The S-Procedure and its Applications in IQC Analysis

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Outline

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References

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Y71 Yakubovich, V. A. (1971) S-procedure in nonlinear control theory. *Vestnik Leningrad University* pp. 62–77. (English translation in Vestnik Leningrad Univ. 4:73-93, 1977).

Y92 Yakubovich, V. A. (1992) Nonconvex optimization problem: The infinite-horizon linear quadratic control problem with quadratic constraints. *Systems and Control Letters* **19**, 13–22.

The S-procedure

Let $\sigma_k : \mathcal{H} \to \mathbf{R}$, $k = 0, 1, \ldots, N$ be real valued functionals.

Consider the following two conditions

S_1 : $\sigma_0(y) \geq 0$ for all $y \in \mathcal{H}$ such that $\sigma_k(y) \geq 0$, $k = 1, \ldots, N$.

S_2 : There exists $\tau_k \geq 0$, $k = 1, \ldots, N$ such that

$$\sigma_0(y) - \sum_{k=1}^{N} \tau_k \sigma_k(y) \geq 0, \quad \forall y \in \mathcal{H}.$$

- We note that S_2 implies S_1. Indeed,

$$S_2 \;\Rightarrow\; \sigma_0(y) \geq \sum_{k=1}^{N} \tau_k \sigma_k(y) \;\Rightarrow\; S_1$$

  since $\tau_k \geq 0$, $k = 1, \ldots, N$. Hence S_2 is sufficient for S_1.
- The S-procedure is the method of verifying S_1 using S_2.
- This is useful since S_2 generally is much simpler to verify than S_1

Example 1. Consider quadratic forms defined over $\mathcal{H} = \mathbf{R}^n$

$$\sigma_k(y) = y^T Q_k y, \quad k = 0, 1, \ldots, N$$

The problem with S_1 is then that

1. σ_0 is not a convex function in general.
2. The constraint set

$$\Omega = \{y \in \mathbf{R}^m : \sigma_k(y) \geq 0,\; k = 1, \ldots, N\}$$

   is not convex in general.

This means that condition S_1 in general corresponds to verifying that the minimum of a nonconvex function over a nonconvex set is positive, i.e., $S_1 \;\Leftrightarrow\; \min_{y\in\Omega} \sigma_0(y) \geq 0$. This problem belongs to NP.

example[cont'd] The problem S_2 corresponds to an LMI since

$$\sigma_0(y) - \sum_{k=1}^{N} \tau_k \sigma_k(y) \geq 0, \qquad \forall y \in \mathbf{R}^n$$

$$\Leftrightarrow \qquad y^T(Q_0 - \sum_{k=1}^{N} \tau_k Q_k) y \geq 0, \qquad \forall y \in \mathbf{R}^n$$

$$\Leftrightarrow \qquad Q_0 - \sum_{k=1}^{N} \tau_k Q_k \geq 0$$

Feasibility of an LMI can be verified using semidefinite programming example

Losslessness

- The two conditions S_1 and S_2 are in general not equivalent
- The S-procedure is called lossless when $S_1 \Leftrightarrow S_2$.
- Under reasonable regularity conditions the S-procedure is lossless when
  - $\mathcal{H} = \mathbf{R}^n$, $\sigma_k(y) = y^T Q_k y$, $k = 0, 1$ (one constraint)
  - $\mathcal{H} = \mathbf{C}^n$, $\sigma_k(y) = y^* Q_k y$, and $k = 0, 1, 2$ (two constraints)
  - $\mathcal{H}$ is a Hilbert space with a special shift-invariance property, $\sigma_k(y) = \langle y, \Phi_k y \rangle \geq 0$, $k = 0, \ldots, N$. I.e., any finite number of constraints.

Condition for Losslessness

Definition 1. Let $\sigma_k : \mathcal{H} \to \mathbf{R}$. The constraint $\sigma_k(y) \geq 0$ for $k = 1, \ldots, N$ is said to be regular if there exists $y^* \in \mathcal{H}$ such that $\sigma_k(y^*) > 0$, $k = 1, \ldots, N$.

