Logic - Tutorial 3

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Reminder

Logical equivalence

Let A_1 and A_2 be two propositions:

- Two formulas A_1 and A_2 are **logically equivalent** if they have the same models.
- Notation: $A_1 \longleftrightarrow A_2$

Equivalent formulations

1. $A_1 \leftrightarrow A_2$
2. $A_1 \equiv A_2$ is valid
3. $A_1 \Rightarrow A_2$ is valid and $A_2 \Rightarrow A_1$ is valid
4. $A_1 \models A_2$ and $A_2 \models A_1$

Exercise 1

Show that $(X \wedge Y) \Rightarrow Z$ et $X \Rightarrow (Y \Rightarrow Z)$ are logically equivalent.

Exercise 1 - Solution

Let denote by the first formula by $A \triangleq (X \wedge Y) \Rightarrow Z$ and the second formula by $B \triangleq X \Rightarrow (Y \Rightarrow Z)$. We have 4 possibilities:

1. Show $A \leftrightarrow B$, i.e. A and B have the same models
2. Show $A \equiv B$ is valid
3. Show $A \Rightarrow B$ is valid and $B \Rightarrow A$ is valid
4. Show $A \models B$ and $B \models A$

We will show the solution with method 4 (feel free to try the other at home).

Exercise 1 - Solution

$A \models B$

Let v be a valuation s.t. $v(A) = T$. Does it imply that $v(B) = T$?
We have: $v(A) = T$ implies $v(X \wedge Y) = F$ or $v(Z) = T$

- Case 1: $v(X \wedge Y) = F$ implies $v(X) = F$ or $v(Y) = F$
  - Case 1.1: $v(X) = F$ implies $v(B) = T \rightarrow$ **OK**
  - Case 1.2: $v(Y) = F$ implies $v(Y \Rightarrow Z) = T$ which implies $v(B) = T \rightarrow$ **OK**
- Case 2: $v(Z) = T$ implies $v(Y \Rightarrow Z) = T$ which implies $v(B) = T \rightarrow$ **OK**

Exercise 1 - Solution

$\underline{B \models A}$

Let v be a valuation s.t. $v(B) = T$. Does it imply that $v(A) = T$?
We have: $v(B) = T$ implies $v(X) = F$ or $v(Y \Rightarrow Z) = T$

- Case 1: $v(X) = F$ implies $v(X \wedge Y) = F$ which implies $v(A) = T \rightarrow$ **OK**
- Case 2: $v(Y \Rightarrow Z) = T$ implies $v(Y) = F$ or $v(Z) = T$
  - Case 2.1: $v(Y) = F$ implies $v(X \wedge Y) = F$ which implies $v(A) = T \rightarrow$ **OK**
  - Case 2.2: $v(Z) = T$ implies $v(A) = T \rightarrow$ **OK**

Exercise 2

Let A, B, X and Y be formulas. If $A \models B$, what can you say, in general, about

- $C \triangleq X \Rightarrow (A \Rightarrow Y)$
- $D \triangleq X \Rightarrow (B \Rightarrow Y)$

($C \models D$? $D \models C$? $C \leftrightarrow D$? No logical consequence?)

Exercise 2 - Solution

A	B	X	Y	$A \Rightarrow Y$	$B \Rightarrow Y$	C	D
T	T	T	T	T	T	T	T
T	T	T	F	F	F	F	F
T	T	F	T	T	T	T	T
T	T	F	F	F	F	T	T
T	F	T	T	T	T	T	T
T	F	T	F	F	T	F	T
T	F	F	T	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	T	T	T	T	T
F	T	T	F	T	F	T	F
F	T	F	T	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	T	T	T	T	T
F	F	F	F	T	T	T	T

Exercise 2 - Solution

A	B	X	Y	$A \Rightarrow Y$	$B \Rightarrow Y$	C	D
T	T	T	T	T	T	T	T
T	T	T	F	F	F	F	F
T	T	F	T	T	T	T	T
T	T	F	F	F	F	T	T
T	F	T	T	T	T	T	T
T	F	T	F	F	T	F	T
T	F	F	T	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	T	T	T	T	T
F	T	T	F	T	F	T	F
F	T	F	T	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	T	T	T	T	T
F	F	F	F	T	T	T	T

$A \not\models B$

Exercise 2 - Solution

Conclusions:

- $C \not\models D$: When $v(A) = v(Y) = F$ and $v(B) = v(X) = T$, we have $v(C) = T$ and $v(D) = F$
- $D \models C$: For every model v for which $v(D) = T$, $v(C) = T$
- $C \nleftrightarrow D$: $C \not\models D$

Exercise 3

Let A, B, X and Y be formulas. If $A \models B$, what can you say, in general, about

- $C \triangleq \neg (X \Rightarrow A) \vee Y$
- $D \triangleq \neg (X \Rightarrow B) \vee Y$

Exercise 3 - Solution

$\underline{C \models D}$

Let v be a valuation s.t. $v(C) = T$. Then $v(\neg(X \Rightarrow A)) = T$ or $v(Y) = T$

- Case 1: $v(Y) = T$ implies $v(D) = T$ → **OK**
- Case 2:
  - $v(\neg(X \Rightarrow A)) = T$ implies $v(X \Rightarrow A) = F$ and thus $v(X) = T$ and $v(A) = F$.
  - But then we can find v such that $v(X) = T$, $v(A) = F$, $v(B) = T$ which implies $v(C) = T$ but $v(D) = F$
  - Therefore $C \not\models D$.

