Therefore, the closed-loop matrices are given by

$$A_c(1) = \begin{bmatrix} 0.4888 & 0.112 \\ -1.512 & -0.388 \end{bmatrix}$$
$$A_c(2) = \begin{bmatrix} 0.2353 & 0.0894 \\ -0.1324 & -0.1353 \end{bmatrix}$$
$$A_c(3) = \begin{bmatrix} 0.092 & 0.016 \\ 1.296 & 0.008 \end{bmatrix}.$$

The necessary and sufficient condition (40) is satisfied for $\delta_1 = 10, \delta_2 = 15, \delta_3 = 10$. One can easily check that all the closed-loop matrices $A_c(\alpha)$, are assigned the same spectrum $\sigma(H_0)$.

On the other hand, the sufficient condition (41) of stochastic stability is also satisfied for $\rho = (\frac{2}{5})$.

VI. Conclusion

In this note, necessary and sufficient conditions for domain $\mathcal{F}$ to be positively invariant w.r.t the system in the closed-loop (5) are established for linear discrete-time systems with Markovian jumping parameters and symmetrical constrained control. A new sufficient condition of stochastic stability is then deduced. These results are obtained by using non quadratic Lyapunov function as is usually the case in the problems with constraints of inequality type. An Algorithm is also presented to compute matrices $H(\alpha)$ together with a simple sufficient condition of stochastic positive invariance which is independent of the probability transition matrix Π.

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Trading the Stability of Finite Zeros for Global Stabilization of Nonlinear Cascade Systems

Rodolphe Sepulchre, Murat Arcak, and Andrew R. Teel

***Abstract*—This note analyzes the stabilizability properties of nonlinear cascades in which a nonminimum phase linear system is interconnected through its output to a stable nonlinear system. It is shown that the instability of the zeros of the linear system can be traded with the stability of the nonlinear system up to a limit fixed by the growth properties of the cascade interconnection term. Below this limit, global stabilization is achieved by smooth static-state feedback. Beyond this limit, various examples illustrate that controllability of the cascade may be lost, making it impossible to achieve large regions of attractions.**

***Index Terms*—Nonlinear cascades, peaking, stabilization.**

I. Introduction

The study of partially linear cascades

$$\begin{aligned} \dot{z} &= f(z) + \Psi(z, \xi, y)y \\ \dot{\xi} &= A\xi + Bu \\ y &= C\xi + Du, \qquad z \in \mathbb{R}^s, \, \xi \in \mathbb{R}^n, \, y \in \mathbb{R}^p, \, u \in \mathbb{R}^m \end{aligned} \tag{1}$$

has been helpful to identify structural obstacles to large regions of attraction (see, e.g., [9], [4], [2], [1], [5], and [11]). The general scenario in these references is that the nonlinear subsystem $\dot{z} = f(z)$ has a globally asymptotically stable equilibrium $z = 0$, so that the local stabilization problem is linear, but that the perturbation $\Psi(z, \xi, y)y$ may cause finite escape time for the solution $z(t)$ if the output $y(t)$ of the linear subsystem (A, B, C, D) is not properly controlled.

Beyond invertibility conditions for the linear system, successive contributions in the literature have revealed the prominent role played by the *zeros* of the linear system in the global stabilizability of the cascade (1).

With their analysis of the peaking phenomenon, Sussmann and Kokotović [9] have shown that the *infinite zeros* of the linear system are the most harmful ones. Because of the large transients that they exhibit during the fast stabilization of the output, the output derivatives must be excluded from the interconnection Ψ to render the global stabilization of the cascade possible. In a subsequent paper [4], Saberi and the same authors showed that if the output derivatives do not enter the interconnection and the zero dynamics of the linear system are Lyapunov stable (the cascade is then said to be "weakly minimum

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phase"), then the stabilizability of the linear system guarantees the global stabilizability of the cascade. This result was further extended in [5] (see also [11] for a different version and [2] for the semiglobal counterpart) to the unstable situation where repeated zeros are allowed on the imaginary axis.

