Systems as operators

Signals as operators

Contents

1. Systems as operators
2. Feedback interconnections of operators
3. Stability analysis of feedback systems

Example: $T$ with inner product $\langle T \rangle = \langle f, g \rangle$

Additional requirements: $H^\infty$ for passivity

$\{ \infty > \| f \|_{L^p(0,1)} \}$

Minimal requirements: $f \in H^1$ normed linear space

Examples:

$L^p(0,1)$

$L^1(0,1)$

Additional requirements: $f \in H^2$ Hilbert space

Example: $L^2(0,1)$ with inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$
Operators

Operators are maps $H$: $L \rightarrow L$ that can be composed and linearly combined. We assume $H(0) = 0$.

Essential properties:

- **Linearity:**
  
  $$H(f + g) = H(f) + H(g)$$
  
  **Finite (linear) gain**
  
  $$k \cdot H(f)$$

- **Causality:**
  
  'Future' of $H(f)$ only depends on 'past' of $f$.

The finite gain property is also called boundedness of the operator (which is different from BIBO!). The 'best' bound

$$\|H\| = \sup_{\|f\| = 1} \|H(f)\|$$

is called the operator gain.

Note:

$$\|H_1 \cdot H_2\| \leq \|H_1\| \cdot \|H_2\|$$

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Linear operators associated to LTI differential systems

Each transfer function $H(s) = \frac{C}{sI - A} - \frac{1}{B + D}$ which has no poles on the imaginary axis ($\forall \omega \in \mathbb{R}$) defines a bounded linear operator on $L^p(-\infty, \infty)$. The impulse response $h(t)$ is the inverse Laplace transform of $H(s)$. The convolution operator

$$(h * g)(t) = \int_{-\infty}^{\infty} h(\tau)g(t-\tau) \, d\tau$$

is an example of the convolution operator and $L^p(-\infty, \infty)$ gain is bounded by

$$\|H\| = \sup_{\|f\| = 1} \|H(f)\|$$

for any $p \in [1, \infty]$. Also, $H$ is bounded by

$$\|H\| = \sup_{\|f\| = 1} \|H(f)\|$$

IQC lecture 25

Gain of operators associated to LTI differential systems

The convolution of the impulse response $h(t)$ with a time-varying bounded gain $g(t)$, where $g(t)$ is the impulse response of the inverse Laplace transform of $H(s)$, is a bounded operator on $L^p(-\infty, \infty)

\int_{-\infty}^{\infty} h(t)g(t) \, dt$$

Other examples of bounded operators

- Sector nonlinearity (the sector needs to be linearly bounded).
- Multiplication by a time-varying bounded gain.
- A sector nonlinearity (the sector needs to be linearly bounded).

IQC lecture 26

Linear operators associated to LTI differential systems

Operators

Operators are maps that can be composed and linearly combined.
The operator is bounded if all the poles are in the (open) left half plane.

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Well-posedness

1. Controllability and observability
2. Passivity and the passivity theorem
3. Relationship to Nyquist criterion
4. Loop transformations and multipliers

But it should be noted that there are still situations where these well-posed situations will be excluded by assumption. Hence, it is important to carefully consider the assumptions.

well-posedness of feedback systems

Feedback interconnections of operators

Systems as operators

Stability analysis of feedback systems
Proof of the small gain theorem

Consider the truncated signals

\[ e_{1T} = u_{1T} - P_T H_2(e_{1T}) \]
\[ e_{2T} = u_{2T} + P_T H_1(e_{1T}) \]

and take norms to show

\[ \| e_{1T} \| \leq 1 - \| H_1 \| \cdot \| H_2 \| \]
\[ \| e_{2T} \| \leq 1 - \| H_1 \| \cdot \| H_2 \| \]

Small gain theorem:

The feedback system \((FS)\) is stable if \(k < 1\) and \(kH_1 < 1\) if \(H_2\) is a contraction.