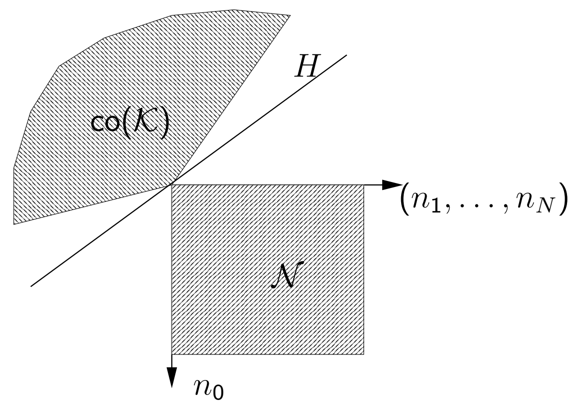
Theorem 1 (Yakubovich). *Let $\sigma_k : \mathcal{H} \to \mathbf{R}$, $k = 0, \ldots, N$ and assume the constraint $\sigma_k(y) \geq 0$, $k = 1, \ldots, N$ is regular. Consider the sets*

$$\mathcal{K} = \{(\sigma_0(y), \sigma_1(y), \ldots, \sigma_N(y)) : y \in \mathcal{H}\},$$
$$\mathcal{N} = \{(n_0, n_1, \ldots, n_N) : n_0 < 0, n_k > 0\}.$$

If $\mathcal{K} \cap \mathcal{N} = \emptyset \Rightarrow \text{co}(\mathcal{K}) \cap \mathcal{N} = \emptyset$ then the S-procedure is lossless, i.e., $S_1 \Leftrightarrow S_2$.

Remark 1. In particular, if $\mathcal{K}$ is a convex set then the S-procedure is lossless.

Proof idea:



Since the two sets are convex and disjoint, there exists a separating hyperplane H. This can be used to prove the result.

Theorem 2. *Assume $\sigma_1(y) = y^T Q_1 y \geq 0$ is regular. Then the following are equivalent*

$S_1 :$ $y^T Q_0 y > 0$, *for all* $y \neq 0$ *such that* $y^T Q_1 y \geq 0$

$S_2 :$ *there exists* $\tau \geq 0$ *such that* $Q_0 - \tau Q_1 > 0$

Proof. We first notice that $S_2 \Rightarrow S_1$ is trivial. For the opposite direction, note that S_1 implies

$$\min_{\{y: y^T Q_1 y \geq 0\}} \frac{y^T Q_0 y}{|y|^2} = \varepsilon$$

for some $\varepsilon > 0$. Hence $\sigma_0(y) = y^T Q_0 y - \varepsilon |y|^2 \geq 0$, $\forall y \in \mathbf{R}^m$ such that $\sigma_1(y) = y^T Q_1 y \geq 0$. $\mathcal{K} = \{(\sigma_0(y), \sigma_1(y)) : y \in \mathbf{R}^m\}$ is convex by a result in, e.g. [Y71]. Hence, by Theorem 1

$$S_1 \quad \Leftrightarrow \quad \exists \tau \geq 0 \text{ s.t. } \sigma_0(y) - \tau \sigma_1(y) \geq 0, \ \forall y \in \mathbf{R}^m$$

The last condition corresponds to the LMI in S_2. □

Example 2 *(Circle criterion).* We will here derive a necessary and sufficient condition for quadratic stability of the system

$$\dot{x} = Ax + Bw, \quad x(0) = x_0$$
$$v = Cx$$

where the input and output satisfies the sector constraint

$$\sigma(v, w) = (\beta v - w)(w - \alpha v) \geq 0,$$

where $\alpha < \beta$ are real numbers.

Let $V(x) = x^T P x$, where $P = P^T > 0$. We require that

$$\dot{V}(x) = 2x^T P(Ax + Bw) < 0, \ \forall (x, w) \neq 0 \text{ s.t. } \sigma(Cx, w) \geq 0. \quad (1)$$

Let us now define the quadratic forms

$$\sigma_0(x, w) = \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

$$\sigma_1(x, w) = 2\sigma(Cx, w) = \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} -2\beta\alpha C^T C & (\beta + \alpha)C^T \\ (\beta + \alpha)C & -2 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

Then condition (1) can then be rewritten as

$$S_1 : \quad \sigma_0(x, w) < 0, \quad \forall (x, w) \neq 0 \text{ s.t. } \sigma_1(x, w) \geq 0$$

The constraint $\sigma_1(x, w) \geq 0$ is regular since $\alpha < \beta$. Hence,

$$S_1 \Leftrightarrow S_2 : \text{there exists } \tau \geq 0 \text{ such that } \quad \sigma_0(x, w) + \tau\sigma_1(x, w) < 0$$

for all $(x, w) \neq 0$. It is easily seen that we need $\tau > 0$ for this to hold.