Exercise 3 - Solution

$\underline{D \models C}$

Let v be a valuation s.t. $v(D) = T$. Then $v(\neg(X \Rightarrow B)) = T$ or $v(Y) = T$

- Case 1: $v(Y) = T$ implies $v(C) = T \rightarrow$ **OK**
- Case 2:
  - $v(\neg(X \Rightarrow B)) = T$ implies $v(X \Rightarrow B) = F$ and thus $v(X) = T$ and $v(B) = F$
  - $A \models B$ and $v(B) = F$ implies $v(A) = F$
  - $v(C) = v(\neg(X \Rightarrow A) \vee Y) = v(\neg(\text{true} \Rightarrow \text{false}) \vee Y) = v(\neg\text{false} \vee Y) = v(\text{true} \vee Y) = T$

$\underline{C \leftrightarrow D}$

No as $C \not\models D$

Exercise 4

Let A, B, X and Y be formulas. If $A \models B$, what can you say, in general, about

- $C \triangleq X \Rightarrow (A \equiv Y)$
- $D \triangleq X \Rightarrow (B \equiv Y)$

Exercise 4 - Solution

The case where $v(X) = F$ is trivial and leads to $v(C) = v(D) = T$.
Similarly, when $v(A) = T$, as $A \models B$, this implies $v(B) = T$ and the two formulas have the same truth value whatever the valuation.
Therefore, let's consider the case where $v(X) = T$ and $v(A) = F$.
Considering $v(Y)$, we have two possibilities:

<u>Case 1:</u> $v(Y) = F$
In that case, we directly see that if $v(B) = T$, then $v(C) = T$ and $v(D) = F$ which means that $C \not modelsD$.

<u>Case 2:</u> $v(Y) = T$
In that case, we directly see that if $v(B) = T$, then $v(C) = F$ and $v(D) = T$ which means that $D \not modelsC$

Therefore $C \not\leftrightarrow D$

Exercise 5

Let A, B, X and Y be formulas. If $A \models B$, what can you say, in general, about

- $C \triangleq X \Rightarrow (\neg A \wedge Y)$
- $D \triangleq X \Rightarrow (\neg B \wedge Y)$

Exercise 5 - Solution

$C \models D$

Let $v(X) = T, v(Y) = T, v(A) = F, v(B) = T$ then $v(C) = T$ but $v(D) = F$. Therefore $C \not\models D$.

$D \models C$

Let v be a valuation s.t. $v(D) = T$. Then $v(X) = F$ or $v(\neg B \wedge Y) = T$.

- Case 1: $v(X) = F$ implies $v(C) = T \rightarrow$ **OK**
- Case 2:
  - $v(\neg B \wedge Y) = T$ implies $v(Y) = T$ and $v(B) = F$
  - $A \models B$ and $v(B) = F$ implies $v(A) = F$.
  - Thus $v(C) = v(X \Rightarrow (\neg A \wedge Y)) = v(X \Rightarrow (\neg \mathrm{false} \wedge \mathrm{true})) = v(X \Rightarrow \mathrm{true}) = T$

  $\rightarrow$ **OK**

Thus $D \models C$

$C \leftrightarrow D$

No as $C \not\models D$.

Exercise 6

Consider a set of five propositional variables $P \triangleq \{a, b, c, d, e\}$.

1. How many formulas, up to logical equivalence, exist that are satisfied by exactly seventeen interpretations?
2. How many formulas, up to logical equivalence, exist that are logical consequence of the formula $a \wedge b$?

Exercise 6 - Solution

a) We have 5 variables, so the truth table has $2^5 = 32$ lines.
For one set of logically equivalent formulas, 17 of these lines must be true.
To find how many different formulas, up to logical equivalence exists, we therefore need to estimate how many ways there are to select 17 lines among 32?
Answer: $C_{32}^{17} = \frac{32!}{17!15!}$

b) This means formulas must be true when $a \wedge b$ is true, i.e. when a is T and b is T. But they can be anything the rest of the time! When a and b are T, this corresponds to $2^3 = 8$ lines in the truth table, thus $32 - 8 = 24$ lines are free, which leads to 2^{24} formulas.