Logic - Tutorial 2

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Reminder

Logical consequence

Let A be a formula and $S = \{A_1, ..., A_n\}$ be a formula set:

- A formula A is a **logical consequence** of a formula set S if every $S - model$ is an $A - model$
- Notation: $S \models A$

Equivalent formulation

1. $S \models A$
2. $S \cup \{\neg A\}$ is inconsistent
3. $A_1 \wedge ... \wedge A_n \Rightarrow A$ is valid
4. $A_1 \wedge ... \wedge A_n \wedge \neg A$ is inconsistent

Exercise 1

Whoever says that I am a man says true. Whoever says that I am stupid says that I am a man. Therefore whoever says that I am stupid says true.

Is the conclusion really a logical consequence of the facts?

Exercise 1 - Solution

1) Atoms:

- m : 'says that I am a man'
- t : 'says true'
- s : 'says that I am stupid'

2) Formulas:

- $H_1 \triangleq m \Rightarrow t$
- $H_2 \triangleq s \Rightarrow m$
- $C \triangleq s \Rightarrow t$

3) Let's show that $\{H_1, H_2, \neg C\}$ is inconsistent

Let's consider that v is a model of that set. As $v(\neg C) = T$, then $v(s) = T$ and $v(t) = F$. Then $v(H_2) = v(s \Rightarrow m) = T$ and $v(s) = T$ implies $v(m) = T$. But $v(H_1) = v(m \Rightarrow t)$ and $v(t) = F$ implies $v(m) = F \rightarrow$ **Contradiction!**

Exercise 2

Exercise 2

If he does not tell her, she will never discover it. If she does not ask him, he will not tell her about it. She discovered it. Therefore she asked about it.

Is the conclusion really a logical consequence of the facts?

Exercise 2 - Solution

1) Atoms:

- t : 'he tells her'
- r : 'she discovers it'
- a : 'she asks him'

2) Formulas:

- $H_1 \triangleq \neg t \Rightarrow \neg d$
- $H_2 \triangleq \neg a \Rightarrow \neg t$
- $H_3 \triangleq d$
- $C \triangleq a$

3) Let's show that $\{H_1, H_2, H_3\} \models C$.

Exercise 2 - Solution

Let's v be a valuation such that $v(H_1) = v(H_2) = v(H_3) = T$. Then

- $v(H_3) = T$ implies $v(d) = T$.
- $v(d) = T$ and $v(H_1) = T$ implies $v(\neg t) = F$
- $v(\neg t) = F$ and $v(H_2) = F$ implies $v(a) = T$

Thus $v(C) = T$ and the conclusion is a consequence of the facts.

Exercise 3

If John hasn't met Peter the other night, this means that Peter is the murderer or that John is a liar. If Peter isn't the murderer, then John hasn't met Peter the other night and the crime happened after midnight. If the crime happened after midnight, then Peter is the murderer or John isn't a liar. Therefore Peter is the murderer.

Is the conclusion really a logical consequence of the facts?

Exercise 3 - Solution

1) Define atoms:

- n : "John hasn't met Peter the other night"
- p : "Peter is the murdered"
- j : "John is a liar"
- m : "The crime happened after midnight"

2) Transform sentences into formulas:

- $H_1 \triangleq n \Rightarrow (p \vee j)$
- $H_2 \triangleq \neg p \Rightarrow (n \wedge m)$
- $H_3 \triangleq m \Rightarrow (p \vee \neg j)$
- $C \triangleq p$

3) Show that $\{H_1, H_2, H_3\} \models C$ (or some equivalent)

Exercise 3 - Solution

We are going to show that $\{H_1, H_2, H_3, \neg C\}$ is inconsistent.

Let's assume this is not the case and there exist a model v of this set, such that $v(H_1) = v(H_2) = v(H_3) = v(\neg C) = T$. Then,

1. $v(\neg C) = T$ implies $v(\neg p) = T$ and $v(p) = F$.
2. $v(H_2) = v(\neg p \Rightarrow (n \wedge m)) = T$ and $v(\neg p) = T$ implies $v(n \wedge m) = T$ and $v(n) = v(m) = T$.
3. $v(H_1) = v(n \Rightarrow (p \vee j)) = T$, $v(n) = T$ implies $v(p \vee j) = T$
4. $v(p \vee j) = T$ and $v(p) = F$ implies $v(p) = T$.

But then, $v(H_3) = v(m \Rightarrow (p \vee \neg j)) = F \neq T \rightarrow$ **Contradiction !**

Then the set $\{H_1, H_2, H_3, \neg C\}$ is inconsistent and the conclusion is a logical consequence of the fact.