The situation of *unstable finite* zeros was considered for the first time by [1]. With simple but illuminating examples, the authors showed that unstable zeros may constitute an obstacle to semiglobal stabilization if they are "too far" to the right. Indeed, their stabilization requires a finite output energy [3] which is sufficient to cause finite escape time for $z(t)$. In the same paper, the authors showed on an example (using discontinuous feedback) that global stabilization might be possible when the zeros are closer to the imaginary axis.

The present note (see [7] for a preliminary version) pursues the analysis of nonlinear cascades in the presence of unstable zeros under the following assumptions.

H1) The linear system $H \equiv (A, B, C, D)$ is square ($m = p$) and has a uniform relative degree $\{r, \ldots, r\}$ so that (1) is feedback equivalent to the normal form

$$\begin{aligned} \dot{z} &= f(z) + \Psi(z, \xi_0, y, \dot{y}, \ldots, y^{(r-1)})y \\ \dot{\xi}_0 &= A\xi_0 + By \\ y^{(r)} &= u, \qquad z \in \mathbb{R}^s, \, \xi_0 \in \mathbb{R}^{n-mr}, \, y \in \mathbb{R}^m, \, u \in \mathbb{R}^m \end{aligned} \tag{2}$$

with new matrices A and B. The pair (A, B) is stabilizable and all the eigenvalues of A (that is, the finite zeros of H) have a real part smaller or equal to $\nu > 0$, i.e., $\max Re\lambda(A) \le \nu$. The functions f and Ψ are locally Lipschitz.

H2) The equilibrium $z = 0$ of $\dot{z} = f(z)$ is globally asymptotically stable. In a neighborhood of the origin, the solutions satisfy the exponential estimate $U(z(t)) \le U(z(0))e^{-\alpha t}$ for some positive constant $\alpha > 0$ and a smooth positive–definite function $U(z)$ with $(\partial^2 U/\partial z^2)(0) > 0$.

H3) The interconnection term Ψ does not depend on the output derivatives, i.e., $\Psi(z, \xi, y) = \Psi(z, \xi_0, y)$, and satisfies the following growth condition: there exist positive constants p, q, C such that, for z sufficiently small

$$\|\Psi(z, \xi, y)\| \le C\|z\|^{q+1}\|(\xi, y)\|^{p-1}. \tag{3}$$

Under the three assumptions above, the results of this note determine a sharp stabilizability boundary in terms of structural parameters of the cascade. The stabilizability condition is expressed as an inequality between two ratios: a *stability* ratio between the local stability of the z-subsystem $\dot{z} = f(z)$ (parameter α) and the instability of the finite zeros (parameter ν), and a *growth* ratio between the growth of the interconnection term $\Psi(z, \xi, y)$ in the variable z (parameter q) and in the variables (ξ, y) (parameter p). The stabilizability limit of the cascade is attained when the stability ratio becomes equal to the growth ratio

$$\frac{\nu}{\alpha} = \frac{q}{p}.$$

Below this limit, the stability of the finite zeros associated with the z-subsystem can be traded with the instability of the finite zeros associated with the ξ-subsystem and we design a smooth feedback control that achieves global stabilization of the origin. Beyond this limit, various examples illustrate the possible loss of global controllability.

Section II describes our main result in the relative degree zero case, that is when $y = u$. Section III provides three examples of loss of global controllability when the stabilizability boundary is attained. Extension to the general cascade (2) is included in Section IV.

II. Main Result

For the sake of clarity, we formulate our main result with further simplifying assumptions that will be removed in Section IV.

Theorem 1: Assume that H1) holds with a relative degree $r = 0$, that H2) holds with a linear z-subsystem, and that H3) holds globally in z. Then the cascade (2) reduces to

$$\begin{aligned} \dot{z} &= Fz + \Psi(z, \xi, u)u \\ \dot{\xi} &= A\xi + Bu \end{aligned} \tag{4}$$

and the equilibrium $(z, \xi) = 0$ of (4) is globally asymptotically stabilizable by smooth state feedback if

$$\frac{\nu}{\alpha} < \frac{q}{p}. \tag{5}$$

Proof: We let

$$U(z) = z^T Q z, \qquad V(\xi) = \xi^T P \xi \tag{6}$$

where the matrix $Q = Q^T > 0$ will be specified and $P = P^T > 0$ is arbitrary, and design a control law which enforces for $U(z)$ the exponential decay

