C_2 \triangleq (C_2' \vee \neg p) \in S$;

  $S := S \cup \{Res(C_1, C_2)\}$

where *Res* is the *resolution rule*: $\frac{A \vee X, B \vee \neg X}{A \vee B}$

The procedure halts in two cases:

- 1) $\Box \in S$: The formula a is inconsistent.
- 2) There is no more $p \in \Pi_S$ respecting the two conditions: The formula is consistent.

<u>Exercise 1</u>

Five people (a, b, c, d, e) have put their money into the same safe. They however have no confidence in each other and decided therefore that the safe can only be opened in the presence of a and b, or b and c, or b, d and e. How many locks does the safe have? How many keys are needed? And who has them?
Hint: Consider the formula

$$\phi\left(p_a, p_b, p_c, p_d, p_e\right) \triangleq \text{"the safe can be opened"},$$

where p_x is true if x is present.

Exercise 1 - Solution

We have that $\phi(p_a, p_b, p_c, p_d, p_e) = (p_a \wedge p_b) \vee (p_b \wedge p_c) \vee (p_b \wedge p_d \wedge p_e)$. We will put this formula in conjunctive normal form. The number of conjunctions will give us the number of keys and the different conjunctions tell us who has which key.

$$\begin{aligned} & (p_a \wedge p_b) \vee (p_b \wedge p_c) \vee (p_b \wedge p_d \wedge p_e) \\ \longleftrightarrow \quad & p_b \wedge (p_a \vee p_c \vee (p_d \wedge p_e)) \\ \longleftrightarrow \quad & p_b \wedge (p_a \vee p_c \vee p_d) \wedge (p_a \vee p_c \vee p_e) \end{aligned}$$

We will have three locks and keys will be distributed in the following way: b has key to lock 1,

- *b* has a key to open lock 1
- *a*, *c* and *d* have keys to open lock 2
- *a*, *c* and *e* have keys to open lock 3

Exercise 2

Give the disjunctive normal form of the following formulas

$$A \triangleq \bigwedge_{1 \leqslant i < n} (p_i \Rightarrow p_{i+1})$$

$$B \triangleq A \wedge (p_n \Rightarrow p_1)$$

$$C \triangleq \bigwedge_{1 \leqslant i,j \leqslant n, i \neq j} (p_i \Rightarrow \neg p_j)$$

$$D \triangleq \bigwedge_{1 \leqslant i \leqslant n} \left(\bigvee_{1 \leqslant j \leqslant n, j \neq i} p_j \right)$$

Exercise 2 - Solution

$$A \triangleq \bigwedge_{1 \leqslant i < n} (p_i \Rightarrow p_{i+1}) \leftrightarrow (p_1 \Rightarrow p_2) \wedge (p_2 \Rightarrow p_3) \wedge \ldots \wedge (p_{n-1} \Rightarrow p_n)$$

1) $A \leftrightarrow \bigwedge_{1 \leqslant i < n} (\neg p_i \vee p_{i=1})$
2) and 3) OK
4)

$$\begin{aligned} A \leftrightarrow \quad & (\neg p_1 \vee p_2) \wedge (\neg p_2 \vee p_3) \wedge \ldots \wedge (\neg p_{n-1} \vee p_n) \\ \leftrightarrow \quad & [\neg p_1 \wedge (\neg p_2 \vee p_3) \wedge \ldots \wedge (\neg p_{n-1} \vee p_n)] \vee \\ & [p_2 \wedge (\neg p_2 \vee p_3) \wedge \ldots \wedge (\neg p_{n-1} \vee p_n)] \\ \leftrightarrow \quad & [\neg p_1 \wedge (\neg p_2 \vee p_3) \wedge \ldots \wedge (\neg p_{n-1} \vee p_n)] \vee \\ & [p_2 \wedge p_3 \wedge \ldots \wedge p_n] \end{aligned}$$

Distributivity and $p \wedge (\neg p \vee q) \leftrightarrow p \wedge q$.

