Lecture I: From absolute stability to IQCs

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Course website: http://www.montefiore.ulg.ac.be/systems/grad04/iqc.htm

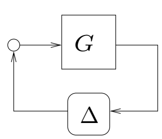
Contents/objectives of the course

- Analysis of feedback systems: closed systems viewed as feedback interconnections of open systems
- Concepts that are central to system theory: dissipativity, passivity, KYP lemma, quadratic forms
- Methods that combine mathematical analysis with efficient numerical tools

This lecture: an informal preview

The general methodology

The feedback interconnection is made of

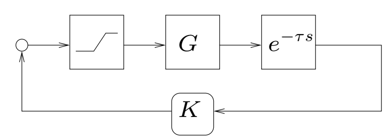


G: the "friendly" part (LTI)

Δ: the "trouble-making" part (nonlinear, time-varying, ∞-dim, uncertain, . . .)

Goal: provide an external characterization of Δ that results in stability criteria ich maximize *tractability* but minimize *conservativeness*.

Example 1

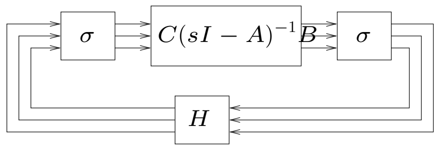


A linear system with a saturation and a delay.

Such feedback loops are frequent in automatic control

Example 2: a neural network

$$\dot{x} = Ax + B\text{sat}(H\text{sat}(Cx))$$

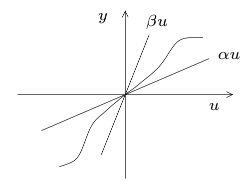


MIMO interconnection of a linear system with the static nonlinearity

$$\Phi(\cdot) = \text{sat}(H\text{sat}(\cdot))$$

Example 3: the Lure problem

$G(s) = c(sI - A)^{-1}b$ is a (SISO) transfer function and Δ is a (SISO) static sector nonlinearity $\phi(\cdot, t)$: $\alpha s^2 \leq \phi(s, t)s \leq \beta s^2$



Notation: $\phi(\cdot, t) \in$ Sector (α, β).

The Lure problem has a very concrete motivation (a basic feedback loop in automatic control). This problem motivated central developments of system theory. It will be the running example of this lecture.

The classical approach

Stability is studied through Lyapunov analysis of the *closed system*

$$\dot{x} = Ax - b\phi(cx)$$

rch for a Lyapunov function : $V(x) > 0$ such that $\dot{V}(x) \leq 0$

Example: $V(x) = x^T P x$ results in $\dot{V} = x^T(PA + A^T P)x - 2x^T Pb\phi(cx)$ cause $\phi(s)s \geq 0$, $\dot{V} \leq 0$ if

$$\begin{aligned} PA + A^T P &\leq 0 \\ Pb &= c^T \end{aligned}$$

Another example: If $\phi(\cdot)$ is time-invariant, one could try the "Lure-type" punov function

$$V(x) = x^T P x + \int_0^{cx} \phi(s)ds$$

Lure problem is hard

- Aizerman conjecture (1946): stability for $k \in (\alpha, \beta)$ is sufficient. (disproved by Pliss (1952))
- Kalman conjecture (1957): Aizerman + $\phi'(s) \in (\alpha, \beta)$ is sufficient. (disproved by Barabanov (1988))
- recent complexity results . . .

+ links to famous conjectures: Markus-Yamabe (1960), Jacobian conjecture (open, 1939)

What this course is about:

An external characterization of G and Δ that will provide a Lyapunov function
the feedback interconnection:

i/o approach: operator theoretic properties ($\Rightarrow$ frequency domain conditions)
for G and Δ (lecture 2)

state-space approach: dissipativity (lecture 3)

The link between the two approaches is the KYP lemma (lecture 4). The
C approach is a unifying treatment that makes the external characterization
merically tractable (lecture 5) and converts the search for a Lyapunov function
o a convex optimization problem (lecture 6).

Outline of lecture 1:

- Part I: Introduction, motivation, examples
- Part II: External characterization for stability analysis of feedback systems: a preview on the Lure problem
- Part III: What makes the problem numerically tractable

Caveat

Precise definitions and mathematical proofs are postponed until lecture 2. In
s lecture, the symbol $'\equiv'$ means that the equivalence will be clarified later and
sibly requires technical conditions.

Dissipativity

Energy stored in an open system (measured by storage $S(x) \geq 0$) can increase only through the external supply:

$$\dot{S} \leq w(u, y)$$

- A central concept for this course (will be the topic of lecture 3)
- The storage function for an open system is analog to the Lyapunov function for a closed system
- Passivity: dissipativity with the supply $u^T y$

Passivity of a linear system

For a linear system, the storage function will always be a quadratic function of the state: $S(x) = x^T P x$, $P = P^T \geq 0$. (an important result !)

