Lecture 2:
eedback systems as interconnections of operators

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Course website: http://www.montefiore.ulg.ac.be/systems/grad04/iqc.htm

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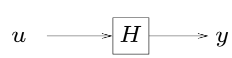
Systems as operators



H 'maps' the signal u to the signal y

H defines an operator from a signal space to a signal space.

? Does it work for a LTI differential system ?

? Does it work for more general situations (nonlinear, time-varying, . . .)?

Signal spaces

This course: enough to view a signal f as a function from $I\!R$ (or $[0,\infty)$) to $I\!R^p$.

Minimal requirements: $f \in \mathcal{L}$ normed linear space
(linear combinations of signals are signals and signals can be 'measured' by a norm)

Examples: $L_p[0,\infty) = \{f \mid \; \| f \| = (\int_0^\infty |f(t)|^p)^{1/p} dt < \infty\}, p = 1, 2, \ldots$
$L_\infty[0,\infty) = \{f \mid \; \| f \| = \sup_{t\geq 0} |f(t)| < \infty\}$

Additional requirements (e.g. for passivity): $f \in \mathcal{H}$ Hilbert space
(norm derives from an inner product + completeness)

Example: $L_2[0,\infty)$, with inner product $< f, g >= \int_0^\infty f(t)^T g(t) dt$

Operators

Operators are maps $H : \mathcal{L} \to \mathcal{L}$ that can be composed and linearly combined. assume $H(0) = 0$.

Essential properties:

linearity: $H(\alpha f + \beta g) = \alpha H(f) + \beta H(g)$

finite (linear) gain γ if $\| H(f) \| \leq \gamma \| f \| \quad \forall f \in \mathcal{L}$

causality: 'future' of $H(f)$ only depends on 'past' of f.

The finite gain property is also called boundedness of the operator (which is erent from BIBO!). The 'best' bound $\gamma = \gamma(H)$ is called the operator gain. te: $\gamma(H_1 H_2) \leq \gamma(H_1)\gamma(H_2)$.

Linear operators associated to LTI differential systems

• Each transfer function $H(s) = C(sI - A)^{-1}B + D$ which has no poles on the imaginary axis ($H \in \mathbf{RL}_\infty^{m\times m}$) defines a bounded linear operator on $L_p(-\infty, \infty)$

$$Hu = h * u = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

where $h(t)$ is the *impulse response*, that is, the inverse Laplace transform of $H(s)$ (restricted to the domain that contains the imaginary axis)

• The operator is causal iff $H(s)$ has no poles in the right-half plane. In this case, $h(t) = 0 \ \ \forall t \leq 0$ and the operator is (also) defined on $L_p[0, \infty)$

ains of operators associated to LTI differential systems

The convolution operator

$$Hu = h * u = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

linear bounded operator on $L_p(-\infty, \infty)$ provided $h \in L_1(-\infty, \infty)$.

For any $p = 1, 2, \ldots$, its L_p gain is bounded by $\| h \|_1$.

Its L_2 gain is $\gamma_2 = \max_{\omega\in I\!R} \sigma_{\max}(H(j\omega)) (= \| H \|_{H_\infty})$.

Its L_∞ gain is $\gamma_\infty = \| h \|_1$.

oof: (1) see lecture notes (based on Hölder inequality); (2,3): homework 1!

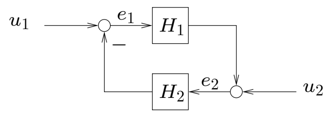
Other examples of bounded operators

- a sector nonlinearity (the sector needs to be linearly bounded);
- multiplication by a time-varying bounded gain.
- a (nonlinear) dissipative system with supply $\gamma|u|^2 - |y|^2$ and zero initial condition:

$$\int_0^t |y(t)|^2 dt \leq \gamma \int_0^t |u(t)|^2 dt - S(x(t))$$

(not much beyond that ...)

