Optimization of IQCs

Theorem 1. Let $G \in L^2[0,1]$, and let $\varphi$, $\psi$ be bounded, causal on $L^2[0,1]$. Suppose

$(i)$ The system is well-posed
$(ii)$ $2 \in \text{IQC}(\mathcal{M})$, $k = 1, \ldots, N$.

Then the system is stable.

Resulting stability test:

$$\sum_{k=1}^{N} 2^k \in \text{IQC}(\mathcal{M})$$

where $\varphi \in L^2[0,1]$, $\varphi = 1, \ldots, N$. If $\varphi \in \text{IQC}(\mathcal{M})$, then also $\varphi \in \text{IQC}(\mathcal{M})$.

We use convex parameterizations.

There are generally many IQCs.

Outline

1. Optimization of IQCs
2. The IQC Toolbox
   - Algorithms
   - Motivation
3. Optimization of IQCs
Ideas for semi-automatic system analysis

- Use IQCs to characterize all uncertain, nonlinear, and time-varying elements.
- Use as many IQCs as possible.
- Not necessary to explicitly construct the LFT $S(C,\forall)$.
- Use as many IQCs as possible.
- Use IQCs to characterize all uncertain, nonlinear, and time-varying elements.

In analysis of systems with many blocks...

...we replace all by IQC relations...

...which leads to the following stability test:

Find $\alpha$ such that

$$0 > \begin{bmatrix} I & (\gamma_1 \theta_1) \\ (\gamma_1 \theta_1) & I \end{bmatrix} \begin{bmatrix} U(s) & V(s) \\ V(s) & U(s) \end{bmatrix}$$

...where $U(s)$ is affine in $\gamma$ and rational in $\theta_i$, i.e.

$$U(s) = U(0) + \sum_{i} \gamma_i U_i$$

and $\theta_i$ is rational in $s$, i.e.

$$\theta_i(s) = \theta_i(0) + \sum_{k} \gamma_k \theta_k(s)$$

is combination of all IQCs.

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$$\theta_i(s) = \theta_i(0) + \sum_{k} \gamma_k \theta_k(s)$$

is combination of all IQCs.
Example 1.

Consider the IQC with $X = \begin{bmatrix} 1 & \lambda \cr \beta & \beta \end{bmatrix}$ and the Hamiltonian

$$0 > \begin{bmatrix} \gamma & S + d \beta \cr S + d \beta & \beta + V + d \beta \end{bmatrix}$$

there exists $d$ such that $(ii)$

$$[\infty, 0] \ni \lambda \Rightarrow \begin{bmatrix} I & S \cr S & \beta \end{bmatrix} \begin{bmatrix} I \\ \beta - (V - I \beta) \end{bmatrix} > 0$$

following are equivalent

**Lemma 1.** Assume $\lambda \neq 0$ and $(A, B)$ stabilizable. Then the

**KYP Lemma**

$$\{ \gamma, \lambda \mid \gamma, \lambda \} \Rightarrow \{ \gamma, \lambda \mid \gamma, \lambda \}$$

where $\gamma, \lambda$ are given. Use the parameterization

$$0 \leq (\gamma, \lambda) x = \begin{bmatrix} \gamma \cr \lambda \cr \lambda \end{bmatrix} x$$

Consider the IQC with

Example 2.

$$0 \leq \begin{bmatrix} (\gamma, \lambda) x & 0 \\ 0 & (\gamma, \lambda) x \end{bmatrix} = \begin{bmatrix} I \\ (\gamma, \lambda) \end{bmatrix}$$

Then

$$\{ \gamma, \lambda \mid \gamma, \lambda \} \Rightarrow \{ \gamma, \lambda \mid \gamma, \lambda \}$$

If we use the parameterizations where

$$\lambda = \lambda$$

then

$$0 \leq \lambda$$

Consider the IQC with

$$0 \leq \begin{bmatrix} X - \lambda \cr \lambda \cr \lambda \end{bmatrix} = \begin{bmatrix} -X \cr \lambda \cr X \end{bmatrix}$$

Example 1. Consider the IQC with

$$(\gamma, \lambda)(I - \mu) + u = \lambda$$

where

$$\lambda = \lambda$$

and

$$0 \leq \lambda$$

Consider the IQC with

$$0 \leq \begin{bmatrix} X - \lambda \cr \lambda \cr \lambda \end{bmatrix} = \begin{bmatrix} -X \cr \lambda \cr X \end{bmatrix}$$