Exercise 2 - Solution

$$
\begin{aligned}
A \leftrightarrow\ & [\neg p_1 \land (\neg p_2 \lor p_3) \land \ldots \land (\neg p_{n-1} \lor p_n)] \lor \\
& [p_2 \land p_3 \land \ldots \land p_n] \\
\leftrightarrow\ & [\neg p_1 \land \neg p_2 \land \ldots \land (\neg p_{n-1} \lor p_n)] \lor \\
& [\neg p_1 \land p_3 \land \ldots \land (\neg p_{n-1} \lor p_n)] \lor \\
& [p_2 \land p_3 \land \ldots \land p_n] \\
\leftrightarrow\ & [\neg p_1 \land \neg p_2 \land \ldots \land (\neg p_{n-1} \lor p_n)] \lor \\
& [\neg p_1 \land p_3 \land \ldots \land p_n] \lor \\
& [p_2 \land p_3 \land \ldots \land p_n] \\
\leftrightarrow\ & \bigvee_{1 \leqslant i < n} \left[\left(\bigwedge_{1 \leqslant j < i} \neg p_j \right) \land \left(\bigwedge_{i < k \leqslant n} p_k \right) \right]
\end{aligned}
$$

$$B \triangleq A \land (p_n \Rightarrow p_1) \leftrightarrow \bigwedge_{1 \leqslant i < n} (p_i \Rightarrow p_{i+1}) \land (p_n \Rightarrow p_1)$$

1) $B \triangleq \bigwedge_{1 \leqslant i < n} (\neg p_i \lor p_{i+1}) \land (\neg p_n \lor p_1)$
2) and 3) OK
4)

$$\begin{aligned}
B \leftrightarrow \quad & (\neg p_1 \lor p_2) \land (\neg p_2 \lor p_3) \land \ldots \land (\neg p_{n-1} \lor p_n) \land (\neg p_n \lor p_1) \\
\leftrightarrow \quad & [\neg p_1 \land (\neg p_2 \lor p_3) \land \ldots \land (\neg p_{n-1} \lor p_n) \land (\neg p_n \lor p_1)] \lor \\
& [p_2 \land (\neg p_2 \lor p_3) \land \ldots \land (\neg p_{n-1} \lor p_n) \land (\neg p_n \lor p_1)] \\
\leftrightarrow \quad & [\neg p_1 \land \neg p_2 \land \ldots \land \neg p_{n-1} \land \neg p_n] \lor \\
& [p_2 \land p_3 \land \ldots \land p_n \land p_1)] \\
\leftrightarrow \quad & \left(\bigwedge_{1 \leqslant i \leqslant n} \neg p_i \right) \lor \left(\bigwedge_{1 \leqslant i \leqslant n} p_i \right)
\end{aligned}$$

Exercise 2 - Solution

$$C \triangleq \bigwedge_{1\leqslant i,j\leqslant n, i\neq j} (p_i \Rightarrow \neg p_j)$$

1) $C \triangleq \bigwedge_{1\leqslant i,j\leqslant n, i\neq j} (\neg p_i \vee \neg p_j)$
2) and 3) OK
4)

$$\begin{aligned} C \leftrightarrow \quad & (\neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee \neg p_3) \wedge \ldots \wedge (\neg p_1 \vee \neg p_n) \wedge \\ & (\neg p_2 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_4) \wedge \ldots \wedge (\neg p_2 \vee \neg p_n) \wedge \\ & \ldots \wedge \\ & (\neg p_{n-1} \vee \neg p_n) \end{aligned}$$

Distributivity and $\neg p \wedge (\neg p \vee q) \leftrightarrow p \wedge q$ with $p \triangleq p_1$ and $p \triangleq p_2$

Exercise 2 - Solution

$$C \leftrightarrow \quad [\neg p_1 \wedge (\neg p_2 \vee \neg p_3) \wedge \ldots \wedge (\neg p_2 \vee \neg p_n) \wedge \ldots (\neg p_{n-1} \vee \neg p_n)] \vee [\neg p_2 \wedge (\neg p_1 \vee \neg p_3) \wedge \ldots \wedge (\neg p_1 \vee \neg p_n) \wedge \ldots (\neg p_{n-1} \vee \neg p_n)]$$