Feedback interconnections of operators



A reasonable notion of stability: existence of a bounded (finite gain) causal erator

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to^{\gamma} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$\approx$ BIBO stability for linear systems

Problem

Requiring *causality* and *boundedness* of H_1 and H_2 seems

- too strong: the theory should include unstable open-loop systems
- too weak: does not guarantee the existence of a causal operator $u \to e$

Extended signal spaces $\mathcal{L}_e$

$f \in \mathcal{L}_e$ if the truncated signal $P_T f \in \mathcal{L}$ for all $T \geq 0$.

$$\begin{aligned} P_T f(t) &= f(t), \quad t \leq T \\ &= 0, \quad\quad t > T \end{aligned}$$

a causal operator H has finite gain (i.e. is bounded) on $\mathcal{L}_e$ if
$\| H \| = \sup_{f \in \mathcal{L} \setminus \{0\}} \frac{\|Hf\|}{\|f\|}$ is finite.

H is causal and bounded on $\mathcal{L}_e$ iff H is causal and bounded on $\mathcal{L}$

Note: causality expresses as $P_T H P_T = P_T H$ for all T.

Linear operators associated to LTI differential systems

Each transfer function $H(s) = C(sI - A)^{-1}B + D$ defines a causal linear operator on $L_{pe}[0, \infty)$

$$Hu = h * u = \int_0^\infty h(\tau) u(t - \tau) d\tau$$

where $h(t)$ is the *impulse response*, that is, the inverse Laplace transform of $H(s)$ (restricted to a RHP that contains no poles of H)

Proof: $e^t \in L_{pe}[0, \infty)$!

The operator is bounded if all the poles are in the (open) left half plane.

Well-posedness

and H_2 causal operators on $\mathcal{L}_e \not\Rightarrow \frac{H_1}{1+H_1H_2}$ and $\frac{H_2}{1+H_2H_1}$ causal operators on $\mathcal{L}_e$.

But it should for any physical system!

$\Rightarrow$ these ill-posed situations will be excluded *by assumption*

Back to the feedback interconnection

Assumptions:

- H_1, H_2 causal operators on $\mathcal{L}_e$
- Well-posedness: the feedback interconnection defines a causal operator in $\mathcal{L}_e$:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rightarrow \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

The feedback system (FS) will be said stable if H has a finite gain.

(Note: Uniqueness of the solution (e_1, e_2) for the input (u_1, u_2) can be relaxed)

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Systems as operators

Feedback interconnections of operators

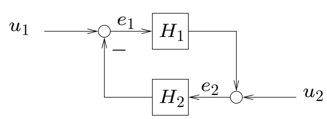
Stability analysis of feedback systems

Stability analysis of feedback systems

1. Contractions and the small gain theorem
2. Passive operators and the passivity theorem
3. Relationship to Nyquist criterion
4. Loop transformations and multipliers

Contractions and the small gain theorem

An operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is a contraction if its gain is smaller than one.



The small gain theorem: the feedback system (FS) is stable if

$$\| H_1 \| . \| H_2 \| < 1$$

Proof of the small gain theorem

Consider the truncated signals

$$e_{1T} = u_{1T} - P_T H_2(e_{2T})$$
$$e_{2T} = u_{2T} + P_T H_1(e_{1T})$$

and take norms to show

$$\| e_{1T} \| \le \frac{1}{1 - \| H_1 \| . \| H_2 \|} \| u_{1T} \| + \frac{\| H_2 \|}{1 - \| H_1 \| . \| H_2 \|} \| u_{2T} \|$$

Passive operators

A causal operator $H : \mathcal{H}_e \to \mathcal{H}_e$ is passive if

$$\langle Hu, u \rangle_T \ge 0 \;\; \forall u \in \mathcal{H}_e, \; \forall T \ge 0$$

It is called output strictly passive if

$$\langle Hu, u \rangle_T \ge \epsilon \, \| P_T H(u) \|^2 \quad \forall u \in \mathcal{H}_e, \; \forall T \ge 0$$

Passivity for LTI differential systems

The causal bounded operator H with transfer function $H(s)$ is passive iff $H(s)$ is positive real, i.e.

$$H(j\omega) + H^*(j\omega) \ge 0 \;\; \forall \omega \in I\!R$$

Proof:
(i) $\langle Hu, u \rangle_T = \langle u, H^* u \rangle_T$ where H^* is the causal operator with transfer function $H^T(-s)$