Example 2.
\[ G(j\omega)I_{375} = 2^{64}(j\omega A) + B^{T}P + S(TR + 375) \]

\[ Q;S;R \text{ are eigenvalues of } (d\omega) \]

\[ \begin{bmatrix} I & (\gamma)R & (\gamma)S \end{bmatrix} \begin{bmatrix} I & (\gamma)R & (\gamma)S \\ d_{1} & (\gamma)S & (\gamma)d + \gamma d + \gamma V \end{bmatrix} \begin{bmatrix} I \end{bmatrix} = 
\]

\[ \begin{bmatrix} I \\ (\gamma!C) & (\gamma!C) & I \end{bmatrix} \]

\[ \text{Use state space realization of } \Pi \text{ and } C \text{ to get } \]

\[ \text{Cutting Plane Algorithms} \]

\[ \text{Fast Algorithms} \]

\[ 2^{64} \rightarrow \text{becomes large, } \text{dim}(d\omega), \text{dim}(\gamma) \text{ increase dimension since } P \]

\[ \text{Computationally demanding if } P \text{ has high dimension. Standard LMI software works well for mid-size problems} \]

\[ 0 > \begin{bmatrix} (\gamma)R & (\gamma)S + d_{1}P \\ (\gamma)S + B^{T}R & (\gamma)d + \gamma d + \gamma V \end{bmatrix} \]

\[ \text{Feasibility test: Find } \gamma \in \mathbb{R} \text{ and } d_{1} \text{ s.t.} \]

\[ \text{Equivalent LMI} \]

\[ \text{Transformation to LMI} \]

\[ \text{An analytic center cutting plane algorithm} \]

\[ \text{Cutting Plane Algorithms} \]

\[ \text{Kelly-type cutting plane algorithm} \]

\[ (a) \text{ C.-Y. Kao. PhD thesis from M.I.T:} \]

\[ \text{http://www.math.kth.se/~/cykao/pub.html} \]

\[ (b) \text{ C.-Y. Kao, A. Megretski and U. Jonsson. A Cutting Plane} \]

\[ \text{Algorithm for Robustness Analysis of Periodic Systems. IEEE} \]

\[ \text{Transactions on Automatic Control, 46(4):759-792, April 2001.} \]

\[ \text{Analytic center cutting plane algorithm} \]

\[ \text{Outer approximation} \]

\[ (a) \text{ C.-Y. Kao. PhD thesis} \]

\[ (b) \text{ C.-Y. Kao, A. Megretski and U. Jonsson. Fast Algorithms for} \]

\[ \text{IQC Feasibility and Optimization Problems. To appear in Automatica.} \]

\[ \text{Fast algorithms to get around the problem with large size } P \text{ matrix:} \]

\[ \text{Interior path following method with barrier function defined by} \]

\[ \text{Exploit structure in the LMI} \]

\[ \text{Special LMI solvers (Hansson, Vandenberghe, Wallin)} \]

\[ \text{Use Hamiltonian matrices to update frequency grid} \]

\[ \text{Optimize in frequency domain over a finite frequency grid} \]

\[ \text{Use Hamiltonian to obtain hyperplane constraints} \]

\[ \text{Cutting plane algorithms (Kao et al.)} \]

\[ \text{Four recent ideas to get around the problem with large size } P \text{ matrix:} \]

\[ \text{Analytic center cutting plane algorithm} \]

\[ \text{Kelly-type cutting plane algorithm} \]

\[ (a) \text{ C.-Y. Kao. PhD thesis from M.I.T:} \]

\[ \text{http://www.math.kth.se/~/cykao/pub.html} \]

\[ (b) \text{ C.-Y. Kao, A. Megretski and U. Jonsson. A Cutting Plane} \]

\[ \text{Algorithm for Robustness Analysis of Periodic Systems. IEEE} \]

\[ \text{Transactions on Automatic Control, 46(4):759-792, April 2001.} \]

\[ \text{Analytic center cutting plane algorithm} \]
Let \( j! \in \mathbb{R} \) and consider \( y \) subject to:

\[
\begin{align*}
\langle y, \mathbf{I} \rangle &\leq 2, \\
\langle y, \mathbf{H} \rangle &> 0.
\end{align*}
\]

If \( y_{\text{opt}} < 0 \) then we have feasibility of IQC.