Distributivity and Distributivity

Exercise 2 - Solution

$$C \leftrightarrow \quad 1)[\neg p_1 \wedge \neg p_2 \wedge (\neg p_3 \vee \neg p_4) \wedge \ldots \wedge (\neg p_3 \vee \neg p_n) \wedge \ldots (\neg p_{n-1} \vee \neg p_n)] \vee$$
$$2)[\neg p_1 \wedge \neg p_3 \wedge (\neg p_2 \vee \neg p_4) \wedge \ldots \wedge (\neg p_2 \vee \neg p_n) \wedge \ldots (\neg p_{n-1} \vee \neg p_n)] \vee$$
$$3)[\neg p_2 \wedge \neg p_1 \wedge (\neg p_3 \vee \neg p_4) \wedge \ldots \wedge (\neg p_3 \vee \neg p_n) \wedge \ldots (\neg p_{n-1} \vee \neg p_n)] \vee$$
$$4)[\neg p_2 \wedge \neg p_3 \wedge (\neg p_1 \vee \neg p_4) \wedge \ldots \wedge (\neg p_1 \vee \neg p_n) \wedge \ldots (\neg p_{n-1} \vee \neg p_n)]$$

1) and 3) are the same so we can keep only one.

Exercise 2 - Solution

$$
\begin{aligned}
C \leftrightarrow \quad & [\neg p_1 \wedge \neg p_2 \wedge (\neg p_3 \vee \neg p_4) \wedge \ldots \wedge (\neg p_3 \vee \neg p_n) \wedge \\
& \qquad \ldots (\neg p_{n-1} \vee \neg p_n)] \vee \\
& [\neg p_1 \wedge \neg p_3 \wedge (\neg p_2 \vee \neg p_4) \wedge \ldots \wedge (\neg p_2 \vee \neg p_n) \wedge \\
& \qquad \ldots (\neg p_{n-1} \vee \neg p_n)] \vee \\
& [\neg p_2 \wedge \neg p_3 \wedge (\neg p_1 \vee \neg p_4) \wedge \ldots \wedge (\neg p_1 \vee \neg p_n) \wedge \\
& \qquad \ldots (\neg p_{n-1} \vee \neg p_n)]
\end{aligned}
$$

Exercise 2 - Solution

If we continued to applied distributivity, we would get:

$$
\begin{aligned}
C \leftrightarrow \quad & [\neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge (\neg p_4 \vee \neg p_5) \wedge \ldots \wedge (\neg p_4 \vee \neg p_n) \wedge \\
& \quad \ldots (\neg p_{n-1} \vee \neg p_n)] \vee \\
& [\neg p_1 \wedge \neg p_2 \wedge \neg p_4 \wedge (\neg p_3 \vee \neg p_5) \wedge \ldots \wedge (\neg p_3 \vee \neg p_n) \wedge \\
& \quad \ldots (\neg p_{n-1} \vee \neg p_n)] \vee \\
& [\neg p_1 \wedge \neg p_3 \wedge \neg p_4 \wedge (\neg p_2 \vee \neg p_5) \wedge \ldots \wedge (\neg p_2 \vee \neg p_n) \wedge \\
& \quad \ldots (\neg p_{n-1} \vee \neg p_n)] \vee \\
& [\neg p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge (\neg p_1 \vee \neg p_5) \wedge \ldots \wedge (\neg p_1 \vee \neg p_n) \wedge \\
& \quad \ldots (\neg p_{n-1} \vee \neg p_n)]
\end{aligned}
$$

where we see that we have a conjunction between formulas where one of the $\neg p_i$ is missing at each time.