(A, B, C) passive $' \equiv' \exists P = P^T \geq 0$:

$$\begin{aligned} PA + A^T P &\leq 0 \\ PB &= C^T \end{aligned}$$

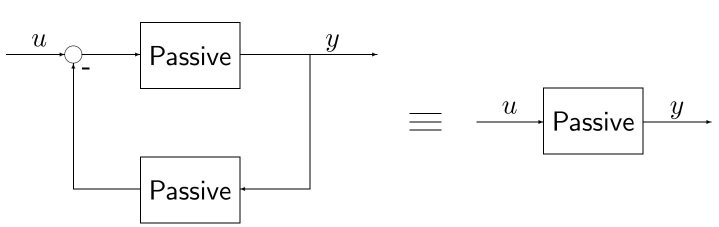
Passivity of a sector nonlinearity

$S = 0$ for a memoryless system.

Passivity reduces to the condition

$$0 \leq uy = u\phi(u, t)$$

Passivity and feedback interconnections



The passivity theorem: Passivity is preserved under (negative) feedback erconnections

Proof: Check the dissipation inequality with $S = S_1 + S_2$

A sufficient condition for the Lure problem

Our try of $V(x) = x^T P x$ as a Lyapunov function for the closed system

$$\dot{x} = Ax - b\phi(cx)$$

is now interpreted as

1) Characterizing $\phi(\cdot, t)$ as a (memoryless) *passive* operator with storage $S_2 = 0$

$$+$$

2) requiring a *passivity* property for the linear system (A, b, c) with storage $S_1 = x^T P x$

Note: $S_1 > 0$ will require a detectability property (see lecture 3).

I/O characterization of passivity (lecture 3)

The characterization of passivity does not require a state-space representation:

Consider the integral form of the dissipativity inequality:

$$S(x(T)) - S(x(0)) \leq \int_0^T u(t)y(t)dt$$

Assuming $x(0) = 0$, this gives: $\int_0^T u(t)y(t)dt \geq 0 \forall T \geq 0$

The integral defines an inner product in a suitable (Hilbert) signal space $\mathcal{H}$. e system (with zero i.c.) defines an input-output operator G. The dissipativity quality becomes a positivity condition for G:

$$\langle u, Gu \rangle_{\mathcal{H}} \geq 0 \quad \forall u \in \mathcal{H}$$

Frequency-domain characterization of passivity

For finite-dimensional LTI causal systems, the i/o characterization translates into a frequency domain condition for the transfer function $G(s)$:

Positivity of the operator becomes positive realness of $G(j\omega)$. For a SISO sytem, this means

$$G(j\omega) + G^*(j\omega) \geq 0 \forall \omega$$

(hint: inner product in $\mathcal{L}_2$ expressed in harmonic basis)

Kalman Yakubovich Popov Lemma

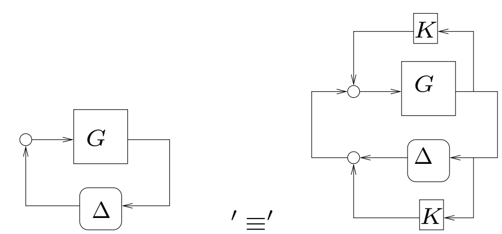
The frequency-domain characterization of passivity is $' \equiv '$ to the state-space racterization of passivity

A (the ?!) fundamental connection between state-space and i/o description of amical systems.

This will be the topic of lecture 4.

Note: a third $' \equiv '$ characterization will be in terms of an optimal control blem.

Loop transformations

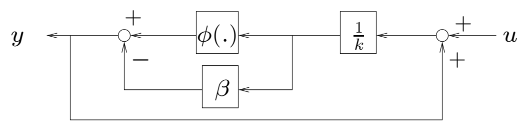


The loop transformation serves to compensate for the shortage of passivity in one channel with the excess of passivity in the other channel.

$\Rightarrow$ broadens the application of the passivity theorem!

Example: the circle criterion

$\phi \in$ sector (α, β) implies passivity of $\frac{\phi-\beta}{\phi-\alpha}$

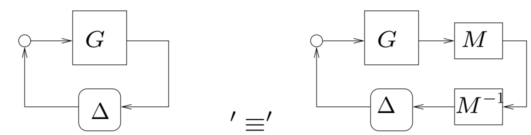


(with $k = \beta - \alpha$)

Consequence for the Lure problem: stability if $\phi \in$ sector (α, β) and $\dfrac{\beta H + 1}{\alpha H + 1}$ sive

Note: we will discuss the frequency interpretation of this condition in lecture 2.

