An Introduction to Integral Quadratic Constraints

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Integral Quadratic Constraints (IQCs)

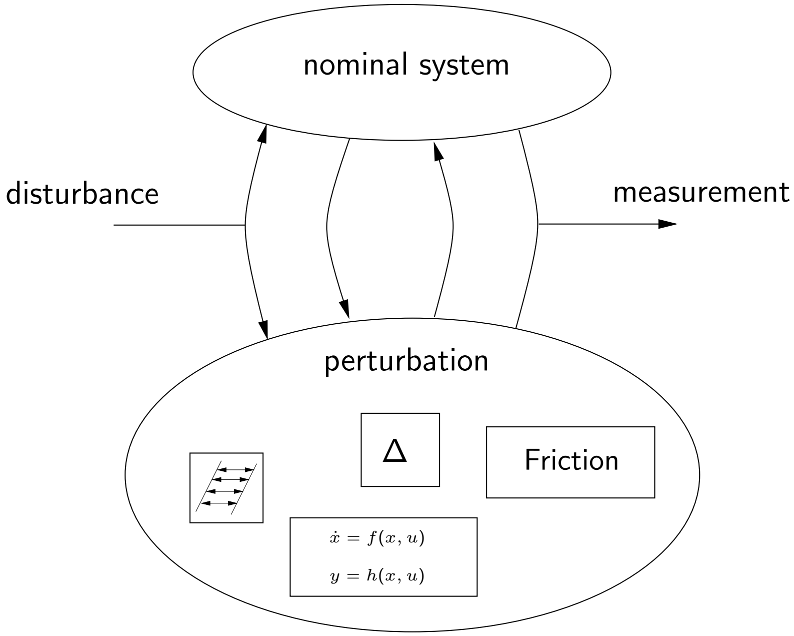
- Unifying framework for systems analysis
  - Generalizes the small gain and passivity theorems
  - Generalizes many results from robust control
- Built upon ideas from optimal control and optimization
- Results in convex optimization problems

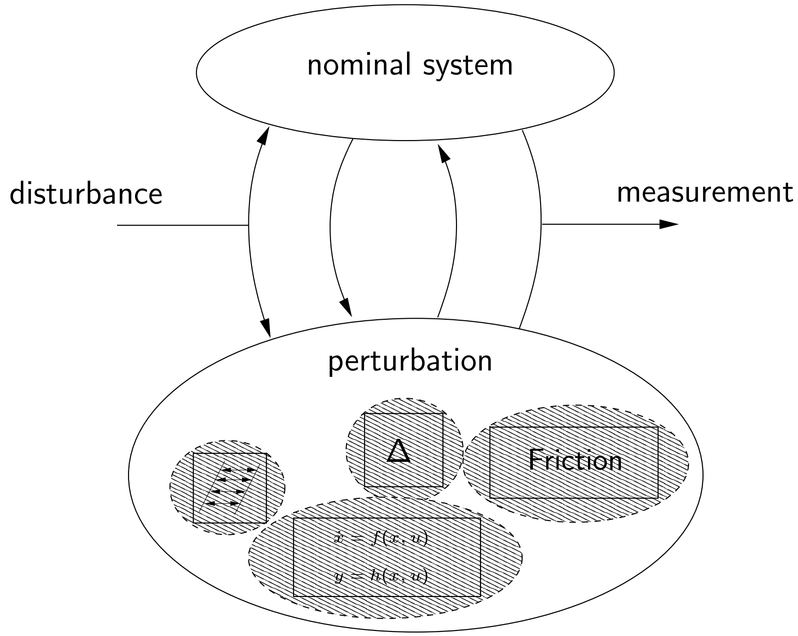
Outline

- Introduction
- Motivation from robust control
- System description
- Integral quadratic constraints (IQC)
- Stability theorem
- Comments and examples