Exercise 2 - Solution

By induction, we therefore obtain that:

$$C \leftrightarrow \bigvee_{1 \leqslant i \leqslant n} \left(\left(\bigwedge_{1 \leqslant j < i} \neg p_j \right) \wedge \left(\bigwedge_{i < j \leqslant n} \neg p_j \right) \right)$$

or

$$C \leftrightarrow \bigvee_{1 \leqslant i \leqslant n} \left(\bigwedge_{1 \leqslant j \leqslant n, i \neq j} \neg p_j \right)$$

Exercise 2 - Solution

$$D \triangleq \bigwedge_{1 \leqslant i \leqslant n} \left(\bigvee_{1 \leqslant j \leqslant n, j \neq i} p_j \right)$$

We have that:

$$\begin{aligned}
\neg C \leftrightarrow & \quad \neg \left[\bigvee_{1 \leqslant i \leqslant n} \left(\bigwedge_{1 \leqslant j \leqslant n, i \neq j} \neg p_j \right) \right] \\
\leftrightarrow & \quad \bigwedge_{1 \leqslant i \leqslant n} \neg \left(\bigwedge_{1 \leqslant j \leqslant n, i \neq j} \neg p_j \right) \\
\leftrightarrow & \quad \bigwedge_{1 \leqslant i \leqslant n} \left(\bigvee_{1 \leqslant j \leqslant n, j \neq i} p_j \right) \\
\leftrightarrow & \quad D
\end{aligned}$$

Exercise 2 - Solution

Then we have:

$$\begin{aligned}
\neg C \leftrightarrow\ & \neg \left[\bigwedge_{1 \leqslant i,j \leqslant n, i \neq j} (p_i \Rightarrow \neg p_j) \right] \\
\leftrightarrow\ & \bigvee_{1 \leqslant i,j \leqslant n, i \neq j} \neg (p_i \Rightarrow \neg p_j) \\
\leftrightarrow\ & \bigvee_{1 \leqslant i,j \leqslant n, i \neq j} (p_i \wedge p_j)
\end{aligned}$$

Thus

$$D \leftrightarrow \bigvee_{1 \leqslant i,j \leqslant n, i \neq j} (p_i \wedge p_j)$$

Exercise 3

Give the conjunctive normal form of ϕ and show that it is inconsistent using the resolution method.

$$\phi \triangleq \neg((q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)))$$

Exercise 3 - Solution

1)

$$
\begin{aligned}
\phi &\triangleq \neg((q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))) \\
&\leftrightarrow \neg\left[\neg(\neg q \vee r) \vee [\neg(\neg p \vee q) \vee (\neg p \vee r)]\right] \\
&\leftrightarrow (\neg q \vee r) \wedge \neg[\neg(\neg p \vee q) \vee (\neg p \vee r)] \\
&\leftrightarrow (\neg q \vee r) \wedge (\neg p \vee q) \wedge \neg(\neg p \vee r) \\
&\leftrightarrow (\neg q \vee r) \wedge (\neg p \vee q) \wedge p \wedge \neg r
\end{aligned}
$$

2) We obtain a CNF which gives the set of clauses
$S = \{\neg q \vee r, \neg p \vee q, p, \neg r\}$

Exercise 3 - Solution

3) Resolution method

1. $\neg q \vee r$
2. $\neg p \vee q$
3. p
4. $\neg r$
5. $\neg q$ (1,4)
6. q (2,3)
7. $\square$ (5,6)

Thus ϕ is inconsistent.

Exercise 4

Using the resolution method, determine whether the following formula is valid, consistent or inconsistent.

$$\phi = (((p \wedge q) \Rightarrow r) \vee ((q \Rightarrow p) \wedge \neg q)) \wedge (\neg p \Rightarrow (q \Rightarrow r))$$

Exercise 4 - Solution

$$
\begin{aligned}
\phi &\triangleq (((p \wedge q) \Rightarrow r) \vee ((q \Rightarrow p) \wedge \neg q)) \wedge (\neg p \Rightarrow (q \Rightarrow r)) \\
&\triangleq ((\neg(p \wedge q) \vee r) \vee ((\neg q \vee p) \wedge \neg q)) \wedge (p \vee (\neg q \vee r)) \\
&\triangleq ((\neg(p \wedge q) \vee r) \vee \neg q) \wedge (p \vee \neg q \vee r) \\
&\triangleq ((\neg p \vee \neg q \vee r) \vee \neg q) \wedge (p \vee \neg q \vee r) \\
&\triangleq (\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r)
\end{aligned}
$$

We obtain a CNF which gives the set of clauses
$S = \{\neg p \vee \neg q \vee r, p \vee \neg q \vee r\}$.

