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Group Coordination and Cooperative Control of Steered Particles in the Plane

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Summary. The paper overviews recent and ongoing efforts by the authors to propose a design methodology to stabilize isolated relative equilibria in a kinematic model of identical particles moving in the plane at unit speed. Isolated relative equilibria correspond to either parallel motion of all particles with fixed relative spacing or to circular motion of all particles about the same center with fixed relative phases.

1 Introduction

Feedback control laws that stabilize collective motions of particle groups have a number of engineering applications including unmanned sensor networks. For example, autonomous underwater vehicles (AUVs) are used to collect oceanographic measurements in formations that maximize the information intake, see e.g. [LPL$^+$05] and the references therein.

In this paper, we consider a kinematic model of identical (pointwise) particles in the plane [JK03]. The particles move at constant speed and are subject to steering controls that change their orientation. In recent work [SPL05], see also [SPL04, PLS05], we proposed a Lyapunov design to stabilize isolated relative equilibria of the model. Isolated relative equilibria correspond to either

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parallel motion of all particles with fixed relative spacing or to circular motion of all particles about a common center with fixed relative phases. The stabilizing feedbacks were derived from Lyapunov functions that prove exponential stability and suggest almost global convergence properties. The results in [SPL05] assume an all-to-all communication topology, that is, the feedback control applied to one given particle uses information about the (relative) heading and position of all other particles.

The objective of the present paper is to relax the all-to-all assumption on the communication topology in different ways. We show how the Lyapunov design of stabilizing control laws can be extended to any constant, bidirectional, and connected communication topology. We provide a unified interpretation of all the Lyapunov functions considered in earlier work as quadratic forms induced by the Laplacian of the graph associated to the communication topology. We then address the more challenging situation of time-varying and unidirectional communication topologies. We briefly review recent results in the literature that address the stabilization of parallel motions and we propose a new control law that stabilizes circular motions with time-varying and unidirectional communication topologies.

The model assumptions are recalled in Section 2. Section 3 introduces the quadratic functions induced by the communication topology. The main Lyapunov functions considered in [SPL05] are then reinterpreted and generalized in Section 4. Section 5 provides a further analysis of the phase potentials used for the design. In Section 6, we address the situation of time-varying and unidirectional communication topologies. A short discussion concludes the paper in Section 7.

2 Particle model and control design

We consider a continuous-time kinematic model of $N > 1$ identical particles (of unit mass) moving in the plane at unit speed [JK03]:

$$\begin{aligned} \dot{r}_k &= e^{i\theta_k} \\ \dot{\theta}_k &= u_k, \end{aligned} \tag{1}$$

where $k = 1, \ldots, N$. In complex notation, the vector $r_k = x_k + iy_k \in \mathbb{C} \approx \mathbb{R}^2$ denotes the position of particle k and the angle $\theta_k \in S^1$ denotes the orientation of its (unit) velocity vector $e^{i\theta_k} = \cos\theta_k + i\sin\theta_k$. We use the boldface variable without index to denote the corresponding N-vector, e.g. $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)^T$. The configuration space consists of N copies of the group $SE(2)$. In the absence of steering control ($\dot{\theta}_k = 0$), each particle moves at unit speed in a fixed direction and its motion is decoupled from the other particles.

We study the design problem of choosing feedback controls that stabilize a prescribed collective motion. The feedback controls are identical for all the particles and depend only on relative orientation and relative spacing, i.e., on the variables $\theta_{kj} = \theta_k - \theta_j$ and $r_{kj} = r_k - r_j,\ j,k = 1,\ldots,N$. Consequently, the closed-loop vector field is invariant under an action of the symmetry group $SE(2)$ and the closed-loop dynamics evolve on a reduced quotient manifold (shape space). Equilibria of the reduced dynamics are called relative equilibria and can be only of two types [JK03]: *parallel* motions, characterized by a common orientation for all the particles (with arbitrary relative spacing), and *circular* motions, characterized by circular orbits of the particles around the same fixed point.

3 Communication topology and Laplacian quadratic forms

The feedback control laws are further restricted by a limited communication topology. The communication topology is defined by an undirected graph $G(V,E)$ with N vertices in $V = \{1,\ldots,N\}$ and e edges $(k,j) \in E$ whenever there exists a communication link between particle k and particle j. We denote by $\mathcal{N}(k) = \{j \mid (j,k) \in E\}$ the set of neighbors of k, that is, the set of vertices adjacent to vertex j. The control u_k is allowed to depend on r_{kj} and θ_{kj} only if $j \in \mathcal{N}(k)$.

Consider the (undirected) graph $G = (V,E)$ and let d_k be the degree of vertex k. The Laplacian L of the graph G is the matrix defined by

$$\begin{aligned} L_{k,j} &= d_k, \quad \text{if } k = j \\ &= -1, \quad \text{if } (k,j) \in E, \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{2}$$

The Laplacian matrix plays a fundamental role in spectral graph theory [Chu97]. Only basic properties of the Laplacian are used in this paper. First, $L\mathbf{1} = 0$, where $\mathbf{1} = (1,\ldots,1)^T \in \mathbb{R}^N$, and the multiplicity of the zero eigenvalue is the number of connected components of the graph. As a consequence, the Laplacian matrix of a connected graph has one zero eigenvalue and $N-1$ strictly positive eigenvalues.