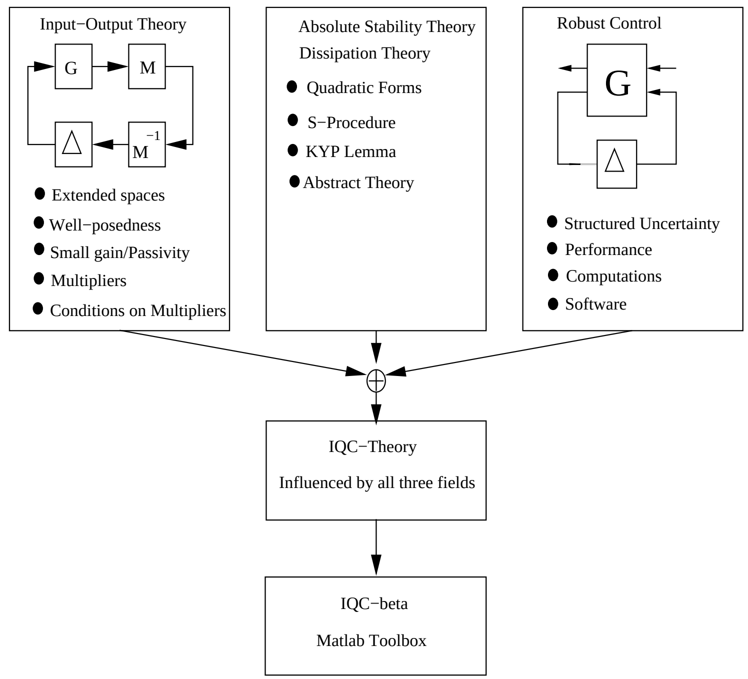
Introduction

- The big picture
- Relation to other methods
- References





- Nominal system suitable for computation
- Quadratic characterization of perturbation
- Use convex optimization to investigate
  - stability
  - performance



We only discuss the main ideas behind IQCs in a "robust control framework". For more details we refer to

A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, 42(6):819–830, June 1997.

Motivation from Robust Control

- System with structured uncertainty
- System with uncertain time-varying parameters
- System with mixed uncertainty (combination of the above)

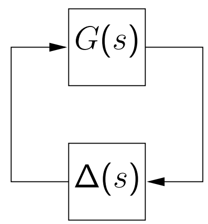
Main point: A need for user friendly theory that covers all these examples.

Notation and Assumptions

- $G \in \mathbf{RH}_\infty^{p\times m}$ means that that G is a stable rational transfer function with m inputs and p outputs.
- $\|\Delta\|_{\mathbf{H}_\infty} = \sup_{\mathrm{Re}\, s\geq 0} \overline{\sigma}(\Delta(s))$, where $\overline{\sigma}(\cdot)$ is the maximal singular value.
- We often use the simplified notation

$$\Upsilon(j\omega) < 0, \quad \forall \omega \in [0,\infty] \quad \Leftrightarrow \quad \Upsilon < 0$$

System with Structured Uncertainty

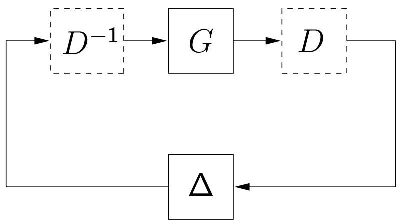


- $G \in \mathbf{RH}_\infty^{m\times m}$ stable rational transfer function.
- Δ structured uncertainty (LTI) $\|\Delta\|_{\mathbf{H}_\infty} \leq 1$

$$\Delta(s) = \mathrm{diag}(\delta_1(s)I_1, \ldots, \delta_M(s)I_M, \Delta_1(s), \ldots, \Delta_N(s))$$

  - $\Delta_k \in \mathbf{RH}_\infty^{m_k\times m_k}$, $\|\Delta_k\|_{\mathbf{H}_\infty} \leq 1$

- D-scaling $D^{-1}\Delta D = \Delta D^{-1} D = \Delta$



- Use the small gain theorem to obtain a stability criterion.

Stable if for all ω (small gain theorem)

$$\overline{\sigma}(D(j\omega)G(j\omega)D(j\omega)^{-1}) < 1, \quad \Leftrightarrow$$
$$G(j\omega)^* \underbrace{D(j\omega)^* D(j\omega)}_{X(j\omega)} G_\omega < \underbrace{D(j\omega)^* D(j\omega)}_{X(j\omega)} \quad \Leftrightarrow$$
$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} X(j\omega) & 0 \\ 0 & -X(j\omega) \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0$$

Alternative Interpretation

If $X(j\omega)$ is chosen such that

$$\begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix}^* \begin{bmatrix} X(j\omega) & 0 \\ 0 & -X(j\omega) \end{bmatrix} \begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix} \geq 0$$

then the system is stable if

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} X(j\omega) & 0 \\ 0 & -X(j\omega) \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0$$

Note that there are many $X(j\omega)$.

An example:

- $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$, $\|\Delta_k\|_{\mathbf{H}_\infty} \leq 1$, $k = 1, 2$.
- $X(j\omega) = \begin{bmatrix} x_1(j\omega)I_1 & 0 \\ 0 & x_2(j\omega)I_2 \end{bmatrix}$ where $x_k(j\omega) \geq 0$, $k = 1, 2$.
- The point with the analysis is to find an appropriate $X(j\omega)$ to prove stability.

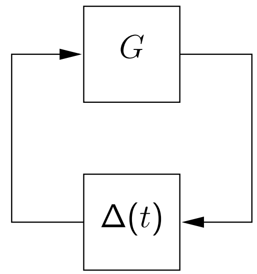
How to Verify in Practice

- Pick sufficiently dense grid $0 = \omega_0 \leq \omega_1 \leq \omega_2 \leq \dots$
- For each k find structured matrix X_k s.t.

$$\begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \begin{bmatrix} X_k & 0 \\ 0 & -X_k \end{bmatrix} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} < 0$$

- How dense is dense enough?