$$U(z(t)) \le U(z(0))e^{-2\overline{\alpha}t} \tag{7}$$

and limits the exponential growth of $V(\xi)$ to

$$V(\xi(t)) \le \overline{\sigma}V(\xi(0))e^{2\overline{\nu}t} \tag{8}$$

for constants $\overline{\alpha} < \alpha, \overline{\nu} > \nu$, and $\overline{\sigma}$ to be designed. Inequalities (7) and (8) imply that the positive–semidefinite function

$$W(z, \xi) := U(z)^{q/2}V(\xi)^{p/2} \tag{9}$$

satisfies the estimate

$$W(z(t), \xi(t)) \le \sigma W(z(0), \xi(0))e^{-\kappa t} \tag{10}$$

where

$$\sigma = \overline{\sigma}^{\,p/2} \quad \text{and} \quad \kappa := \overline{\alpha}q - \overline{\nu}p. \tag{11}$$

To ensure that $W(z, \xi)$ converges to zero exponentially, we select $\overline{\alpha} < \alpha$ and $\overline{\nu} > \nu$ sufficiently close to α and ν, respectively, so that $\kappa > 0$ because of (5).

Global asymptotic stabilization of (4) will then be achieved if the design is such that, when W is sufficiently small, the control law becomes a linear feedback $u = K\xi$ which stabilizes the ξ-subsystem.

A first lemma puts conditions on the control law to guarantee the exponential decay of $U(z)$.

Lemma 1: For any $\overline{\alpha} < \alpha$, there exists $\epsilon_0 > 0$ such that the exponential estimate (7) holds if

$$\|u\| \le \gamma(W)\|\xi\| \tag{12}$$

with $\gamma(\cdot): \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ a bounded function such that $\gamma(W)W \le \epsilon_0$ for all $W \ge 0$.

Proof of Lemma 1: Let $\alpha' \in (\overline{\alpha}, \alpha)$, and let $Q = Q^T > 0$ in (6) be such that

$$F^T Q + QF \le -2\alpha' Q. \tag{13}$$

From (4), (3), and (13) $U(z)$ satisfies

$$\begin{aligned} \dot{U} &\le -2\alpha' U + 2z^T Q\Psi(z, \xi, u)u \\ &\le -2\alpha' U(z) + 2C\|Q\|\,\|z\|^{q+2}\|(\xi, u)\|^{p-1}\|u\|. \end{aligned}$$

If u satisfies (12), then there exists a constant $c_1 > 0$ such that

$$\dot{U} \le -2\alpha' U + c_1 U\gamma(W)W \le -2\alpha' U + \epsilon_0 c_1 U \tag{14}$$

thus (7) holds if we select ϵ_0 small enough to satisfy $\overline{\alpha} \le \alpha' - \epsilon_0(c_1/2)$. [An explicit calculation yields

$c_1 = 2C\|Q\|\lambda_{\min}^{-(p/2)}(P)\lambda_{\min}^{-(1+q/2)}(Q)(1+\overline{\gamma}^2)^{(p-1)/2}$ where $\overline{\gamma}$ is an upper bound on $\gamma(W)$]. □

To design a control law that ensures (8) for the ξ-subsystem of (4), we denote by $\beta_\epsilon(\cdot):\ \mathbb{R}\rightarrow[0,\,1]$, for a given constant $\epsilon>0$, a smooth monotone function satisfying

$$\beta_\epsilon(W)=\begin{cases}0 & \text{when } |W|\geq\epsilon\\ 1 & \text{when } |W|\leq\epsilon/2\end{cases} \tag{15}$$

and we make use of the following lemma proved in the Appendix.

Lemma 2: Let A_i be a matrix such that $Re\,\lambda(A_i)\leq\nu$, and let l denote the number of its eigenvalues in the closed right-half plane. If the pair $(A_i,\,B)$ is stabilizable, then there exists a matrix P_i such that

1) $A-BB^TP_i$ has at most $l-1$ eigenvalues in the closed right-half plane;
2) Given any $P=P^T>0$ and $\overline{\nu}>\nu$, we can find a constant $\overline{\sigma}_i>0$ such that, for any constant $\epsilon_i>0$, the solutions of the system

$$\dot{\xi}=(A_i-\beta_{\epsilon_i}(W(z,\,\xi))BB^TP_i)\xi \tag{16}$$

satisfy the exponential estimate (8) where $V(\xi)=\xi^TP\xi$ as in (6).