(ii) $\langle Hu, u \rangle_T = \frac{1}{2} \langle (H + H^*)u, u \rangle_T = \frac{1}{2} \langle (H(j\omega) + H^*(j\omega) \hat{u}_T(j\omega), \hat{u}_T(j\omega) \rangle$

Passivity does not imply boundedness

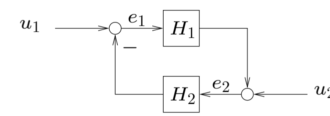
Example: $H(s) = \frac{1}{s}$. Let $U(t) = \int_0^t u(\tau)d\tau$. Then

$$\langle Hu, u\rangle_T = \int_0^T U dU = \frac{1}{2}U(T)^2 \geq 0$$

Note: the link between passivity and positive realness will be extended to nsfer functions with poles on the imaginary axis in lecture 3.

The passivity theorem

Consider the feedback system (FS) with H_1 and H_2 defined on $\mathcal{H}_e$ and $u_2 = 0$.



Passivity theorem: If H_1 is output strictly passive and H_2 is passive, then the feedback system is stable, that is, the operator $u_1 \to e_2$ has finite gain.

Link with the Nyquist criterion

Let $H_1(s)$ and $H_2(s)$ two stable (SISO) transfer functions.

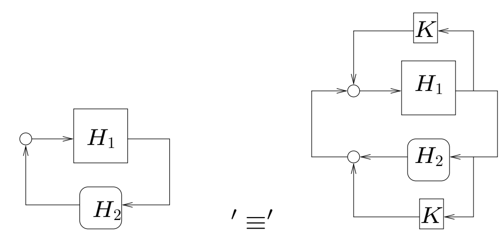
Nyquist criterion requires no root for

$$1 + H_1(j\omega)H_2(j\omega) = 0, \omega \in I\!R$$

Disregarding the phase information, a sufficient condition is $(j\omega)H_2(j\omega)| < 1 \; \forall\omega$ ($\Leftarrow$ small-gain theorem)

Disregarding the amplitude information, a sufficient condition is $H_1(j\omega)H_2(j\omega)) = \angle H_1(j\omega) + \angle H_2(j\omega) < 180 \deg \; \forall\omega$ ($\Leftarrow$ passivity theorem)

Loop transformations

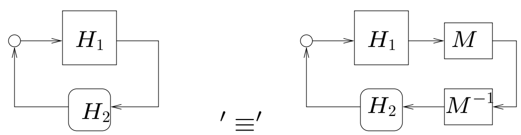


The loop transformation serves to compensate for the shortage of passivity (resp. contraction) in one channel with the excess of passivity (resp. contraction) in the other channel.

The loop transformation is well posed if $(I + KH_1)^{-1}H_1$ is a well defined causal operator on $\mathcal{H}_e$.

Note: H_1 does not need to be bounded for the (loop transformed) feedback system to satisfy the assumptions of small gain or passivity theorem.

Mutlipliers



Same idea as loop transformations: broadens the application of the
sivity/small gain theorems.

Equivalence if M and its inverse M^{-1} are bounded causal operators.

Loop transformations, mutlipliers, and IQCs

- Causality of the multiplier can be relaxed if M can be factored into $M = M_- M_+$ with M_-^*, M^+ and their inverses causal and bounded. The use of noncausal multipliers is important in practice. (This is a result of Zames).
- The IQC framework incorporates loop transformations and multipliers and simplifies the (tricky) question of equivalence of feedback loops.
- The invertibility assumption on K and the factorization condition on M are replaced by a (usually easily satisfied) well-posedness assumption on the feedback system.

A link between passivity and contraction

If $I + H$ is invertible on $\mathcal{H}$, then

passivity of H is equivalent to contraction of $S = (H - I)(H + I)^{-1}$

This is a generalization of the conformal mapping $\frac{s-1}{s+1}$ between the right half
ne and the unit circle.

Summary of lecture

- Treating systems as operators provide a strong generalization of Nyquist criterion
- Feedback interconnections of operators do not always define operators
- causality of the operators is an important issue in stability analysis of interconnections
- passivity and contraction are fundamental properties for stability of interconnections