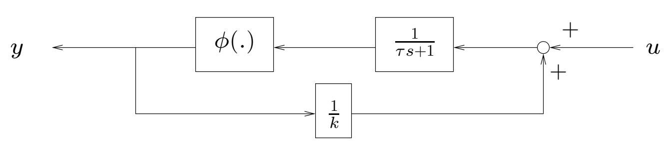
Mutlipliers



Same idea as loop transformations: broadens the application of the passivity theorem

Example: the Popov multiplier

$\phi(\cdot) \in [0, k]$ implies passivity of



State-space $\tau\dot{\zeta} = -\zeta + u + \frac{1}{k}\phi(\zeta)$ Storage: $S_2(\zeta) = \int_0^{\zeta} \phi(s)ds$

Consequence for the Lure problem: stability if $\phi \in$ sector $(0, \beta)$ and $G(s)(\tau s + 1)$ passive

Note: Lyapunov function will be of "Lure type": $S_1 + S_2 = x^T P x + \int_0^y \phi(s)ds$

Summary

- Passivity (a special form of dissipativity) is fundamental to relate external characterization of systems to stability of feedback interconnections
- Conservatism of stability conditions is reduced by refining the external characterization of Δ
- Lure problem: excess of passivity of $\varphi(\cdot) \in$ sector (α, β) characterized through loop transformations (circle criterion) and multipliers (Popov criterion)

Outline of lecture 1:

Part I: Introduction, motivation, examples

Part II: External characterization for stability analysis of feedback systems: a preview on the Lure problem

Part III: What makes the problem numerically tractable: quadratic forms, LMIs, S procedure, and IQCs

LMI characterization of passivity

Passivity of (A, B, C) means the existence of $P = P^T \geq 0$ such that

$$x^T(PA + A^TP)x + 2x^TPBu \leq x^TC^Tu$$

or equivalently

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0$$

Finding $P = P^T$ such that

$$\begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & 0 \end{bmatrix} \leq 0$$

is a Linear Matrix Inequality (LMI) in the unknown P.

LMIs and passivity

Importance of LMI characterization of passivity: There exists efficient (recent) algorithms (including Matlab toolbox) to solve LMIs

⇒ Quadratic storage functions for linear systems are found by numerical methods

⇒ Thanks to KYP lemma, frequency-domain positivity conditions can be converted into LMIs as well !

Quadratic forms for external characterizations

We have just seen that the passivity characterization of the "friendly" part (G) can be expressed as a LMI.

What about the "trouble-making" part Δ ?

Search for quadratic constraints on Δ !

Quadratic characterization of a sector condition

$u = \varphi(y)$, $y = Cx$, and $\alpha s^2 \le \phi(s)s \le \beta s^2$

can be expressed as

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} -2\alpha\beta C^T C & (\alpha+\beta)C^T \\ (\alpha+\beta)C & -2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \ge 0$$

The circle criterion revisited

$V(x) = x^T P x$ will be a Lyapunov function for the feedback system if

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \le 0$$

whenever

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} -2\alpha\beta C^T C & (\alpha+\beta)C^T \\ (\alpha+\beta)C & -2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \ge 0$$

Geometric interpretation: the passivity LMI for G must hold only in a cone characterized by the sector condition.

S-procedure (lecture 5)

A powerful 'trick' to incorporate conic relations into an LMI problem.

Example: Finding $S = S^T > 0$ such that $z^T S z \le 0$ whenever $z^T Q z \ge 0$ is verted into finding $S = S^T > 0$ and $\tau \ge 0$ such that

$$\tau Q + S \le 0$$

s is a new LMI in P and τ!

The circle criterion LMI

Find $P = P^T > 0$ and $\tau \ge 0$ such that

$$\begin{bmatrix} A^T P + PA - 2\tau\alpha\beta C^T C & PB + \tau(\alpha+\beta)C^T \\ B^T P + \tau(\alpha+\beta)C & -2\tau \end{bmatrix} \le 0$$

$\Rightarrow$ the loop transformation is incorporated in the passivity LMI thanks to the characterization of Δ through a quadratic positivity condition $\Rightarrow$ conservatism is reduced.

The passivity IQC

In a Hilbert space, the passivity condition

$$\langle u, y \rangle_{\mathcal{H}} \geq 0 \ \ \forall \, u, y \in \mathcal{H}$$

rites as a quadratic positivity condition

$$\left\langle \begin{bmatrix} u \\ y \end{bmatrix}, \Pi \begin{bmatrix} u \\ y \end{bmatrix} \right\rangle_{\mathcal{H}} \geq 0 \ \ \forall \, u, y \in \mathcal{H}$$

h $\Pi = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

This is an example of Integral Quadratic Constraint (IQC). Π is the multiplier t defines the IQC.

The IQC approach (lecture 5)

Search for external characterizations in the form of Integral Quadratic Constraints

$$\left\langle \begin{bmatrix} u \\ y \end{bmatrix}, \Pi \begin{bmatrix} u \\ y \end{bmatrix} \right\rangle_{\mathcal{H}} \geq 0 \, \forall u, y \in \mathcal{H}$$

The characterization of Δ by means of one or several IQCs is quite general! (Π will be allowed to be any bounded and self-adjoint operator).

Through the KYP lemma, each IQC verified by Δ will provide an additional conic relation for the LMI to be satisfied by $G = (A, B, C)$ and therefore reduce the conservatism of the stability criterion.

Summary of lecture

The course is about external characterizations of open systems in view of internal analysis of closed systems

External characterization is based on *dissipativity* conditions (state-space, lecture 3) or *positivity* conditions (i/o, lecture 2).

The link between dissipativity and positivity is made through the KYP lemma.(lecture 4)

Quadratic conditions can be converted into LMIs, which makes the problem computationally tractable (lecture 5 and 6).