We denote by $< \cdot,\cdot >$ the standard inner product in $\mathbb{C}^N$. The quadratic form $Q(\boldsymbol{z}) =< \boldsymbol{z}, L\ \boldsymbol{z} >$, where L is the Laplacian of a connected graph, vanishes only when $\boldsymbol{z} = \mathbf{1}z_0$. It defines a norm on the shape space $\mathbb{C}^N/\mathbb{C}$ induced by the action of the group of rigid displacements $\boldsymbol{z} \to \boldsymbol{z} + \mathbf{1}z_0$.

Consider the valence matrix $D = \mathrm{diag}(\mathbf{d})$, the adjacency matrix A, and the incidence matrix $B \in \mathbb{R}^{N\times e}$ associated to the graph G. One easily shows that $L = D - A$. Using the property $L = BB^T$ for a bidirectional graph, an alternative expression for the quadratic form $Q(\boldsymbol{z})$ is

$$Q(\boldsymbol{z}) = \sum_{(k,j)\in E} |z_k - z_j|^2.$$

In words, $Q(\boldsymbol{z})$ is thus the sum of the squared lengths of the edges connecting communicating vertices z_k.

Example 1. Let $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^N$ and $I = \mathrm{diag}(\mathbf{1})$. An all-to-all communication topology with N vertices corresponds to a complete graph, K_N, with the Laplacian $L = NI - \mathbf{1}\mathbf{1}^T = NP$ where $P = I - \frac{1}{N}\mathbf{1}\mathbf{1}^T$ is the projector orthonormal to the vector $\mathbf{1}$. Using the property $P^2 = P$, the quadratic form $Q(\boldsymbol{z})$ then takes the form

$$Q(\boldsymbol{z}) = N \parallel P\boldsymbol{z} \parallel^2,$$

which is (N times) the sum of the squared distances of vertices z_k, $k = 1, \ldots, N$, to their centroid $\frac{1}{N}\sum_{j=1}^{N} z_j$.

Example 2. A meaningful generalization of the all-to-all communication topology is the topology corresponding to a circulant graph in which each node is connected to n other nodes, for a fixed n where $2 \leq n \leq N-1$. Each column of the Laplacian is a cyclic shift of the vector v with

$$\begin{aligned} v_1 &= n, \\ v_k &= -1, \quad k = 2, \ldots, n+1 \\ v_k &= 0, \quad k = n+2, \ldots, N. \end{aligned} \tag{3}$$

4 Collectives specified by synchrony

4.1 Parallel and circular motion

Under the constant control $u_k = \omega_0$, $\omega_0 \neq 0$, the particle k rotates on a circle of radius $\rho_0 = 1/|\omega_0|$ centered at $c_k = r_k + i\omega_0^{-1}e^{i\theta_k}$. Achieving a circular formation amounts to synchronizing all the particle centers. This prompts us to define

$$s_k = i\omega_0 c_k = -e^{i\theta_k} + i\omega_0 r_k \tag{4}$$

and to specify a circular formation by the synchronization condition $\boldsymbol{s} = \mathbf{1}s_0$ for an arbitrary $s_0 \in \mathbb{C}$, or, equivalently,

$$(I - \frac{1}{N}\mathbf{1}\mathbf{1}^T)\boldsymbol{s} = 0. \tag{5}$$

Note that for $\omega_0 = 0$, we have $s_k = e^{i\theta_k}$ and the condition (5) thus specifies the phase synchrony $\boldsymbol{\theta} = \mathbf{1}\theta_0$. One concludes that the sync condition (5) either specifies a parallel motion ($\omega_0 = 0$), or a circular motion of radius $\rho_0 = |\omega_0|^{-1}$.

The dynamics

$$\dot{s}_k = -ie^{i\theta_k}(u_k - \omega_0) \tag{6}$$

shows that each s_k is invariant under the constant control $u_k = \omega_0$. The Lyapunov function

$$U(\boldsymbol{s}) = KQ(\mathbf{s}) = K < \boldsymbol{s}, L\boldsymbol{s} >, \quad K > 0,$$

where L is the Laplacian of a connected graph, thus has the property of reaching its global minimum (only) when the sync condition (5) is satisfied. $U(\boldsymbol{s})$ is invariant under the constant control $u_k = \omega_0$:

$$\dot{U} = -2K \sum_{k=1}^{N} < ie^{i\theta_k}, L_k \boldsymbol{s} > (u_k - \omega_0). \tag{7}$$

The (dissipation) control

$$u_k = \omega_0 + K < ie^{i\theta_k}, L_k \boldsymbol{s} >, \quad K > 0 \tag{8}$$

ensures that U evolves monotonically along the closed-loop solutions since $\dot{U} = -1/2 \parallel \frac{\partial U}{\partial \boldsymbol{\theta}} \parallel^2 \leq 0$. Moreover, it satisfies the restrictions imposed by the communication topology.

Proposition 1 *Let L be the Laplacian matrix of a bidirectional, connected graph. Consider the model (1) with the shape control law (8). Then for $\omega_0 < 0$ (resp. $\omega_0 > 0$), the set of clockwise (resp. counter-clockwise) circular motions of radius $|\omega_0|^{-1}$ is globally asymptotically stable and locally exponentially stable. For $\omega_0 = 0$, the set of parallel motions is locally exponentially stable; moreover, each solution converges to the set of critical points of $U(e^{i\boldsymbol{\theta}})$.*

Proof. This is proved by a straightforward adaptation of the corresponding proof in [SPL05] where it was applied to the particular case $L = N(I - \frac{1}{N}\mathbf{1}\mathbf{1}^T)$.