In view of Lemmas 1 and 2, we select the control law

$$u=-\beta_{\epsilon_1}(W)B^TP_1\xi-\cdots-\beta_{\epsilon_N}(W)B^TP_N\xi \tag{17}$$

where $\beta_{\epsilon_i}(\cdot)$s are smooth functions defined as in (15), and $N\leq l$ where l denotes the number of closed right-half plane eigenvalues of A.

The design of the matrices P_i, $1\leq i\leq N$, uses the construction of Lemma 2 applied to $A_1=A$, and then iteratively to $A_{i+1}=A_i-BB^TP_i$, $i\geq 1$, until a matrix A_N is obtained that is Hurwitz. By construction, the number of eigenvalues in the closed right-half plane decreases at each iteration, so that N does not exceed the number of eigenvalues of A in the closed right-half plane. With this construction, the control law (17) is a linear stabilizing feedback for the ξ-subsystem in the region defined by $W(z,\,\xi)\leq\epsilon_N/2$.

The design of the parameters $\epsilon_N<\epsilon_{N-1}<\cdots<\epsilon_1$ is made to guarantee that $W(z(t),\,\xi(t))$ indeed decreases to zero along any solution. The parameter ϵ_1 is selected to guarantee the exponential decay (7) with the control law (17). Because $u=0$ when $W\leq\epsilon_1$ and because $\|u\|\leq K\|\xi\|$ (with $K=\sum_{i=1}^N\|B^TP_i\|$), (12) holds with a function $\gamma(\cdot)$ satisfying $\gamma(W)W\leq K\epsilon_1$. Thus, Lemma 1 ensures that (7) holds if ϵ_1 is selected such that $K\epsilon_1<\epsilon_0$.

Having selected ϵ_1 and using the matrix P_1 constructed according to Lemma 2, (8) holds for some $\overline{\sigma}_1$ whenever $u=-\beta_{\epsilon_1}(W)B^TP_1\xi$, which is the control (17) in the region where $W\geq\epsilon_2$.

The parameter ϵ_2 is now selected to guarantee that for any initial condition $(z_0,\,\xi_0)$, there exists a finite time T such that $W(z(t),\,\xi(t))\leq\epsilon_1/2$ for all $t\geq T$. Using the estimate (10) (with $\sigma_1=\overline{\sigma}_1^{p/2}$) whenever $W\geq\epsilon_2$, such a T will exist if $\sigma_1\epsilon_2<\epsilon_1/2$. From time T on, and as long as $W(z(t),\,\xi(t))\geq\epsilon_3$, the closed-loop system reads

$$\dot{\xi}=(A_2+\beta_{\epsilon_2}(W)BB^TP_2)\xi$$

thus, by Lemma 2, the estimate (8) holds in the region defined by $W(z,\,\xi)\geq\epsilon_3$. This construction is iterated for $i=3,\,\ldots,\,N$ by selecting ϵ_i such that $\sigma_{i-1}\epsilon_i<\epsilon_{i-1}/2$. With $\overline{\sigma}$ selected as $\overline{\sigma}_N$, the estimate (8) then holds along every closed-loop solution.

With the ϵ_i parameters so constructed, $W(z(t),\,\xi(t))$ exponentially converges to zero along any solution of the closed-loop system. After a finite time, $W(z(t),\,\xi(t))$ must remain smaller than $\epsilon_N/2$, and from this time on, $\xi(t)$ exponentially converges to zero, which concludes the proof. □

The next theorem deals with the situation where the inequality (5) becomes an equality.

Theorem 2: With all the remaining assumptions being unchanged, assume that the assumption (5) of Theorem 1 is replaced by

$$\frac{\nu}{\alpha}=\frac{q}{p}. \tag{18}$$

Then the equilibrium $(z,\,\xi)=0$ of (4) is globally stabilizable by smooth state feedback if the matrices $A-\nu I$ and $F+\alpha I$ are Lyapunov stable and $\Psi(z,\,\xi,\,0)=0$.