Exercise 4

Brown, Jones and Smith are three Irish salesmen in New York. They are on trial for the fabrication and sale of alcohol (during prohibition). They swore on the Bible and declared:

- Brown: "Jones is guilty, and Smith est innocent."
- Jones: "If Brown is guilty then Smith is guilty as well."
- Smith: "I am innocent, but at least one of the two others is guilty."

Let b, j, s be the tree propositions "Brown is innocent", "Jones is innocent", "Smith is innocent".

(a) Give a logical formula for each of the statements.
(b) Is this set of statements consistent?
(c) One statement is a logical consequence of another one. Which one?
(d) If everyone is innocent, who made a false declaration?
(e) As the Irish are very religious people, one could assume that they tell the truth. In this case, who is guilty?
(f) If the innocent tell the truth and the guilty lie, who is guilty?

Exercise 4 - Solution

a) Formulas:

- $B \triangleq \neg j \wedge s$
- $J \triangleq \neg b \Rightarrow \neg s$
- $S \triangleq s \wedge (\neg b \vee \neg j)$

b) Consistent?

For $v(B)$ to be T, we need $v(j) = F$ and $v(s) = T$. Then if $v(b) = T$, we have $v(J) = v(S) = T$. Therefore v is a model of $\{B, J, S\}$ and the set is consistent.

Exercise 4 - Solution

c) Which statement is a logical consequence of the other?

b	j	s	$B \triangleq \neg j \wedge s$	$J \triangleq \neg b \Rightarrow \neg s$	$S \triangleq s \wedge (\neg b \vee \neg j)$
T	T	T	F	T	F
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	F	F	T
F	T	F	F	T	F
F	F	T	T	F	T
F	F	F	F	T	F

As $v(S) = T$ for all v where $v(B) = T$, $B \models S$.

Exercise 4 - Solution

d) If everyone is innocent, $v(b) = v(j) = v(s) = T$, then $v(B)$ and $v(S)$ are F. Therefore, Brown and Smith lied.

e) If everyone tells the truth, $v(B) = v(J) = v(S) = T$, then $v(b) = T$, $v(j) = F$ and $v(s) = T$. Therefore, Smith is guilty.

f) To model the fact that the innocent tells the truth and the guilty lie, we define the formulas:

- $B' \triangleq b \equiv \neg j \wedge s$
- $J' \triangleq j \equiv \neg b \Rightarrow \neg s$
- $S' \triangleq s \equiv s \wedge (\neg b \vee \neg j)$

and sets them all to true: $v(B') = v(J') = v(S') = T$. We now need to find the valuation of b, j and s that verify this assumption.

Exercise 4 - Solution

Let's analyze the possible cases $v(j) = F$ and $v(j) = T$.

Case 1: $v(j) = F$

- $v(J') = T$ and $v(j) = F$ implies $v(\neg b) = T$ and $v(\neg s) = F$.
- But then $v(B') = v\,(\text{false} \equiv (\text{true} \wedge \text{true})) = F$ which is not possible.

Case 2: $v(j) = T$

- $v(B') = T$ and $v(j) = T$ implies $v(b) = v(\neg js) = F$.
- $v(J') = T, v(j) = T$ implies that $v(\neg b \Rightarrow \neg s) = T$
- This and $v(b) = F$, implies $v(s) = F$

This second case verifies the original hypothesis and therefore Brown and Smith are guilty.

Exercise 5

Exercise 5

Logic is intriguing me. Everything that is comprehensible never intrigues me. Therefore logic is incomprehensible.

Is the conclusion really a logical consequence of the facts?

Exercise 5 - Solution

1) Atoms:
   - i : 'Logic is intriguing'
   - c : 'logic is comprehensible'
2) Formulas:
   - $H_1 \triangleq i$
   - $H_2 \triangleq c \Rightarrow \neg i$
   - $C \triangleq \neg c$
3) Does $\{H_1, H_2\} \models C$?

Let's consider a model v of $\{H_1, H_2\}$. If $v(H_1) = v(i) = T$, then $v(\neg i) = F$. As $v(H_2) = T$, $v(c) = F$. Therefore $v(C) = v(\neg c) = T$ and v is a model of C.

<u>Exercise 6</u>

He said that he would come if it does not rain. It rains. Therefore he does not come.

Is the conclusion really a logical consequence of the facts?

Exercise 6 - Solution

1) Atoms:

- c : 'he comes'
- r : 'it rains'

2) Formulas:

- $H_1 \triangleq \neg r \Rightarrow c$
- $H_2 \triangleq r$
- $C \triangleq \neg c$

3) Let's show that $\{\neg r \Rightarrow c, r, c\}$ is inconsistent.

We have v such that $v(r) = v(c) = T$. But then $v(\neg r \Rightarrow c) = T$ and therefore the set is consistent. Thus the conclusion is not a consequence of the facts.