Using the equality $L = D - A = BB^T$, one has several equivalent expressions for $Q(\mathbf{s})$ with $\omega_0 = 0$:

$$\begin{aligned} Q(\dot{\mathbf{r}}) &= < e^{i\boldsymbol{\theta}}, Le^{i\boldsymbol{\theta}} > & (9)\\ &= \mathrm{tr}D - < e^{i\boldsymbol{\theta}}, Ae^{i\boldsymbol{\theta}} > & (10)\\ &= \mathrm{tr}D - 2\mathbf{1}_{\boldsymbol{e}}{}^T \cos(B^T\boldsymbol{\theta}). & (11)\end{aligned}$$

Likewise, one has the equivalent expressions for the derivative

$$\frac{\partial Q}{\partial \boldsymbol{\theta}} = 2B\sin(B^T\boldsymbol{\theta}) \tag{12}$$

and

$$\frac{\partial Q}{\partial \theta_k} = 2 \sum_{j \in \mathcal{N}(k)} \sin(\theta_k - \theta_j). \tag{13}$$

The quadratic function $Q(\dot{\boldsymbol{r}})$ reaches its minimum when $\dot{\boldsymbol{r}} = \mathbf{1}e^{i\theta_0}$, that is, when all phases synchronize, which corresponds to a parallel motion. The control $\mathbf{u} = -B\sin(B^T\boldsymbol{\theta})$ is proposed in [JMB04] to achieve synchronization in the phase model $\dot{\boldsymbol{\theta}} = \mathbf{u}$. It generalizes to arbitrary communication topologies the all-to-all sinusoidal coupling encountered in Kuramoto model [Kur84]. For the K_N topology (all-to-all communication), the quadratic function Q becomes

$$Q(\dot{\boldsymbol{r}}) = N \parallel Pe^{i\boldsymbol{\theta}} \parallel^2 = N^2(1 - |\frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}|^2). \tag{14}$$

Up to a constant and a change of sign, it coincides with the phase potential $U(\boldsymbol{\theta}) = |p_\theta|^2$ used in [SPL05], where $p_\theta = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}$ denotes the centroid of particle phases, or equivalently, the average linear momentum $\dot{R} = \frac{1}{N}\sum_{k=1}^{N} \dot{r}_k$. The parameter $|p_\theta|$ is a classical measure of synchrony of the phase variables $\boldsymbol{\theta}$ [Kur84, Str00]. It is maximal when all phases are synchronized (identical). It is minimal when the phases balance to result in a vanishing centroid. In the particle model (1), synchronization of the phases corresponds to a parallel formation: all particles move in the same direction. In contrast, balancing of the phases corresponds to collective motion with a fixed center of mass.

4.2 Isolated circular relative equilibria

For $\omega_0 \neq 0$, the sync condition (5) specifies a circular relative equilibrium but the phase arrangement of the particles is arbitrary. A given phase arrangement $\boldsymbol{\theta} = \bar{\boldsymbol{\theta}} + \mathbf{1}(\omega_0 t + \theta_0)$ (with $\theta_0 \in S^1$ arbitrary) is specified by means of the additional sync condition

$$(I - \frac{1}{N}\mathbf{1}\mathbf{1}^T)e^{i(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}})} = 0. \tag{15}$$

To enforce this additional sync condition by feedback, we use the augmented Lyapunov function

$$V(\boldsymbol{r},\boldsymbol{\theta}) = K < \boldsymbol{s}, L\boldsymbol{s} > + K_1 < e^{i(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}})}, Le^{i(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}})} >$$

with $K > 0$ and $K_1 \geq 0$ arbitrary positive constants, and, accordingly, the augmented feedback control

$$u_k = \omega_0 + K < ie^{i\theta_k}, L_k\boldsymbol{s} > - K_1 < ie^{i(\theta_k - \bar{\theta}_k)}, L_k e^{i(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}})} > \tag{16}$$

Proposition 2 *Let L be the Laplacian matrix of a bidirectional, connected graph. Then the shape feedback control (16) exponentially stabilizes the isolated relative equilibrium determined by a circular motion of radius $|\omega_0|^{-1}$ ($\omega_0 \neq 0$) with a phase arrangement satisfying the synchrony condition (15). Moreover, every solution of the closed-loop system converges to a critical point of $Q(e^{i(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}})})$.*

4.3 Isolated parallel relative equilibria

For $\omega_0 = 0$, the sync condition (5) specifies a parallel relative equilibrium but the distance between particles is arbitrary. A fixed vector $\bar{\boldsymbol{r}} \in \mathbb{C}^N$ specifies an isolated parallel relative equilibrium via the condition