Proof: Using the strengthened assumptions on A and F, we let $Q=Q^T>0$ and $P=P^T>0$ be such that

$$F^TQ+QF\leq-2\alpha Q \tag{19}$$

$$A^TP+PA\leq 2\nu P \tag{20}$$

and construct U, V, and W as in the proof of Theorem 1.

Augmenting the control law (17) with the additional term

$$u_0=-\frac{2}{1+l(W)}\,B^TP\xi \tag{21}$$

we will construct the function $l(\cdot)$ in such a way that all solutions converge in finite time to an invariant region where the proof of Theorem 1 can be applied. Our first requirement on $l(\cdot)$ will be that

$$\|B^TP\|\,\frac{2W}{1+l(W)}<\epsilon_0 \tag{22}$$

so that the conclusion of Lemma 1 applies.

The time-derivative of W satisfies

$$\dot{W}\leq\frac{W}{V}\left(q\,\frac{V}{U}\,z^TQ\Psi+p\xi^TPB\right)u. \tag{23}$$

From (3) and $\Psi(z,\,\xi,\,0)=0$, there exists a constant $c_2=c_2(C)>0$ such that

$$\|u\|\leq C\|\xi\|\Rightarrow V(\xi)\|\Psi(z,\xi,u)\|\leq c_2\|z\|^{q+1}\|\xi\|^p\|u\|. \tag{24}$$

This also implies, for some other constant $c_3=c_3(C)$

$$\|u\|\leq C\|\xi\|\Rightarrow\frac{V(\xi)}{U(z)}\,\|z^TQ\Psi(z,\,\xi,\,u)\|\leq c_3W\|u\|. \tag{25}$$

From (23), (25), and with the control law given by (21), we obtain for $c_4=2qc_3$

$$\dot{W}\leq\frac{W}{V}\left(c_4\,\frac{2W}{1+l(W)}-p\right)\|B^TP\xi\|^2 \tag{26}$$

which yields the estimate

$$\dot{W}\leq-\frac{p}{2}\,\|B^TP\xi\|^2 \tag{27}$$

if, in addition to (22), $l(W)$ is chosen such that $c_4(2W/(1+l(W)))\leq p/2$ and $l(W)=1$ for $W\leq\epsilon_0$ with ϵ_0 small enough to have $c_4\epsilon_0<p/4$.

We now prove that W decays to zero exponentially along any solution of the closed-loop system with $u=u_0$. By contradiction, suppose that, for some initial condition, $W(z(t),\,\xi(t))$ does not converge to zero. Then, (27) implies that $y(t):=B^TP\xi(t)$ is in $L_2(0,\,\infty)$ while $\xi(t)$ is solution of the differential equation

$$\begin{aligned}\dot{\xi}&=A\xi-\frac{2}{1+l(W)}\,BB^TP\xi\\&=(A-BB^TP)\xi+\left(1-\frac{2}{1+l(W)}\right)y(t).\end{aligned} \tag{28}$$

Note that all the eigenvalues of $A-BB^TP$ have a real part smaller than or equal to some $\overline{\nu}<\nu$. The solutions of (28) satisfy the estimate

$$\|\xi(t)\|\leq Me^{\overline{\nu}t}\left(\|\xi(0)\|+\int_0^t e^{-\overline{\nu}s}\|y(s)\|\,ds\right).$$

Along a solution such that $y(t)$ is in $L_2(0, \infty)$, this implies

$$\|\xi(t)\| \le M' e^{\overline{\nu} t} \|\xi(0)\|$$

where the constant M' is allowed to depend on $z(0)$. Then, such a solution also satisfies

$$W(z(t), \xi(t)) \le \sigma e^{-\kappa t} W(z(0), \xi(0)) \tag{29}$$

with $\kappa = \alpha q - \overline{\nu} p > 0$, which contradicts the assumption that W does not decay exponentially to zero.