---

Let \( q(\cdot) = \inf \{ y : \langle y, \mathbf{g} \rangle \leq 0 \} \) be the max spectral value function (convex). Idea: Use polyhedral approximation of \( q(\cdot) \).

Update polyhedral function successively until convergence:

\[
\begin{align*}
\text{pick test point } (y_k; k) \\
\text{if } \langle y_k, \mathbf{I} \rangle < 0 \text{ then generate new hyperplane} \\
\text{else } \text{pick test point } (y_k; k) \\
\text{repeat}
\end{align*}
\]

New upper bound

\[
q(\cdot) = \inf \{ y : \langle y, \mathbf{g} \rangle \leq 0 \}.
\]

Several ways to generate new test point:

1. \( \{ y : \langle y, \mathbf{h} \rangle = 0 \} \) gives the cutting hyperplane

\[
\langle c + \mathbf{A}^T \mathbf{y}, \mathbf{y} \rangle = \langle \mathbf{A}^T \mathbf{y}, \mathbf{y} \rangle - \langle \mathbf{A}^T \mathbf{y}, \mathbf{y} \rangle 
\]

If \( y \in \text{eig}(\mathbf{H}) \) then \( \langle \mathbf{h}, \mathbf{H} \rangle \) has no eigenvalues on the imaginary axis (here \( \mathbf{Q}(\cdot) \) e.t.c.).

\[
\begin{bmatrix}
\mathbf{B} & \mathbf{Y} \\
\mathbf{Y}^T & \mathbf{S}^T - \mathbf{Y}^T \mathbf{Y} \\
\mathbf{S} & \mathbf{H}^T \mathbf{E} - \mathbf{Y}^T \mathbf{Y} \\
\mathbf{E}^T & \mathbf{H} - \mathbf{Y}^T \mathbf{Y} - \mathbf{S} \\
\end{bmatrix} = \mathbf{H}
\]

and \( 0 > (\mathbf{h}, \mathbf{H}) \).\( \mathbf{h} \in \text{eig}(\mathbf{H}) \).

---

If \( y_{\text{opt}} > 0 \) then we have feasibility of IQC.

\[
\begin{bmatrix}
\mathbf{I} \\
\mathbf{H} \\
\end{bmatrix} y_{\text{opt}} \in \text{eig}(\mathbf{H}) \}
\]

\( y \) subject to \( y \in \text{eig}(\mathbf{H}) \).

Let and consider:

\[
\begin{bmatrix}
\mathbf{I} \\
\mathbf{H} \\
\end{bmatrix} y_{\text{opt}} \in \text{eig}(\mathbf{H}) \}
\]

\( y \) subject to \( y \in \text{eig}(\mathbf{H}) \).

---

Kelly-type Cutting Plane Algorithm.
otherwise the oracle generates a cutting hyperplane:

\[ q = \chi L^0 \]

- If the oracle detects that \( \mathcal{J}(\lambda, m) > 0 \) then introduce the cut

- Exists oracle such that

Polyhedral outer bound

\[ \bigl(a^T z, y \bigr) \]

\[ \mathcal{O} \]

**Main Ideas behind the ACCPA**

\[
\begin{bmatrix}
I \\
(m^T \mathcal{J})^T
\end{bmatrix}
\begin{bmatrix}
(m^T \mathcal{J} \lambda) & 0 \\
0 & (m^T \mathcal{J} m + I \gamma) \\
(m^T \mathcal{J} \lambda) & 0 \\
0 & (m^T \mathcal{J} m + I \gamma)
\end{bmatrix}
\begin{bmatrix}
I \\
(m^T \mathcal{J})^T
\end{bmatrix}
= (m^T \mathcal{J}) \lambda
\]

When \( \mathcal{J}(\lambda, m) \) is defined on the next slide

Optimal performance problem

\[
\begin{bmatrix}
\infty \\
0
\end{bmatrix} \in \mathcal{O} \quad \text{subject to} \quad \mathcal{J}(\lambda, m) > 0 \\
m \\
\lambda
\]

\[
\begin{bmatrix}
\infty \\
0
\end{bmatrix} \in \mathcal{O} \quad \text{subject to} \quad \mathcal{J}(\lambda, m) > 0 \\
m \\
\lambda
\]