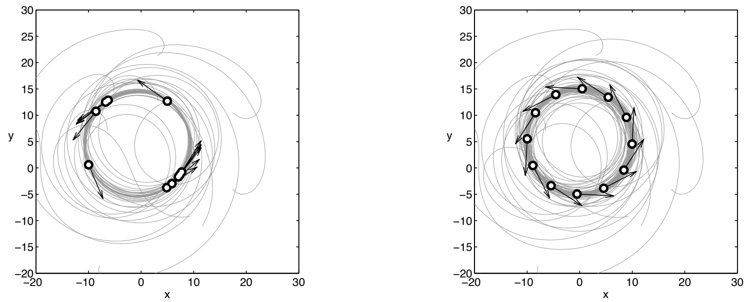

Fig. 1. Simulations of the circular control law (16) for a 4-circulant topology connecting $N = 12$ particles. The particle positions and headings are indicated by black circles and arrows. Both simulations use $\omega_0 = 0.1$, $K = 0.04$, and were simulated for the duration $10\frac{2\pi}{\omega_0}$. (a) Control (16) with $K_1 = 0$; (b) control (16) with $K_1 = 0.01$ and $\bar{\theta}$ corresponding to a splay state).

$$\boldsymbol{r} = \bar{\boldsymbol{r}} + \mathbf{1}r_0(t) \tag{17}$$

where $r_0(t) = r_0 + te^{i\theta_0}$ and where the constants $\theta_0 \in S^1$ and $r_0 \in \mathbb{C}$ are arbitrary. Motivated by the previous sections, the following proposition specifies this parallel equilibrium as a synchrony condition.

Proposition 3 *Let L be the Laplacian matrix of a bidirectional, connected graph. Let $\bar{\boldsymbol{r}}$ be an arbitrary vector in $\mathbb{C}^N$ such that $L_k\boldsymbol{r} \neq -1$, $k = 1, \ldots, N$. Define the vector*

$$\boldsymbol{t} = (I + \bar{D})^{-1}(L\boldsymbol{r} + e^{i\boldsymbol{\theta}}), \quad \bar{D} = \mathrm{diag}(L\bar{\boldsymbol{r}}). \tag{18}$$

Then the isolated relative equilibrium (17) is uniquely determined by the synchrony conditions

$$L\boldsymbol{t} = 0 \tag{19}$$

$$Le^{i\boldsymbol{\theta}} = 0. \tag{20}$$

Proof. The synchrony conditions (19) and (20) impose $\boldsymbol{t} = \mathbf{1}t_0$ and $e^{i\boldsymbol{\theta}} = \mathbf{1}e^{i\theta_0}$ for some fixed t_0 and $e^{i\theta_0}$. By definition of $\boldsymbol{t}$, this yields

$$(I + \bar{D})\mathbf{1}t_0 = L\boldsymbol{r} + \mathbf{1}e^{i\theta_0}. \tag{21}$$

Left multiplication of both members by $\mathbf{1}^T$ yields $t_0 = e^{i\theta_0}$ since $\mathbf{1}^T\bar{D}\mathbf{1} = \mathbf{1}^T L\bar{\mathbf{r}} = 0$. But then (21) implies $L\boldsymbol{r} = \bar{D}\mathbf{1} = L\bar{\boldsymbol{r}}$, that is,

$$\boldsymbol{r}(t) = \bar{\boldsymbol{r}} + \mathbf{1}r_0(t).$$

Differentiating both members, we obtain $\dot{\boldsymbol{r}}(t) = e^{i\boldsymbol{\theta}} = \mathbf{1}\dot{r}_0(t)$ which implies $\dot{r}_0 = e^{i\theta_0}$ under the synchrony assumption (20). This concludes the proof.

To enforce the conditions (19) and (20), we choose the Lyapunov function

$$V(\boldsymbol{s}, \boldsymbol{\theta}) = K < \boldsymbol{t}, L\boldsymbol{t} > +(1+K) < e^{i\boldsymbol{\theta}}, Le^{i\boldsymbol{\theta}} > . \tag{22}$$

The importance of specifying the parallel equilibrium through the synchrony conditions (19) and (20) is that the Lyapunov function (22) can be rendered nonincreasing along the closed-loop solutions by means of a feedback control that satisfies the required communication topology.

Proposition 4 *Consider the particle model (1) with the control law*

$$u_k = -(1+K) < ie^{i\theta_k}, L_k e^{i\boldsymbol{\theta}} > -K < L_k \mathbf{t}, (1 + L_k \bar{\boldsymbol{r}})^{-1} ie^{i\theta_k} >, \quad K > 0. \tag{23}$$

The parallel relative equilibrium defined by (17) is Lyapunov stable and a global minimum of the Lyapunov function (22). Moreover, for every $K > 0$, there exists an invariant set in which the Lyapunov function is nonincreasing along the solutions. In this set, solutions converge to a parallel equilibrium that satisfies

$$< (1 + L_k \bar{\mathbf{r}}) L_k \boldsymbol{t}, ie^{i\theta_0} >= 0 \tag{24}$$

for some $\theta_0 \in S^1$ and for $k = 1, \ldots, N$.

Proof. This is proved by a straightforward adaptation of the corresponding proof in [SPL05] where it was applied in the particular case $L = N(I - \mathbf{1}\mathbf{1}^T)$.

The phase control (8) stabilizes the set of parallel equilibria, which is of dimension $2(N-1)$ in the shape space. Away from singularities, the N algebraic constraints (24) are independent. As a result, the control law (23) isolates a subset of parallel equilibria of dimension $N-2$ in the shape space. However, it does not isolate the desired parallel equilibrium for $N > 2$. A simple calculation indeed shows that the Jacobian linearization of (1) at the parallel equilibrium (17) possesses $N-2$ uncontrollable spatial modes with zero eigenvalue. This means that the Jacobian linearization of the closed-loop system will possess $N-2$ zero eigenvalues for any smooth static state feedback. For $N > 2$, no smooth static state feedback can achieve exponential stability of an isolated relative parallel equilibrium.