Normalize such that $\tau = 1$ and $P/\tau \to P$. This proves that quadratic stability is equivalent to feasibility of the LMI:

$\exists P = P^T > 0$ such that

$$\begin{bmatrix} A^T P + PA - 2\beta\alpha C^T C & PB + (\beta + \alpha)C^T \\ B^T P + (\beta + \alpha)C & -2 \end{bmatrix} < 0.$$

This LMI corresponds to the circle criterion.

Losslessness in Hilbert space

Assumption 1. Let the quadratic forms $\sigma_k : \mathcal{H} \to \mathcal{H}$ be defined as

$$\sigma_k(y) = \langle y, \Phi_k y \rangle$$

where $\Phi_k = \Phi_k^*$ are bounded operators on $\mathcal{H}$. Let $S_\tau : \mathcal{H} \to \mathcal{H}$ be a shift operator. We assume

(i) if $f \in \mathcal{H}$ then $S_\tau f \in \mathcal{H}$ for all $\tau \geq 0$

(iia) $\langle S_\tau f_1, \Phi_k f_2 \rangle \to 0$ as $\tau \to \infty$

(iib) $\langle f_1, \Phi_k S_\tau f_2 \rangle \to 0$ as $\tau \to \infty$

(iii) $\sigma_k(S_\tau f) = \sigma_k(f)$ for all $\tau \geq 0$ and all $f \in \mathcal{H}$

Example 3. If $\Phi = \Phi^* \in \mathbf{RL}_\infty^{m\times m}$, $\mathcal{H} = \mathbf{L}_2[0,\infty)$, $(S_\tau y)(t) = y(t-\tau)$, and $\sigma(y) = \langle y, \Phi y\rangle$ then all the above properties hold due to the time-invariance of Φ and the standard properties of the $\mathbf{L}_2$ integrals.
Example 4. If $\Phi = \Phi^* \in \mathbf{RL}_\infty^{m\times m}$, $\mathcal{H} = l_2[0,\infty)$, $S_\tau y_k = y_{k-\tau}$, and $\sigma(f) = \langle y, \Phi y\rangle$ then all the above properties hold due to the time-invariance of Φ and the standard properties of the l_2 space.

Theorem 3 (S-Procedure Lossless Theorem). *Assume the quadratic form satisfies the properties in Assumption 1 and that there exists $f^* \in \mathcal{H}$ such that $\sigma_k(f^*) > 0$ for $k = 1,\dots,N$. Then the S-procedure is lossless, i.e., the following are equivalent*

S_1 : $\sigma_0(f) \le 0$ *for all* $f \in \mathcal{H}$ *such that* $\sigma_k(f) \ge 0$, $k = 1,\dots,N$.

S_2 : *There exists* $\tau_k \ge 0$, $k = 1,\dots,N$ *such that*

$$\sigma_0(f) + \sum_{k=1}^N \tau_k\sigma_k(f) \le 0, \quad \forall f \in \mathcal{H}.$$

Proof. Define

$$\mathcal{K} = \{(\sigma_0(f), \sigma_1(f), \dots, \sigma_N(f)) : f \in \mathcal{H}\},$$
$$\mathcal{N} = \{(n_0, n_1, \dots, n_N) : n_k > 0,\ k = 0,1,\dots,N\}.$$

From the assumptions it follows that the closure of $\mathcal{K}$ is convex. The result now follows as in Theorem 1. □

Remark 2. Note that condition S_2 can be replaced by the feasibility test: Find $\tau_k \ge 0$ such that

$$\Phi_0 + \sum_{k=1}^N \tau_k\Phi_k \le 0$$

Various versions of the so-called Kalman-Yakubovich-Lemma can be used to verify this condition in practical situations.

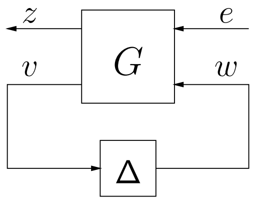
Necessity in IQC Analysis

The S-procedure can be used to prove necessity in IQC analysis for some special cases. We refer to

A. Megretski. Necessary and sufficient conditions of stability: a multiloop generalization of the circle criterion Automatic Control, IEEE Transactions on ,Volume: 38 , Issue: 5 , May 1993 Pages:753 - 756