Exercise 4 - Solution

<u>Resolution method</u>

1. $\neg p \vee \neg q \vee r$
2. $p \vee \neg q \vee r$
3. $\neg q \vee r \qquad (1, 2)$

Thus ϕ is consistent. Let's see if it is valid.

Exercise 4 - Solution

$$
\begin{aligned}
\neg\phi &\leftrightarrow \neg[(\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r)] \\
&\leftrightarrow \neg(\neg p \vee \neg q \vee r) \vee \neg(p \vee \neg q \vee r) \\
&\leftrightarrow (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \\
&\leftrightarrow (q \wedge \neg r) \wedge (p \vee \neg p) \\
&\leftrightarrow q \wedge \neg r
\end{aligned}
$$

We obtain a CNF which gives the set of clauses $S = \{q, \neg r\}$. We cannot derive $\square$ from this set so $\neg\phi$ is consistent and therefore ϕ is not valid.

Exercise 5

Using the resolution method, determine whether the following formula is valid, consistent or inconsistent.

$$\phi \triangleq ((p \land q) \Rightarrow (\neg q \land r)) \Rightarrow (((q \Rightarrow r) \Rightarrow (p \land r)) \Rightarrow ((p \land q) \Rightarrow (p \land r)))$$

We will first try by analyzing $\neg\phi$ because applying the negation over the implication will directly give us conjunction.

Exercise 5 - Solution

$\phi \triangleq$

$$\neg\,[[(p \wedge q) \Rightarrow (\neg q \wedge r)] \Rightarrow [((q \Rightarrow r) \Rightarrow (p \wedge r)) \Rightarrow ((p \wedge q) \Rightarrow (p \wedge r))]]$$

$$\leftrightarrow [(p \wedge q) \Rightarrow (\neg q \wedge r)] \wedge \neg[((q \Rightarrow r) \Rightarrow (p \wedge r)) \Rightarrow ((p \wedge q) \Rightarrow (p \wedge r))$$

$$\leftrightarrow [(p \wedge q) \Rightarrow (\neg q \wedge r)] \wedge ((q \Rightarrow r) \Rightarrow (p \wedge r)) \wedge \neg((p \wedge q) \Rightarrow (p \wedge r))$$

$$\leftrightarrow [(p \wedge q) \Rightarrow (\neg q \wedge r)] \wedge ((q \Rightarrow r) \Rightarrow (p \wedge r)) \wedge (p \wedge q) \wedge \neg(p \wedge r)$$

$$\leftrightarrow [\neg(p \wedge q) \vee (\neg q \wedge r)] \wedge (\neg(\neg q \vee r) \vee (p \wedge r)) \wedge (p \wedge q) \wedge \neg(p \wedge r)$$

$$\leftrightarrow [\neg p \vee \neg q \vee (\neg q \wedge r)] \wedge [(q \wedge \neg r) \vee (p \wedge r)] \wedge p \wedge q \wedge (\neg p \vee \neg r)$$

$$\leftrightarrow (\neg p \vee \neg q) \wedge [(q \wedge \neg r) \vee p] \wedge [(q \wedge \neg r) \vee r)] \wedge p \wedge q \wedge (\neg p \vee \neg r)$$

$$\leftrightarrow (\neg p \vee \neg q) \wedge (q \vee p) \wedge (\neg r \vee p) \wedge (q \vee r) \wedge p \wedge q \wedge (\neg p \vee \neg r)$$

We obtain a CNF which gives the set of clauses
$S = \{\neg p \vee \neg q, q \vee p, \neg r \vee p, q \vee r, p, q, \neg p \vee \neg r\}$.

Exercise 5 - Solution

Resolution

1. $\neg p \vee \neg q$
2. $q \vee p$
3. $\neg r \vee p$
4. $q \vee r$
5. p
6. q
7. $\neg p \vee \neg r$
8. $\neg q \qquad (1, 5)$
9. $\square \qquad (6, 8)$

Thus $\neg\phi$ is inconsistent, whence ϕ is valid.