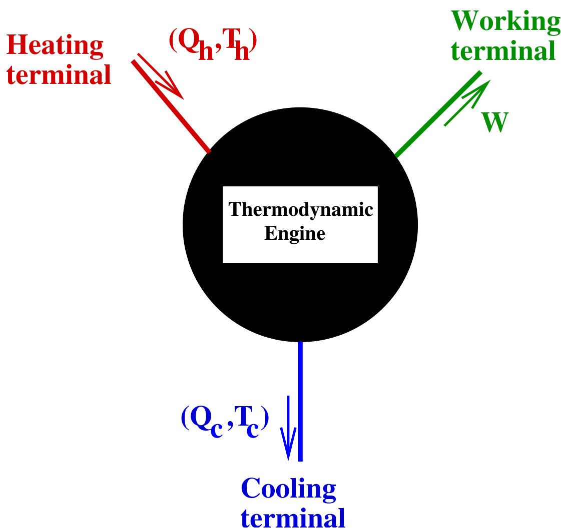
ANALYSIS of FEEDBACK SYSTEMS: THEORY and COMPUTATION

Homework 2

For all exercises, assume time-invariance, connectedness, observability, $s(w) \in \mathcal{L}^{\text{loc}}$.

Exercise 1 (Thermodynamics) Consider a thermodynamic engine, see the figure below.



The engine has 3 terminals, through which it interacts with the outside world:

1. The *heating terminal*, through which it is connected to a source.
   This delivers heat to the engine at a rate Q_h at (absolute) temperature T_h.
2. The *cooling terminal*, through which it is connected to a heat sink.
   This absorbs heat from the engine at a rate Q_c at temperature T_c.
3. The *work terminal*, through which it delivers work at power rate W. This could be in the form of motion, electrical power, whatever the customer demands.

Assume $Q_h \geq 0, Q_c \geq 0$. The positive directions of the heat and work flows are shown on the figure. The 1st and 2nd laws of thermodynamics state:

1. Conservation of energy: The system is conservative w.r.t. $Q_h - Q_c - W$.
2. Second law: The system is dissipative w.r.t. $-\frac{Q_h}{T_h} + \frac{Q_c}{T_c}$.

Assume that the engine cycles through a periodic motion. Call the period T. Define

$$\begin{aligned}
\mathbb{Q}_+ &= \oint Q_h(t)\ dt, \\
\mathbb{Q}_- &= \oint Q_c(t)\ dt, \\
\mathbb{W} &= \oint W(t)\ dt, \\
\mathbb{T}_+ &= \text{supremum}\{T_h(t) \mid t \in [0,T]\}, \\
\mathbb{T}_- &= \text{infimum}\{T_c(t) \mid t \in [0,T]\}.
\end{aligned}$$

1. Assume $\mathbb{W} = 0$. Prove that $\mathbb{T}_+ < \mathbb{T}_-$ implies $\mathbb{Q}_+ = \mathbb{Q}_-$. We cannot transport heat from a cold source to a hot sink. This is unfortunate: we cannot heat our houses by simply cooling the outside air, or, in the summer, cool our houses by warming the outside air.

2. Prove that
$$\mathbb{W} \leq \frac{\mathbb{T}_+ - \mathbb{T}_-}{\mathbb{T}_+}\ \mathbb{Q}_+.$$
Explain this in terms of the efficiency of a thermodynamic engine. Efficient thermodynamic engines require hot sources and cool sinks. The former is limited (materials could start melting), the latter also, since cooling usually uses ambient air or water.

3. Prove that
$$\mathbb{Q}_- \geq \frac{\mathbb{T}_-}{\mathbb{T}_+ - \mathbb{T}_-}\ \mathbb{W}.$$
So, the fact that the environment around power stations is heated is unavoidable, not a matter of inefficiency of the production process.

The second law of thermodynamics is a serious matter.

Exercise 2 (positive realness) For which $A \in \mathbb{R}$ and $\mathtt{k} \in \mathbb{Z}$ is the rational function

$$A\xi^{\mathtt{k}}$$

positive real? Use the very definition of positive realness.

Exercise 3 (Conservative systems) Let Σ be conservative w.r.t. s.

1. State and prove the analogue of the basic theorem.

2. Prove that conservativeness implies
$$V_{\text{available}} = V_{\text{required}}.$$

3. Prove or give a counterexample to the converse implication.

4. Let $\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $s(\mathrm{u}, \mathrm{y}) = \mathrm{u}^\top \mathrm{y}$. State the (LMI) and the conditions on the transfer function such that this system is conservative (explain both the case that the storage function is arbitrary or non-negative)? In the SISO case, what does this mean (both cases) in terms of the location of the poles and the zeros? The condition is quite neat. Don't quit before you have it.

Exercise 4 (LMI) Consider

$$\frac{s+2}{(s+1)(s+3)}.$$

Is this transfer function p.r.? Give a controllable and observable state representation for it. Consider the supply rate $u * y$ and compute all solutions of the relevant (LMI). What are K_+, K_-? Is the set convex and compact?