4.4 Symmetries of the closed-loop vector field

All the control laws in this paper are shape control laws, i.e. they result in a closed-loop vector field that is invariant under the action of the group $SE(2)$. For an all-to-all communication topology, we have shown in [SPL05] how to break this symmetry with the help of an extra virtual particle that acts as a leader. This result easily extends to the fixed, bidirectional, and connected topologies considered in this paper and the details are omitted.

The control law (8) is further invariant under the (discrete) group of permutations, that is, there is no differentiation among particles. This symmetry property might be a desirable feature of the design but is lost for the

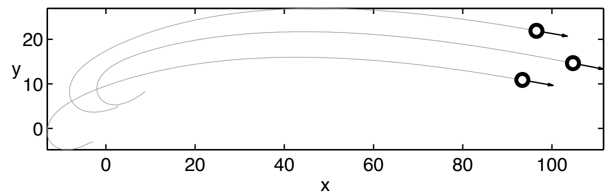

Fig. 2. Simulation of the parallel control law (23) for a complete graph connecting $N = 3$ particles. The particle positions and headings are indicated by black circles and arrows. The simulation uses $K = 1$ and $\bar{r} = 2e^{i\bar{\theta}}$ where $\bar{\theta}$ is a splay state.

control laws (16) and (23) that aim at stabilizing *isolated* relative equilibria of the model. An issue of interest is whether isolated relative equilibria can be stabilized with control laws that retain the permutation symmetry. In [PLS05, SPL05], we have addressed this question for the stabilization of isolated circular relative equilibria with certain symmetric phase arrangements. The results assume an all-to-all topology and make use of a linear combination of higher-harmonics potentials $U(e^{im\boldsymbol{\theta}})$, $m \in \{1, \ldots, \lfloor \frac{N}{2} \rfloor\}$. The generalization of such results to arbitrary communication topologies remains to be addressed.

5 Critical points of phase potentials

Several results in the previous section provide global convergence results to a critical point of the phase potential $Q(e^{i\boldsymbol{\theta}})$. In this section we further investigate the structure of the critical points of these phase potentials.

Lemma 1 *Let L be the Laplacian matrix of a bidirectional, connected graph. If $\bar{\boldsymbol{\theta}}$ is a critical point of the phase potential $Q(e^{i\boldsymbol{\theta}}) =< e^{i\boldsymbol{\theta}}, Le^{i\boldsymbol{\theta}} >$ then there exists a nonnegative, real vector $\boldsymbol{\alpha} \in \mathbb{R}^N$ such that*

$$(L - \mathrm{diag}(\boldsymbol{\alpha}))e^{i\bar{\boldsymbol{\theta}}} = 0 \tag{25}$$

and

$$\boldsymbol{\alpha}^T e^{i\bar{\boldsymbol{\theta}}} = 0. \tag{26}$$

The Hessian of the phase potential evaluated at these critical points is given by

$$H(e^{i\bar{\boldsymbol{\theta}}}) = 2(\tilde{L}(\bar{\boldsymbol{\theta}}) - \mathrm{diag}(\boldsymbol{\alpha})) \tag{27}$$

where $\tilde{L}(\boldsymbol{\theta})$ *is given by*

$$\tilde{L}_{kj} = L_{kj} < e^{i\theta_k}, e^{i\theta_j} > . \tag{28}$$

.

Proof. Critical points of the phase potential are characterized by

$$< ie^{i\bar{\theta}_k}, L_k e^{i\bar{\boldsymbol{\theta}}} >= 0. \tag{29}$$

If $\bar{\boldsymbol{\theta}}$ is a critical point, then

$$L_k e^{i\bar{\boldsymbol{\theta}}} = \alpha_k e^{i\bar{\theta}_k},\ \alpha_k \in \mathbb{R}, \tag{30}$$

for $k = 1, \ldots, N$. The condition (25) is the matrix form of (30). Left-multiplying (25) by $\mathbf{1}^T$ and using $\mathbf{1}^T L = 0$ gives the second condition (26). In words, the α_k are the weights for which the weighted centroid of the phasors, $e^{i\bar{\theta}_k}$, vanishes. To see that $\boldsymbol{\alpha}$ is nonnegative, we solve (30) for α_k to obtain

$$\alpha_k = d_k - \sum_{j \in \mathcal{N}_k} < e^{i\bar{\theta}_k}, e^{i\bar{\theta}_j} > . \tag{31}$$

Equation (31) gives the bounds $0 \leq \alpha_k \leq 2d_k$.

The Hessian of $Q(e^{i\boldsymbol{\theta}})$ is determined by

$$\frac{\partial^2 Q}{\partial \theta_k^2} = 2(d_k - < e^{i\theta_k}, L_k e^{i\boldsymbol{\theta}} >) \tag{32}$$

and, for $j \neq k$,

$$\frac{\partial^2 Q}{\partial \theta_j \partial \theta_k} = \begin{cases} -2 < e^{i\theta_k}, e^{i\theta_j} >, & \text{if } j \in \mathcal{N}_k, \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

Using (32) and (33) we can express the Hessian as

$$H(e^{i\boldsymbol{\theta}}) = 2(\tilde{L}(\boldsymbol{\theta}) - \mathrm{diag}(< e^{i\theta_1}, L_1 e^{i\boldsymbol{\theta}} >, \ldots, < e^{i\theta_N}, L_N e^{i\boldsymbol{\theta}} >)) \tag{34}$$

where $\tilde{L}(\boldsymbol{\theta})$ is given by (28). Evaluating (34) at a critical point and using (30), we obtain (27), which completes the proof.