IQC lecture 216

Passivity for LTI differential systems

The causal bounded operator \(H\) with transfer function \(H(s)\) is passive if \(H(s)\) is positive real, i.e.

\[ \exists A \quad 0 \leq A(\omega) + (H(s) + H^H(\omega)) A(\omega) \leq A \]

Proof:

(i) \(\langle H(u,v), \gamma \rangle = \langle (u, H^H(v)), \gamma \rangle \) where \(H^H\) is the causal operator with transfer function

\[ H^H(s) = \frac{1}{s} \left( H(s) + H^H(\omega) \right) \]

(ii) \(\langle H(u,v), \gamma \rangle = \langle H(u), \gamma \rangle - \langle H^H(\omega)(v), \gamma \rangle \)

IQC lecture 217

Passive operators

A causal operator \(H: \mathcal{H}_c \rightarrow \mathcal{H}_c\) is passive if

\[ \langle H(u,v), \gamma \rangle \leq 0 \quad \forall (u,v) \in \mathcal{H}_c \times \mathcal{H}_c \]

It is called output strictly passive if

\[ \langle H(u,v), \gamma \rangle \leq 0 \quad \forall (u,v) \in \mathcal{H}_c \times \mathcal{H}_c \]

IQC lecture 218

The small gain theorem: the feedback system \((FS)\) is stable if

\[ \| H_1 \| \cdot \| H_2 \| < 1 \]

IQC lecture 219
The passivity theorem

Consider the feedback system \((FS)\) with \(H_1\) and \(H_2\) defined on \(H\) and \(\mathcal{H}\) respectively.

Passivity theorem: If \(H_1\) is output strictly passive and \(H_2\) is passive, then the feedback system is stable, that is, the operator \(H = (I + KH_1)^{-1}H_2\) is a well-defined causal operator on \(\mathcal{H}\). The loop transformation is well posed if \(H\) in one channel (resp. contraction) in one channel with the excess of passivity (resp. contraction) in the other channel.

The loop transformation serves to compensate for the shortage of passivity.

Loop transformations

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Note: \(H_1\) does not need to be bounded for the (loop transformed) feedback system to satisfy the assumptions of small gain or passivity theorem.

Note: the link between passivity and positive-realness will be extended to transfer functions with poles on the imaginary axis in Lecture 3.

Consider the feedback system \((S,F)\) with \(S\) and \(F\) defined on \(H\) and \(\mathcal{H}\) respectively.

The passivity theorem

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The loop transformation

\[ \begin{bmatrix} Y \\ \dot{H} \end{bmatrix} = \begin{bmatrix} \hat{H} & I \\ 0 & \hat{H} \end{bmatrix} \begin{bmatrix} X \\ \dot{Y} \end{bmatrix} \]
Loop transformations, multipliers, and IQCs

A link between passivity and contraction

If \((I + H)(I - H) = S\) is equivalent to contraction of \(H\), then \(H + I\) is invertible on \(H\).

System equivalence if \(H\) and its inverse \(H^{-1}\) are bounded causal operators. Small gain theorems.

Same idea as loop transformations: broadens the application of the IQC framework incorporating loop transformations and multipliers.

The invertibility assumption on \(K\) and the factorization condition on \(M\) are replaced by a (usually easily satisfied) well-posedness assumption on the feedback loop.

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The invertibility assumption on \(K\) and the factorization condition on \(M\) are replaced by a (usually easily satisfied) well-posedness assumption on the feedback loop.

Example: a generalization of the conformal mapping between the right half plane and the unit circle.

If a feedback interconnection of operators do not always define operators.

Causality of the operators is an important issue in stability analysis of interconnections.

Passivity and contraction are fundamental properties for stability of interconnections.

Summary of lecture: 

- Loop transformations, multipliers, and IQCs

- System equivalence if \(H\) and its inverse \(H^{-1}\) are bounded causal operators.

- Small gain theorems.

- Same idea as loop transformations: broadens the application of the IQC framework incorporating loop transformations and multipliers.

- The invertibility assumption on \(K\) and the factorization condition on \(M\) are replaced by a (usually easily satisfied) well-posedness assumption on the feedback loop.

- Causality of the multipliers can be relaxed if \(M\) can be factored into \(M = M_+ M_-\).

- Causality of the operators is an important issue in stability analysis of interconnections.

- Passivity and contraction are fundamental properties for stability of interconnections.

- Feedback interconnections of operators do not always define operators.

- Treating systems as operators provides a strong generalization of Nyquist criterion.