Exercise 7

If Jordan or Algeria joins the union then, if Syria or Kuwait boycott the union then, even though Iraq does not boycott the union, Yemen does. If Iraq or Morocco do not boycott the union, Egypt will join the union. Therefore, if Jordan joins the union then, if Syria boycotts the union, Egypt will join the union.

Is the conclusion really a logical consequence of the facts?

Exercise 7 - Solution

1) Atoms:

- j : "Jordan joins the union"
- a : "Algeria joins the union"
- s : "Syria joins the union"
- k : "Kuwait joins the union"
- i : "Iraq joins the union"
- y : "Yemen joins the union"
- m : "Morocco joins the union"
- e : "Egypt joins the union"

2) Formulas:

- $H_1 \triangleq (j \vee a) \Rightarrow [(s \vee k) \Rightarrow (i \wedge \neg y)]$
- $H_2 \triangleq (i \vee m) \Rightarrow e$
- $C \triangleq j \Rightarrow (\neg s \Rightarrow e)$

Exercise 7 - Solution

3) Let's show that $\{H_1, H_2, \neg C\}$ is inconsistent.

Let's consider that there exist a model v of the set. Then:

1. $v(H_1) = v(H_2) = v(\neg C) = T$
2. $v(\neg C) = T$ implies $v(j) = T$ and $v(\neg s \Rightarrow e) = F$.
3. $v(\neg s \Rightarrow e) = F$ implies $v(s) = T$ and $v(e) = F$
4. $v(H_2) = T$ and $v(e) = F$ implies $v(i \vee m) = F$
5. $v(i \vee n) = F$ implies $v(i) = F$ and $v(m) = F$

But then

$$
\begin{aligned}
v(H_1) &= v((j \vee a) \Rightarrow [(s \vee k) \Rightarrow (i \wedge \neg y)]) \\
&= v((\mathrm{true} \vee a) \Rightarrow [(\mathrm{true} \vee k) \Rightarrow (\mathrm{false} \wedge \neg y)]) \\
&= v(\mathrm{true} \Rightarrow (\mathrm{true} \Rightarrow \mathrm{false}) = F \rightarrow \textbf{Contradiction!}
\end{aligned}
$$

Therefore, $\{H_1, H_2, \neg C\}$ is inconsistent.

Exercise 8

If one considers that the people that study extrasensory perceptions are honest, then one must admit the existence of such perceptions. Further, if one puts to test extrasensory perceptions, one needs to seriously consider clairvoyance. To admit the existence of extrasensory perceptions will push us to put them to test and to explain them.

Clairvoyance needs to be seriously considered if one is willing to consider seriously occult phenomena. And if one is willing to consider seriously those phenomena, one need to respect psychic media. Further, if we respect these people, we also need to take seriously their ability to talk to the deceased. Finally, if we take seriously their ability to talk to the deceased, we must believe in ghosts.

Therefore, considering that the people who study extrasensory perceptions are honest forces us to believe in ghosts.

Exercise 8 - Solution

1) Atoms:

- h : 'consider these people are honest'
- a : 'admit the existence of such perceptions'
- t : 'put to test such perceptions'
- c : 'consider clairvoyance'
- e : 'explain such perceptions'
- o : 'consider occult phenomena'
- m : 'respect psychic media'
- d : 'take seriously their ability to talk to the deceased'
- g : 'believe in ghosts'

Exercise 8 - Solution

2) Formulas:

- $H_1 \triangleq h \Rightarrow a$
- $H_2 \triangleq t \Rightarrow c$
- $H_3 \triangleq a \Rightarrow (tc)$
- $H_4 \triangleq o \Rightarrow c$
- $H_5 \triangleq o \Rightarrow m$
- $H_6 \triangleq m \Rightarrow d$
- $H_7 \triangleq d \Rightarrow g$
- $C \triangleq h \Rightarrow g$

Exercise 8 - Solution

3) Let's show that $S = \{H_1, H_2, H_3, H_4, H_5, H_6, H_7, \neg C\}$ is inconsistent.
Ad absurdum
Let's consider a model v of the set S. We have that:

- $v(\neg C) = T$ implies that $v(h) = T$ and $v(g) = F$
- $v(H_1) = T$ and $v(h) = T$ implies $v(a) = T$
- $v(H_3) = T$ and $v(a) = T$ implies $v(t) = T$ and $v(c) = T$
- $v(H_2) = T$ and $v(t) = T$ implies $v(c) = T$
- $v(H_7) = T$ and $v(g) = F$ implies $v(d) = F$
- $v(H_6) = T$ and $v(d) = F$ implies $v(m) = F$
- $v(H_5) = T$ and $v(m) = F$ implies $v(o) = F$

Then $v(H_4) = v(o \Rightarrow c) = T$. Thus the set S is consistent and the conclusion is not a consequence of the facts.