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There are three special sets of critical points of the phase potential $Q(e^{i\boldsymbol{\theta}})$. If $\boldsymbol{\alpha} \in \mathrm{span}\{\mathbf{1}\}$, then $\boldsymbol{\alpha} = \lambda \mathbf{1}$ where λ is an eigenvalue of the graph Laplacian with the eigenvector $e^{i\bar{\boldsymbol{\theta}}}$. *Synchronized critical points* correspond to $\lambda = 0$ and $e^{i\bar{\boldsymbol{\theta}}} = e^{i\theta_0}\mathbf{1}$, where $\theta_0 \in S^1$. The set of synchronized critical points exists for any graph and is a global minimum of the phase potential. *Balanced critical points* correspond to $\boldsymbol{\alpha} = \lambda\mathbf{1}$ where $\lambda > 0$ and $\mathbf{1}^T e^{i\bar{\boldsymbol{\theta}}} = 0$. A sufficient condition for balanced critical points to exist is that the graph is circulant, that is, the Laplacian is a circulant matrix. The eigenvectors of any circulant matrix other than $\mathbf{1}$ are symmetric phase patterns on the unit circle centered at the origin of the complex plane.

The only critical points for which $\boldsymbol{\alpha} \notin \mathrm{span}\{\mathbf{1}\}$ that we have identified are *unbalanced* $(2, N)$*-patterns*. These patterns have two phase clusters that are separated by π and contain an unequal number of phases. If $e^{i\bar{\boldsymbol{\theta}}}$ is an unbalanced $(2, N)$-pattern, then $\mathbf{1}^T e^{i\bar{\boldsymbol{\theta}}} \neq 0$ and $\mathbf{1}^T e^{i2\bar{\boldsymbol{\theta}}} = N$. Using (13), we observe that the unbalanced $(2, N)$-patterns are critical points of the phase potential for any Laplacian since $\sin\bar{\theta}_{kj} = 0$ for all $(j, k) \in E$.

Lemma 2 *An equivalent expression to (27) for the Hessian of the phase potential $Q(e^{i\boldsymbol{\theta}})$ is the weighted Laplacian,*

$$H(\bar{\boldsymbol{\theta}}) = BW(\bar{\boldsymbol{\theta}})B^T. \tag{35}$$

The weight matrix is defined by

$$W(\bar{\boldsymbol{\theta}}) = 2\mathrm{diag}(\cos(B^T\bar{\boldsymbol{\theta}})) \in \mathbb{R}^{e\times e}. \tag{36}$$

Proof. Using (27) and (31), we obtain

$$H_{kj} = \begin{cases} 2\sum_{j\in\mathcal{N}_k} < e^{i\bar{\theta}_k}, e^{i\bar{\theta}_j} >, & \text{if } j = k, \\ 2 < e^{i\bar{\theta}_k}, e^{i\bar{\theta}_j} >, & \text{if } j \in \mathcal{N}_k, \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Equation (37) is equivalent to the weighted Laplacian (35). Let $f \in \{1, \ldots, e\}$ be the index of the edge $(j, k) \in E$. The corresponding weight is $W_{ff} = 2 < e^{i\bar{\theta}_k}, e^{i\bar{\theta}_j} >= 2\cos\bar{\theta}_{kj}$ in agreement with (36).

Next, we give sufficient conditions for asymptotic stability and instability of critical points of the phase potential that are isolated in the shape manifold T^N/S^1.

Proposition 5 *Let L be the Laplacian matrix of a bidirectional, connected graph. The potential $Q(e^{i\boldsymbol{\theta}}) =< e^{i\boldsymbol{\theta}}, Le^{i\boldsymbol{\theta}} >$ reaches its global minimum when $\boldsymbol{\theta} = \theta_0\mathbf{1}$, $\theta_0 \in S^1$ (synchronization). The gradient control $\dot{\boldsymbol{\theta}} = K_1\frac{\partial Q}{\partial\boldsymbol{\theta}}$ forces convergence of all solutions to the critical set of $Q(e^{i\boldsymbol{\theta}})$. If $\bar{\boldsymbol{\theta}}$ is a critical point that is isolated in the shape manifold T^N/S^1, then a sufficient condition for asymptotic stability of $\bar{\boldsymbol{\theta}}$ is $K_1W(\bar{\boldsymbol{\theta}}) < 0$ where $W(\bar{\boldsymbol{\theta}})$ is given by (36). If $KW(\bar{\boldsymbol{\theta}}) > 0$, then $\bar{\boldsymbol{\theta}}$ is unstable.*

Proof. Pursuant to Lemma 2, if $L = BB^T$ is the Laplacian of a connected graph and $W(\bar{\boldsymbol{\theta}})$ is definite, then $H(\bar{\boldsymbol{\theta}})$ has rank $N-1$ with the zero eigenvector $\mathbf{1}$. The stability result follows from the fact that the Jacobian of the gradient control is equal to the Hessian of the phase potential. Definiteness of $W(\bar{\boldsymbol{\theta}})$ implies that all eigenvalues of $H(\bar{\boldsymbol{\theta}})$ other than $\lambda = 0$ are positive.