Because W exponentially decays to zero with $u = u_0$ and because $u_0 = -B^T P \xi$ for $W \le \epsilon_0$, all solutions converge in finite time to a region where $\dot{\xi} = (A - BB^T P)\xi + B(u - u_0)$. With A replaced by $A - BB^T P$ and ν replaced by $\overline{\nu}$, the design of $u - u_0$ is pursued as in the proof in Theorem 1 because $\overline{\nu}/\alpha < p/q$. □

III. Obstacles to Controllability

In this section, we show that relaxing any of the conditions of Theorems 1 and 2 leads to situations in which the cascade (4) is no longer globally asymptotically controllable to the origin. (Global asymptotic controllability to the origin is obviously a necessary condition for semiglobal stabilization).

Our first example, adapted from [1], illustrates a situation of uncontrollability when the inequality (5) is reversed.

Example 1 [1]: Consider the cascade

$$\begin{aligned} \dot{z} &= -\alpha z + z^{q+1} u^2 \\ \dot{\xi} &= \nu \xi + u, \qquad z \in \mathbb{R},\, \xi \in \mathbb{R},\, u \in \mathbb{R} \end{aligned} \tag{30}$$

and suppose that $\nu - \alpha q/2 := \kappa > 0$. With an argument similar to the one in [1], one shows that, if $\xi(t)$ converges to zero, the finite escape time of $z(t)$ can be avoided only for initial conditions that satisfy the constraint

$$z^q(0)\xi(0)^2 \le \frac{1}{2q\kappa}. \tag{31}$$

Initial conditions that violate (31) are uncontrollable to zero.

Our second example illustrates that the requirement for $A - \nu I$ and $F + \alpha I$ to be Lyapunov stable in Theorem 2 cannot be weakened to the condition $\max \mathcal{R}e\lambda(A) \le \nu$.

Example 2: Consider the cascade

$$\begin{aligned} \dot{z} &= -\alpha z + z^{q+1} \xi_1^4 u^2 \\ \dot{\xi}_1 &= \nu \xi_1 + \xi_2 \\ \dot{\xi}_2 &= \nu \xi_2 + u, \qquad z \in \mathbb{R},\, \xi \in \mathbb{R}^2,\, u \in \mathbb{R} \end{aligned} \tag{32}$$

and assume that $6\nu = \alpha q$. Note that the condition $\Psi(z, \xi, 0) = 0$ is satisfied but that $A - \nu I$ is the (unstable) double integrator. Defining $\tilde{\xi}_i = e^{-\nu t}\xi_i$ and $\tilde{u} = e^{-\nu t} u$, we rewrite the linear part in the form

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 &= \tilde{u}. \end{aligned} \tag{33}$$

Let $\sigma = z^{-q}$ and $\sigma(0) > 0$. To avoid the finite escape time of $z(t)$, the following condition must hold:

$$\int_0^\infty e^{-\alpha q s} \xi_1^4(s) u^2(s)\, ds = \int_0^\infty \tilde{\xi}_1^4(s) \tilde{u}^2(s)\, ds < \frac{\sigma(0)}{q}. \tag{34}$$

We now show that, for large initial conditions, it is not possible to ensure the convergence of $\tilde{\xi}(t)$ while satisfying the conditions (34). Let $\tilde{\xi}_2(0) > 1$ and $\tilde{\xi}_1(0) \ge 1$. If $\tilde{\xi}_2(t)$ asymptotically converges to zero, then there exists a finite $T > 0$ such that $\tilde{\xi}_2(T) = 1$ and $\tilde{\xi}_2(t) > 1$ on the interval $[0, T]$. Thus

$$\left| \int_0^T \tilde{u}(s)\, ds \right| = \tilde{\xi}_2(0) - 1 \tag{35}$$

and $\tilde{\xi}_1(t) = \tilde{\xi}_1(0) + \int_0^t \tilde{\xi}_2(s) ds \ge 1 + t$ on the interval $[0, T]$ and, hence, (34) implies

$$\int_0^T \tilde{u}^2(s)(s+1)^4\, ds < \frac{\sigma(0)}{q}. \tag{36}$$

Using Holder's inequality, we have

$$\begin{aligned} \int_0^T \tilde{u}(s)\, ds &= \int_0^T \left(\tilde{u}(s)(s+1)^2\right) \frac{1}{(s+1)^2}\, ds \\ &\le \left(\int_0^T \tilde{u}^2(s)(s+1)^4\, ds \right)^{1/2} \left(\int_0^T \frac{1}{(s+1)^4}\, ds \right)^{1/2} \\ &< K \left(\int_0^T \tilde{u}^2(s)(s+1)^4\, ds \right)^{1/2} \end{aligned}$$