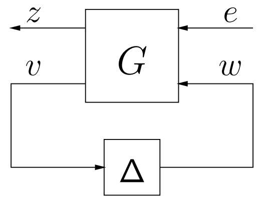
K. Poola and A. Tikku. Robust performance against time-varying structured perturbations Automatic Control, IEEE Transactions on ,Volume: 40 , Issue: 9 , Sept. 1995 Pages:1589 - 1602

Robust Performance Analysis



- $G \in \mathbf{RH}_\infty^{(q+m)\times(q+m)}$
- Δ belongs to a class of bounded causal operators
- Robust performance means that
  - The closed loop system is robustly stable
  - A worst case performance criterion is satisfied

Linear Fractional Transformation



If $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ then the (lower) LFT with respect to Δ is defined as

$$\mathcal{S}_l(G,\Delta) = G_{11} + G_{12}\Delta(I - G_{22}\Delta)^{-1}G_{21}.$$

Robust performance means

(i) $(I - G_{22}\Delta)^{-1}$ is causal and bounded.

(ii) $w \mapsto z = \mathcal{S}_l(G,\Delta)e$ satisfies some norm bound

IQC for Signals

Definition 2. A signal set $\mathcal{E} \subset \mathbf{L}_2^q[0,\infty)$ satisfies the IQC defined by $\Psi = \Psi^* \in \mathbf{RL}_\infty^{q\times q}$ ($\mathcal{E} \in \mathrm{IQC}(\Psi)$) if

$$\sigma_\Psi(e) = \int_{-\infty}^{\infty} \hat{e}(j\omega)^*\Psi(j\omega)\hat{e}(j\omega)d\omega \geq 0 \qquad (2)$$

for all $e \in \mathcal{E}$.

Dominant Harmonics: If supp $\hat{e} \in [-b,-a] \cup [a,b]$, then we use

$$\Psi(j\omega) = \begin{cases} 0, & |\omega| \in [a,b], \\ -NI, & \text{otherwise} \quad N >> 1. \end{cases}$$

or an rational approximation of this function.

Signals with Given Spectral Characteristic: Consider a signal with *spectrum*

$$|\hat{e}(j\omega)|^2 = \frac{\|e\|^2}{\|H\|_2^2}|H(j\omega)|^2 \tag{3}$$

where H is a given transfer function. Such signals can be used to model *filtered deterministic "white noise"*. If Ψ satisfies

$$\int_{-\infty}^{\infty} \Psi(j\omega)|H(j\omega)|^2\, d\omega \geq 0$$

then the IQC (2) holds for all signals with spectrum (3). This follows since

$$\int_{-\infty}^{\infty} \Psi(j\omega)|\hat{e}(j\omega)|^2 d\omega = \frac{\|e\|^2}{\|H\|_2^2}\int_{-\infty}^{\infty} \Psi(j\omega)|H(j\omega)|^2\, d\omega \geq 0.$$

Performance Criterion

The most common performance criterion is the $\mathbf{L}_2$-gain of the system

$$\sigma_P(z,e) = \int_0^{\infty}(|z(t)|^2 - \gamma^2|e(t)|^2)dt \leq 0.$$

Other performance measures are

- $\mathbf{L}_2 \to \mathbf{L}_\infty$ gain
- weighted sensitivity measures

Definition 3. Assume $e \in \mathcal{E} \subset \mathbf{L}_2^q[0,\infty)$. Then the system

$$\begin{bmatrix} z \\ v \end{bmatrix} = G \begin{bmatrix} e \\ w \end{bmatrix} \tag{4}$$

$$w = \Delta(v) \tag{5}$$

has robust performance with respect to the performance IQC σ_P if

(i) the system is stable

(ii) $\sigma_P(z,e) \leq 0$ for all $z = \mathcal{S}_l(G,\Delta)e$, $e \in \mathcal{E}$.

Proposition 1. *Assume that $\mathcal{E} \in$ IQC(Ψ) and $\Delta \in$ IQC(Π). Then the system (4) has robust $\mathbf{L}_2$-gain γ if*

(i) *it is stable*

(ii) *the frequency domain inequality*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & \Pi_{11}(j\omega) & 0 & \Pi_{12}(j\omega) \\ \hline 0 & 0 & -\gamma^2 I + \Psi(j\omega) & 0 \\ 0 & \Pi_{12}^*(j\omega) & 0 & \Pi_{22}(j\omega) \end{array}\right] \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq 0,$$

holds for all $\omega \in [0,\infty]$.

Furthermore, if the system is well posed, $\Pi_{11} \geq 0$, $\Pi_{22} \leq 0$, then stability is implied if the frequency domain inequality above holds strictly.