Proposition 5 addresses phase arrangements in which $\cos\theta_{kj}$ has the same sign for all $(j,k) \in E$. Thus, synchronized critical points are asymptotically stable for $K_1 < 0$ and unstable otherwise since $W(e^{i\theta_0}\mathbf{1})$ is the identity matrix in $\mathbb{R}^{e\times e}$. The weight matrix is definite for some balanced critical points as well. For circulant matrices, it is sufficient to check that all $\cos\theta_{1j}$ have the same sign for $j \in \mathcal{N}_1$. A complete characterization of balanced critical points and the unbalanced $(2,N)$-patterns is the subject of ongoing work and will be presented in a separate paper.

6 Time-varying and unidirectional topologies

The stabilization results of the previous section require a communication graph which is time-invariant, undirected, and connected. It is of interest, both from the theoretical and practical viewpoint, to investigate which of these assumptions can be relaxed. In the rest of the paper, $L(t)$ denotes the Laplacian matrix of a *time-varying directed* graph $G(t)$.

The simplest situation to analyze is the parallel control (8), with $\omega_0 = 0$, which only involves the phase dynamics

$$\dot{\theta}_k = u_k = < ie^{i\theta_k}, L_k(t)e^{i\boldsymbol{\theta}} > \tag{38}$$

Several synchronization results for pure integrator dynamics have recently appeared in the literature [JLM02, JMB04, OSM04, Mor05], both for continuous-time and discrete-time models. Convergence results are obtained for time-varying and undirected communication graphs, under relaxed connectedness assumptions. These convergence results are not global: they require all the initial phase differences to be in the in the open interval $(-\pi/2, \pi/2)$. Under this assumption, the phase dynamics (38) can be mapped to the Euclidean state-space $\mathbb{R}^N$ through a change of coordinates. Then, a key observation for the convergence analysis in [Mor05] is that the convex hull of the states can only contract along the solutions.

We now propose an extension of the results in [Mor05] for the stabilization of circular motions ($\omega_0 \neq 0$) by considering the synchronization of the "centers" (4) which obey the dynamics

$$\dot{s}_k = -ie^{i\theta_k}(u_k - \omega_0) \tag{39}$$

for $k = 1, \ldots, N$. Following [Mor05], we adopt $\mathrm{co}\{\mathbf{s}\}$ as a set-valued Lyapunov function and modify the control (8) such that this function is nonincreasing. Using (4), one can interpret this modification as contracting the convex hull of the centers of the particles' circular orbits.

Denote by $\mathrm{conhull}(\mathcal{N}_k(t))$ the conic hull spanned by the neighbors of s_k at time t, that is,

$$\mathrm{conhull}(\mathcal{N}_k(t)) = \{\sum_j \beta_j(s_j - s_k) \mid j \in \mathcal{N}_k(t),\ \beta_j \geq 0\}$$

and consider the sets $S_k^+(t) = s_k + \text{conhull}(\mathcal{N}_k(t))$, $S_k^-(t) = s_k - \text{conhull}(\mathcal{N}_k(t))$, and $S_k(t) = S_k^+(t) \cup S_k^-(t)$. To ensure that co$\{\mathbf{s}\}$ contracts along the solutions of the particle model, we modify the circular control (8) as

$$u_k = \omega_0 + K_k(\theta_k, S_k(t))|L_k(t)\boldsymbol{s}| \tag{40}$$

with the gain $K_k(\theta_k, S_k(t))$ a Lipschitz continuous function that satisfies

$$K_k(\theta_k, S_k(t)) \begin{cases} > 0 \text{ if } -ie^{i\theta_k} \in S_k^+(t) \\ < 0, \text{ if } -ie^{i\theta_k} \in S_k^-(t) \\ 0, \qquad \text{otherwise} \end{cases} \tag{41}$$

Indeed, if $\dot{s}_k \in S_k^+(t)$ (resp. $\dot{s}_k \in S_k^-(t)$), then $\dot{s}_k$ (resp. $-\dot{s}_k$) points inwards co$\{\mathbf{s}\}$ at s_k. As a consequence, the control law (40) renders the convex hull co$\{\mathbf{s}\}$ invariant, from which we obtain the following result.

Proposition 6 *Let $L(t)$ be the time-dependent Laplacian matrix of a time-varying directed graph. Assume that $L(t)$ is T-periodic and that the following condition is satisfied for all $\mathbf{s} \neq \mathbf{1}s_0$:*

$$\nexists\, \theta_0 \in S^1, \quad k \in \{1, \ldots, N\} : -ie^{i(\omega_0 t + \theta_0)} \in \mathbb{C} \backslash S_k(t) \quad \forall t > 0. \tag{42}$$

Then the shape feedback control (40) uniformly asymptotically stabilizes the set of circular motions of radius $|\omega_0|^{-1}$ ($\omega_0 \neq 0$).