where the constant $K > 0$ is independent of T. Using (35) and (36), we obtain the constraint

$$\tilde{\xi}_1(0) \ge 1 \quad \Rightarrow \quad \tilde{\xi}_2(0) - 1 < K \left(\frac{|\sigma(0)|}{q} \right)^{1/2} \tag{37}$$

which implies that initial conditions of (32) violating (37) cannot be controlled to the origin.

Our last example illustrates the necessity of the condition $\Psi(z, \xi, 0) = 0$ in Theorem 2.

Example 3: The cascade

$$\begin{aligned} \dot{z} &= -3z - z^2 \xi^2 u \\ \dot{\xi} &= \xi + u, \qquad z \in \mathbb{R},\, \xi \in \mathbb{R},\, u \in \mathbb{R} \end{aligned} \tag{38}$$

satisfies all the conditions of Theorem 1 except that $\Psi(z, \xi, 0) = -z^2\xi^2 \neq 0$. (The equality $p\nu = \alpha q$ is satisfied with $\alpha = p = 3$, and $\nu = q = 1$). It is easily verified that $z\xi^3 = 3$ is an invariant manifold regardless of the choice of u because

$$\left. \frac{d}{dt}(z\xi^3) \right|_{z\xi^3 = 3} = u z \xi^2 (-z\xi^3 + 3)|_{z\xi^3=3} = 0. \tag{39}$$

Hence, initial conditions satisfying $z(0)\xi(0)^3 = 3$ cannot be controlled to the origin.

IV. Discussion on the General Case

The results of Section II, proven for the particular cascade (4), are retained under the more general assumptions H1)–H3).

First, observe that the linearity assumption on the z-subsystem is easily relaxed to H2) and that H3) needs not hold globally, as it is assumed in Section II, but only locally. The extension of the results to this situation is straightforward because the proof of the theorems only relies on *local* properties of the z-subsystem. As a consequence, it is sufficient to multiply the constructed control laws by a gain function $\beta(\|z\|)$ which is zero for $\|z\| \ge \delta_-$ and which is equal to one for $\|z\| \le \delta$, where δ_- and δ are sufficiently small positive constants.

Next, relaxing a relative degree zero assumption to an arbitrary relative degree r is standard using Lyapunov backstepping of the relative degree zero control law through m chains of integrators [4]. Strictly speaking, standard backstepping requires the knowledge of a Lyapunov function for the relative degree zero subsystem, and such a construction is not provided in the present note. Nevertheless, it is not difficult to show that the backstepping procedure can be accommodated with the positive (semidefinite) functions U, V, and W, that were used to construct the relative degree zero control law in order to construct a smooth gobally stabilizing control law in higher relative degree situations.

The results of the present paper thus extend previous results in the literature on global stabilization of relative degree r partially linear cascades, which did not allow for (finite) zeros in the open right-half plane. The particular case of all zeros in the closed left half-plane with possibly repeated zeros on the imaginary axis is also of interest: previous results do not require local exponential stability of the z-subsystem but the unstable states of the ξ-subsystem are excluded from the interconnection [5] because of the slow peaking phenomenon [6]. Theorem 1 yields different conditions. The unstable states are no longer excluded from the interconnection but the z-subsystem must be locally exponentially stable and the interconnection must be at least quadratic in z near $z = 0$ [the inequality (5) is then satisfied for any values of α, p, and q, because ν can always be selected in the interval $(0, \alpha q/p)$]. Thus, a form of local input-to-state stability property for the nonlinear subsystem is sufficient to overcome the slow peaking phenomenon in the ξ-subsystem.