Proof. The result follows from the S-procedure. Let

$$\mathcal{H} = \left\{ (z, v, e, w) \in \mathbf{L}_2^{2m+2q}[0,\infty) : \begin{bmatrix} z \\ v \end{bmatrix} = G \begin{bmatrix} e \\ w \end{bmatrix} \right\}.$$

A sufficient condition is

$$\sigma_P(z, e) \leq 0, \text{ for all } (z, v, w, e) \in \mathcal{H} \text{ such that } \sigma_\Psi(e) \geq 0,\ \sigma_\Pi(v, w) \geq 0.$$

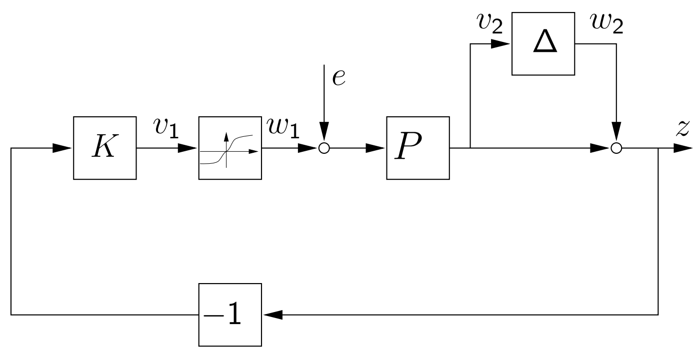
This is clearly the case if there exists $\tau_1, \tau_2 \geq 0$ such that $\sigma(z, v, e, w) := \sigma_P(z, e) + \tau_1 \sigma_\Psi(e) + \tau_2 \sigma_\Pi(v, w) \leq 0$ for all $(z, v, w, e) \in \mathcal{H}$. It is no restriction to assume that τ_1 and τ_2 are included in Π and Ψ, respectively. The corresponding IQCs are still valid.

Using that $(z, v) = G(e, w)$ gives the equivalent statement

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{e} \\ \hat{w} \end{bmatrix}^* \begin{bmatrix} G \\ I \end{bmatrix}^* \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & \Pi_{11} & 0 & \Pi_{12} \\ \hline 0 & 0 & -\gamma^2 I + \Psi & 0 \\ 0 & \Pi_{12}^* & 0 & \Pi_{22} \end{array}\right] \begin{bmatrix} G \\ I \end{bmatrix} \begin{bmatrix} \hat{e} \\ \hat{w} \end{bmatrix} d\omega \leq 0$$

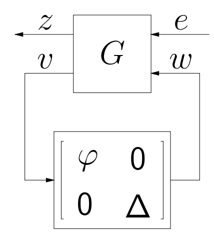
for all $(e, w) \in \mathbf{L}_2^{m+q}[0, \infty)$. This is equivalent to the frequency domain inequality in (ii). The last claim is easy to verify. □

Example 5.



- A saturation nonlinearity φ and dynamic uncertainty Δ.
- Transfer function P and K are stable.
- Estimate the $\mathbf{L}_2$ gain from e to z

LFT formulation



$$G = \begin{bmatrix} P & P & 1 \\ -KP & -KP & -K \\ P & P & 0 \end{bmatrix}.$$

We can use familiar IQCs from Lecture 1.

The system is stable and the $\mathbf{L}_2$-gain is less than γ if

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & \Pi_{11}(j\omega) & 0 & \Pi_{12}(j\omega) \\ \hline 0 & 0 & -\gamma^2 I & 0 \\ 0 & \Pi_{12}^*(j\omega) & 0 & \Pi_{22}(j\omega) \end{array}\right] \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq 0,$$

where

$$\Pi(j\omega) = \left[\begin{array}{cc|cc} 0 & 0 & 1 + H(j\omega) & 0 \\ 0 & x(j\omega) & 0 & 0 \\ \hline 1 + H(j\omega)^* & 0 & -\frac{2}{k}(1 + \text{Re}\, H(j\omega)) & 0 \\ 0 & 0 & 0 & -x(j\omega) \end{array}\right]$$

where $\|H\|_1 = \int_{-\infty}^{\infty} |h(t)| dt \leq 1$ and $x(j\omega) \geq 0$.

Concluding Remarks

- The S-procedure provides a means of relaxing hard problems in analysis
- The relaxed problem is convex
- The S-procedure is exact under special conditions
- Useful in LMI analysis and IQC analysis