Proof. By design, the set-valued Lyapunov function co$\{\mathbf{s}\}$ is nonexpanding with the control (40). Moreover, it is strictly contracting when $u_k - \omega_0 \neq 0$ for some k. This means that the limit set of each solution is a set where $\mathbf{u} \equiv \mathbf{1}\omega_0$. In this set, $\dot{\boldsymbol{s}} = 0$ and $\dot{\boldsymbol{\theta}} = \mathbf{1}\omega_0$. Invariance of the limit sets implies that all solutions must converge to a constant $\boldsymbol{s}$ that satisfies

$$\forall\, t : K_k(\theta_0 + \omega_0 t, S_k(t)) = 0,\ k = 1, \ldots, N$$

for some $\theta_0 \in S^1$. This set reduces to $\mathbf{1}s_0$ under the condition (42).

The result of Theorem (6) holds without the periodicity assumption on $L(t)$ but the proof is more technical and not detailed in the present paper. The convergence result is illustrated by the simulations shown in Figure 3 and Figure 4.. In the simulations of Figure 3, we used a fixed unidirectional connected 2-circulant topology (each particle is connected to two neighbors). The simulation illustrates a situation where the control (8) that stabilizes the circular motion with a fixed bidirectional, connected topology fails to achieve the same stabilization when the communication becomes unidirectional. In contrast, the modfied gain (41) achieves stabilization. In the simulations of Figure 3, we used a unidirectional, weakly connected topology in which the neighbor sets, $\mathcal{N}_k$, $k = 1, \ldots, 12$, have identical cardinality, $d_k = 4$, but whose members are chosen randomly with frequency $\frac{\omega_0}{2\pi}$. The simulation illustrates

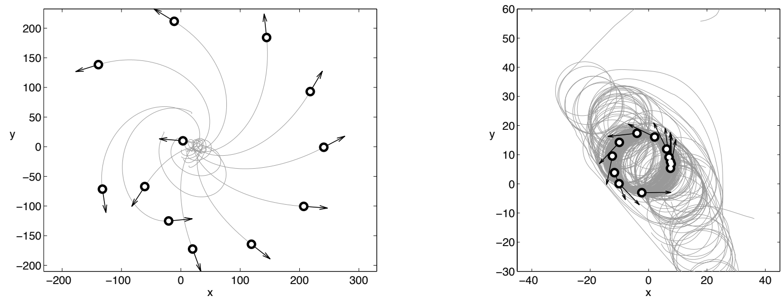

Fig. 3. Simulations of a unidirectional, connected 2-circulant topology. The particle positions and headings are indicated by black circles and arrows. Both simulations have identical initial conditions, use $\omega_0 = 0.1$, and were simulated for the duration $30\frac{2\pi}{\omega_0}$. (left) Control (8) with $K = 1$; (right) control (40) with gain (41).

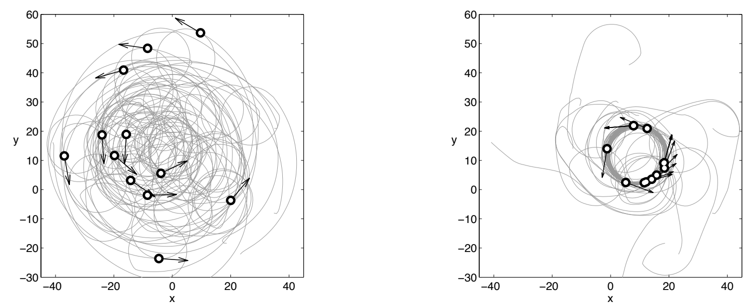

Fig. 4. Simulations of a unidirectional, weakly connected topology in which the neighbor sets, $\mathcal{N}_k$, $k = 1, \ldots, 12$, have identical cardinality, $d_k = 2$, but whose members are chosen randomly with frequency $\frac{\omega_0}{2\pi}$. The particle positions and headings are indicated by black circles and arrows. Both simulations have identical initial conditions and randomly switching Laplacian $L(t)$, use $\omega_0 = 0.1$, and were simulated for the duration $10\frac{2\pi}{\omega_0}$. (left) Control (8) with $K = 1$; (right) control (40) with gain (41).

a situation where the control (8) that stabilizes the circular motion with a fixed bidirectional, connected topology fails to achieve the same stabilization when these assumptions are relaxed.

It is of interest to note that, for the simulation described in Figure 4, the control (8) is not stabilizing for $K = 1$ but does stabilize the circular formation for smaller values of the gain. The convergence for sufficiently weak coupling gain $K = \epsilon > 0$ suggests to apply averaging analysis to the closed-loop system

$$\begin{aligned} \dot{s}_k &= -\epsilon i e^{i(\omega_0 t + \tilde{\theta}_k)} < i e^{i(\omega_0 t + \tilde{\theta}_k)}, L_k(t)\boldsymbol{s} > \\ \dot{\tilde{\theta}}_k &= \epsilon < i e^{i(\omega_0 t + \tilde{\theta}_k)}, L_k(t)\boldsymbol{s} > \end{aligned} \tag{43}$$

where we have used the coordinate change $\tilde{\theta}_k = \theta_k - \omega_0 t$. For a switching (piecewise-constant) communication topology with switching times synchronized with the rotation period $\frac{2\pi}{\omega_0}$, the averaged system is $\dot{\boldsymbol{s}} = -(\epsilon I + 0(\epsilon^2))L(t)\boldsymbol{s}$ and the convergence results in [Mor05] apply, that is, synchronization is achieved under a weak connectedness assumption on $L(t)$.

7 Discussion

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