Analytic Center Cutting Plane Algorithm
1. Solve LMI problem
2. Solve LMI problem
3. Use KYP lemma to check feasibility. If feasible, then stop otherwise add new frequencies to the grid.
4. If $\gamma \leq 0$ then feasible. Stop.
5. Go to step 2.
   \[ \{\sigma\} \cup U = \emptyset \]

\[ (\gamma, \sigma) \in \mathbb{R} \times \mathbb{R}^+ \]

Path Following Algorithm

Structure Preserving Interior Point Algorithms

Outer Approximation
The IQC Toolbox

Background

1. Early versions
   IQCTools (94-96) by Jonsson and Rantzer, LTH. An interface inspired by the MuToolbox.
   Toolbox (around 94-96) by Megretski and Cyganek. Equipped with a graphical user interface (GUI).

2. The IQC environment
   Command line mode
   Graphical user interface (GUI)

3. The IQC toolbox ICQ

4. Predefined IQC blocks

5. Optimization Code

6. Examples

History

Environments

Tutorial on the IQC part of the toolbox

Abstract Environments

Tutorial

I. Jonsson, KTH
Louvain-la-Neuve, Feb 16-19, 2004

Abstract Environments

1. Standard LMI problems

\[ \inf_{\mathbf{X}} \mathbf{X} \text{s.t.} \mathbf{F}(\mathbf{X}) = \mathbf{F}_0 + \sum_{i=1}^{m} \mathbf{F}_i \mathbf{X}_i \mathbf{X}_i^T \geq 0 \]

2. Frequency dependent LMI problems

\[ \inf_{\mathbf{X}} \mathbf{X} \text{s.t.} \mathbf{F}_0 + \sum_{i=1}^{m} \mathbf{F}_i \mathbf{X}_i \mathbf{X}_i^T \geq 0 \]

3. Robustness analysis in the IQC framework

\[ \infty \text{sup} \mathbf{X} \text{s.t.} \mathbf{H}(\mathbf{X}) = \mathbf{H}_0 + \sum_{i=1}^{m} \mathbf{H}_i \mathbf{X}_i \mathbf{X}_i^T \geq 0 \]

U. Jonsson, KTH
Louvain-la-Neuve, Feb 16-19, 2004
Compute $L_2$-gain $r$!

Use Popov IQC

Example

The IQC Environment and its "$abst" Types

Initialization: "$abst"-type $\tilde{t}q$

Inputs (external input): "$abst" type input

Matrix variables: "$abst" type $\text{var}$table

Variables: "$abst" type constant

\textbf{Detection}

\textbf{GUI}

Using GUI for our example
Definition of IQCs

Example: Computed gain $f \rightarrow v$ of $v = G(w + f)$

We can use the IQC

Example of LTI types to define descriptions

Integral quadratic constraint (IQC)

Frequency dependent LMI

Linear matrix inequalities (LMI)
The block definition as a Matlab function:

```matlab
function w = iqc_uncertainty(v, a)

w = signal(size(v, 1));
X = symmetric(n);
for k = 1:length(a)
    Xk = symmetric(n);
    X = X + Xk * (1/(s + a(k)));
end
if X > 0;
    v' * X * v - w' * X * w > 0;
end

U.Jonsson, KTH, Louvain-la-Neuve, Feb 16-19, 2004
```

This is used either as

```matlab
w == iqc_uncertainty(v, a);
```

or

```matlab
[wsl, X] == iqc_uncertainty(v, a);
w == wsl;
```

U.Jonsson, KTH, Louvain-la-Neuve, Feb 16-19, 2004

---

Predefined IQC blocks:

