KUNGL TEKNISKA HÖGSKOLAN

Matematiska institutionen
Avdelningen för Optimeringslära och systemteori

Homework 3, Spring 2004

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1. Let $\Delta : \mathbf{L}_2^m[0,\infty) \to \mathbf{L}_2^m[0,\infty)$ be the operator defined by multiplication by $\cos(\omega_0 t)$ in the time domain, i.e. $(\Delta(v))(t) = \cos(\omega_0 t)v(t)$. Show that Δ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} \frac{1}{2}(X(j(\omega+\omega_0)) + X(j(\omega-\omega_0))) & 0 \\ 0 & -X(j\omega) \end{bmatrix}$$

where $X(j\omega) = X(j\omega)^* \geq 0$.
Remark: This IQC is due to J. C. Willems.

2. Consider the control system in Figure 1. We assume that we have an ideal saturation nonlinearity

$$\varphi(x) = \begin{cases} x, & |x| \leq 1 \\ \text{sign}(x), & |x| > 1 \end{cases}$$

The plant is $G(s) = \dfrac{1}{s^2 + 0.5s + 1}$ and the controller is $K = 2$. Prove stability for this system by using the IQC theorem and the IQC for the ideal saturation nonlinearity that was derived in the lecture notes.

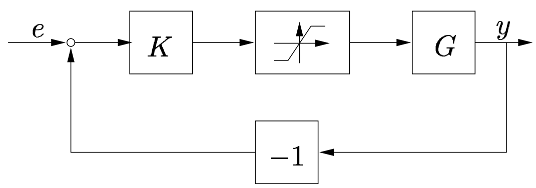

Figure 1: Control system with saturation.

3. Consider the discrete time nonlinear system

$$v_{k+2} + 0.5v_{k+1} + 0.9\varphi(v_k) = 0.9e_k, \quad v_k = 0,\ k \leq 0,$$

where φ is a nonlinear function that satisfies the sector condition $0 \leq \varphi(x)x \leq x^2$. Prove that the system is l_2 stable, i.e. there exists a constant $c > 0$ such that $\sum_{k=0}^{N} |v_k|^2 \leq c \sum_{k=0}^{N} |e_k|^2$ for all $N \geq 0$.

4. Consider two nondegenerate ellipsoids (i.e., there exists interior points)

$$\mathcal{E}_k = \{x \in \mathbf{R}^n : x^T P_k x + 2x^T b_k + c_k \leq 0\}, \quad k = 1, 2.$$

Prove that

$$\mathcal{E}_1 \subset \mathcal{E}_2 \;\Leftrightarrow\; \exists \tau \geq 0 \quad \text{s.t.} \quad \begin{bmatrix} P_2 & b_2 \\ b_2^T & c_2 \end{bmatrix} \leq \tau \begin{bmatrix} P_1 & b_1 \\ b_1^T & c_1 \end{bmatrix}$$

You can now choose to do either problem 5, which is a theoretical excercise or problem 6, which is an easy application with IQCbeta. Pick the one you like the most!!

5. Let $\Delta : \mathbf{L}_2^m[0,\infty) \to \mathbf{L}_2^m[0,\infty)$ be the operator defined by $(\Delta(v))(t) = \delta(t)v(t)$, where[1] $0 \leq \delta(t) \leq U < \infty$ and $\dot{\delta}(t) \leq 0$. In other words, Δ corresponds to multiplication in the time domain by a positive nonincreasing scalar. Show that Δ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & H(j\omega)^* \\ H(j\omega) & 0 \end{bmatrix},$$

where $H(j\omega)^* + H(j\omega) \geq 0$.

6. Consider the control system in Figure 6. Let

$$P(s) = \frac{1}{s^2 + 0.2s + 1}$$

$$K(s) = \frac{2500(s + 0.2}{(s+5)(s+100)}$$

$$W(s) = \frac{0.1}{s+1}$$

The nonlinearity is a saturation

$$\varphi(x) = \begin{cases} x, & |x| \leq 1 \\ \text{sign}(x), & |x| > 1 \end{cases}$$

and $|\Delta\|_{\mathbf{H}_\infty} \leq 1$. Prove stability and compute a bound on the $\mathbf{L}_2$-gain from e to z using IQCbeta.
Hint: Use the IQCs in `iqc_slope` for the nonlinearity and the IQC `iqc_ltiunmod` for the uncertainty.

[1] U is just some positive constant

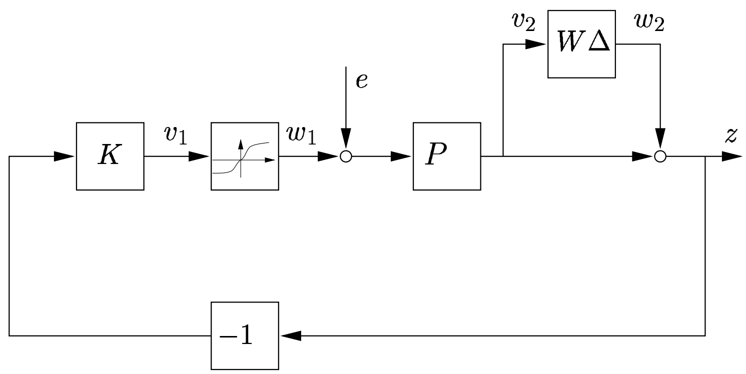

Figure 2: Control system with saturation and uncertainty.