APPENDIX
PROOF OF LEMMA 2

Take one of the closed right half-plane eigenvalues of A_i, and let $\nu' \le \nu$ denote its real part. Then, a coordinate transformation $\tilde{\xi} = T_i \xi$ exists such that

$$T_i^{-1} A_i T_i = \begin{bmatrix} A_{11} & A_{12} \\ 0 & J \end{bmatrix} \quad T_i^{-1} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where either $J = \nu'$ or

$$J = \begin{bmatrix} \nu' & w \\ -w & \nu' \end{bmatrix}. \tag{40}$$

With the choice

$$P_i = (T_i^T)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & k_i I \end{bmatrix} T_i^T$$

where $k_i > 0$ is to be specified, we obtain

$$T_i^{-1}(A_i - BB^T P_i)T_i = \begin{bmatrix} A_{11} & A_{12} - k_i B_1 B_2^T \\ 0 & J - k_i B_2 B_2^T \end{bmatrix}. \tag{41}$$

To prove the first part of the Lemma, we show that k_i can be selected such that

$$J - k_i B_2 B_2^T$$

has at least one eigenvalue in the open left half-plane. If J is scalar then $J - k_i B_2 B_2^T$ is negative for large enough k_i. If J is as in (40), we rewrite the matrix $J - k_i B_2 B_2^T$ as

$$J = \begin{bmatrix} \nu' & w \\ -w & \nu' \end{bmatrix} - k_i \begin{bmatrix} b_1^T b_1 & b_1^T b_2 \\ b_1^T b_2 & b_2^T b_2 \end{bmatrix} \tag{42}$$

where b_1 and b_2 are the transposes of the rows of B_2. Thus, the sum of the two eigenvalues of (42) is $2\nu' - k_i(b_1^T b_1 + b_2^T b_2)$ which is negative for large enough k_i. This means that at least one eigenvalue moves to the open left-half plane by selecting k_i large.

To prove the second part of the Lemma we rewrite (16) in the $[\tilde{\xi}_1^T \;\; \tilde{\xi}_2^T]^T = T_i \xi$ coordinates

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= A_{11}\tilde{\xi}_1 + [A_{12} - k_i \beta_{\epsilon_i}(W) B_1 B_2^T]\tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 &= [J - k_i \beta_{\epsilon_i}(W) B_2 B_2^T]\tilde{\xi}_2. \end{aligned} \tag{43}$$

Because $Y = \tilde{\xi}_2^T \tilde{\xi}_2$ satisfies

$$\begin{aligned} \frac{d}{dt} Y &= \tilde{\xi}_2^T (J + J^T - 2k_i \beta_{\epsilon_i}(W) B_2 B_2^T)\tilde{\xi}_2 \\ &\le \tilde{\xi}_2^T (J + J^T)\tilde{\xi}_2 \le 2\nu Y \end{aligned} \tag{44}$$

and, hence, $\|\tilde{\xi}_2(t)\| \le e^{\nu t}\|\tilde{\xi}_2(0)\|$. Because $\max Re\,\lambda(A_{11}) \le \nu$, for any $\overline{\nu} > \nu$, the $\tilde{\xi}_1$ subsystem driven by $\tilde{\xi}_2$ satisfies $\|\tilde{\xi}_1(t)\| \le \sigma_i e^{\overline{\nu} t}\|(\tilde{\xi}_1(0), \tilde{\xi}_2(0))\|$ for some $\sigma_i > 0$. □

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Adaptive Observer for Multiple-Input–Multiple-Output (MIMO) Linear Time-Varying Systems

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***Abstract*—For *joint state-parameter estimation* in linear time-varying (LTV) multiple-input–multiple-output (MIMO) systems, a new approach to the design of adaptive observers is proposed in this note. It is conceptually simple and computationally efficient. Its *global exponential convergence* is established for noise-free systems. In the presence of noises, it is proved that the estimation errors are bounded and converge in the mean to zero if the noises are bounded and have zero means. Potential applications are fault detection and isolation, and adaptive control.**

***Index Terms*—Adaptive observer, continuous-time system, linear time-varying (LTV) system, multiple-input–multiple-output (MIMO), state and parameter estimation.**

I. INTRODUCTION

In this note, we mainly consider linear time-varying (LTV) multiple-input–multiple-output (MIMO) state-space systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \Psi(t)\theta \tag{1a}$$
$$y(t) = C(t)x(t) \tag{1b}$$

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