<table>
<thead>
<tr>
<th>Block name</th>
<th>M-file name</th>
</tr>
</thead>
<tbody>
<tr>
<td>iqc</td>
<td>domharmonic.m</td>
</tr>
<tr>
<td>iqc</td>
<td>whitenoiseperformance.m</td>
</tr>
<tr>
<td>iqc</td>
<td>repeateddiagonalslope-restrictednonlinearity.m</td>
</tr>
<tr>
<td>iqc</td>
<td>oddnonlinearityversionofdomharmonic.m</td>
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<tr>
<td>iqc</td>
<td>encapsulateddeadzone.m</td>
</tr>
<tr>
<td>iqc</td>
<td>encapsulatedodddeadzone.m</td>
</tr>
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<td>iqc</td>
<td>L2norm-boundedgeneralblock.m</td>
</tr>
<tr>
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<td>monotonic.m</td>
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<tr>
<td>iqc</td>
<td>ratelimiter.m</td>
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<tr>
<td>iqc</td>
<td>popov.m</td>
</tr>
<tr>
<td>iqc</td>
<td>sector.m</td>
</tr>
<tr>
<td>iqc</td>
<td>sector+popov.m</td>
</tr>
<tr>
<td>iqc</td>
<td>vect.m</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Block name</th>
<th>M-file name</th>
</tr>
</thead>
<tbody>
<tr>
<td>iqc</td>
<td>cdelay.m</td>
</tr>
<tr>
<td>iqc</td>
<td>uncertaindelay.m</td>
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<tr>
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<td>harmonicoscillation.m</td>
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<td>unknownconstant.m</td>
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<td>iqc</td>
<td>uncertaintydelay.m</td>
</tr>
<tr>
<td>iqc</td>
<td>uncertaintydelay(simple).m</td>
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<tr>
<td>iqc</td>
<td>uncertaintydelay(arr).m</td>
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<td>iqc</td>
<td>polytope.m</td>
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<td>iqc</td>
<td>stvp.m</td>
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<td>iqc</td>
<td>slowtv.m</td>
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<td>iqc</td>
<td>tvscalar.m</td>
</tr>
<tr>
<td>iqc</td>
<td>window.m</td>
</tr>
</tbody>
</table>

U.Jonsson, KTH, Louvain-la-Neuve, Feb 16-19, 2004

---

The block definition as a Matlab function:

```matlab
function X = IQCvector(V, lambda, gamma, delta)

m = 2*V;
M = 1/2*V;

w = IQCUncertainty(V, lambda, gamma, delta);

if w > 0
    X = w^ M - lambda * X + delta * A';
end

for k = 1:size(A, 1)
    X = X + A(k) * X + A(k)' * X + A(k)';
end

X = symmetric(n); % Matrix X is symmetric

function w = IQCUncertainty(V, lambda, gamma, delta)

This is used either as

```matlab
w == IQCUncertainty(V, lambda, gamma, delta);
```

or

```matlab
[wsl, X] == IQCUncertainty(V, lambda, gamma, delta);
w == wsl;
```

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```
Optimization Code

The SDP solver in the current version of the Matlab LMI toolbox is too slow for many realistic examples.

Special purpose codes are being developed:
1. Cutting plane algorithms by C.-Y. Kao
2. Path following algorithms by C.-Y. Kao
3. Various structure exploiting SDP solvers by Hansson, Vandenberghe and Wallin

Numerical Example

Spaceshuttle lateral axis control system

Adapted from an example in the -tools manual.

The example is taken from
Kao, Megretski, Jonsson, Rantzer A MATLAB Toolbox for Robustness Analysis
where it was adapted from Kao’s Ph.D. thesis. Jonsson, Rantzer, A MATLAB Toolbox for Robustness Analysis.

The example is taken from

Adapted from an example in the -tools manual.

Space shuttle lateral axis flight control system


Analytica: Department of Mechanical Engineering, MIT.
C.-Y. Kao “Efficient Computational Methods for Robustness where it was adapted from Robustness Analysis
Kao. Megretski, Jonsson, Rantzer A MATLAB Toolbox for Robustness Analysis.

Numerical Example

W.

Numerical Example
Choice of basis for the IQC is important for feasibility.

Advanced IQC were used for the multiple nonlinearity. If
\[(x) = \sum_{u}^{f(t)} y \left| x \right| + \sum_{u}^{f(t) \neq f(t) = f(t)} y \geq \eta \]

where \( \rho \) be a symmetric matrix valued function s.t.
\( \rho \in \mathbb{R}^{2 \times 2} \) has non-positive off diagonal elements,
\( \rho \in \mathbb{R}^{2 \times 2} \) is positive definite diagonal elements,
where \( \eta = 0 \).

Then
\[
\begin{bmatrix}
(x)
\end{bmatrix}
\]

Concluding Remarks

We do not need to rearrange the system on the \( G \)-form before.