Time-varying Uncertainty



- $\Delta(t)$ time-varying structured uncertainty

$$\Delta(t) = \begin{bmatrix} \delta_1(t)I_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \delta_N(t)I_N \end{bmatrix}$$

$-1 \leq \delta_k(t) \leq 1.$

- Find matrices Q, S and R such that

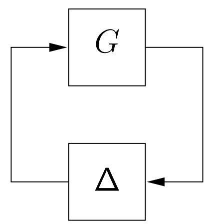
$$\begin{bmatrix} I \\ \Delta(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} I \\ \Delta(t) \end{bmatrix} \geq 0$$

- Stable if

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0$$

- Proof: Lyapunov theory + S-procedure + KYP-Lemma
- How to verify in practice?
  - Sufficiently dense frequency gridding
  - or KYP lemma (see later)

Mixed Uncertainty



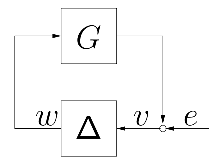
- Mixed uncertainty $\Delta = \begin{bmatrix} \Delta_1(s) & 0 \\ 0 & \Delta_2(t) \end{bmatrix}$
  - Δ_1 structured uncertainty (LTI)
  - Δ_2 time-varying structured uncertainty

- One would guess the stability criterion

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} X(j\omega) & 0 & 0 & 0 \\ 0 & Q & 0 & S \\ 0 & 0 & -X(j\omega) & 0 \\ 0 & S^T & 0 & R \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0 \qquad (1)$$

- Need more theory to prove this!
- Not practical to verify using gridding!
- How do we verify using KYP lemma?

System Description



1. Signals and operator spaces
2. Well-posedness
   - Extended spaces
   - Causality
3. Adjoint operators and quadratic forms

Signal Spaces and Operators

- Signals are represented as functions from normed vector spaces
  - The norm allow us to measure the size of the signals
  - We consider Hilbert spaces which contains additional structure because of the inner product. This allow us to define quadratic forms that characterizes relationship between signals
- The systems are represented as operators on the signal space
  - The induced norm allows us to measure the signal amplification

Signal Spaces

We assume the signals belongs to a Hilbert space

$$\mathcal{H} = \{v : \mathcal{T} \to \mathcal{V} : \|v\| < \infty\}$$

- $\mathcal{T}$ is a time axis, e.g.
  - $\mathbf{R}_+ = [0, \infty)$ continuous time systems
  - $Z_+ = \{0, 1, 2, \ldots\}$ discrete time systems
- $\mathcal{V}$ is a normed vector space, e.g.
  - $\mathcal{V} = (\mathbf{R}^n, |\cdot|)$ where $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$ will be used
- $\mathcal{H}$ is equipped with an inner product, i.e. a bilinear function $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbf{R}$ satisfying
  1. $\langle v, v \rangle \geq 0$, $\forall v \in \mathcal{V}$ and $\langle v, v \rangle = 0$ iff $v = 0$.

2. $\langle \alpha_1 v_1 + \alpha_2 v_2, v_3 \rangle = \alpha_1 \langle v_1, v_3 \rangle + \alpha_2 \langle v_2, v_3 \rangle$
3. $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$

- The norm on $\mathcal{H}$ is defined in terms of the inner product $\|v\| = \langle v, v \rangle$

Two main examples will be considered

$\mathbf{L}_2[0,\infty)$ The space of square integrable functions with inner product

$$\langle v, w \rangle = \int_0^\infty v(t)^T w(t) dt$$

The corresponding norm is $\|v\| = \sqrt{\int_0^\infty |v(t)|^2 dt}$.

$l_2(Z_+)$ The space of square summable sequences with inner product

$$\langle v, w \rangle = \sum_0^\infty v_k^T w_k$$

The corresponding norm is $\|v\| = \sqrt{\sum_0^\infty |v_k|^2}$.

Plancherel-Parceval Theorem

Let the Fourier transforms of v and w be defined as

$$\widehat{v}(j\omega) = \lim_{T\to\infty} \int_0^T v(t) e^{-j\omega t} dt, \ \omega \in \mathbf{R}$$

$$\widehat{v}(j\omega) = \lim_{N\to\infty} \sum_{k=0}^N v_k e^{-j\omega k}, \quad \omega \in [-\pi, \pi]$$

Then we have the Plancherel-Parceval relation

$$\langle v, w \rangle = \int_0^\infty v(t)^T w(t) dt = \frac{1}{2\pi} \int_{-\infty}^\infty \widehat{v}(j\omega)^* \widehat{w}(j\omega)\, d\omega \quad \mathbf{L}_2^m[0,\infty) \quad (2)$$

$$\langle v, w \rangle = \sum_{i=0}^\infty v_i^T w_i = \frac{1}{2\pi} \int_{-\pi}^\pi \widehat{v}(j\omega)^* \widehat{w}(j\omega)\, d\omega \quad l_2^m(Z_+) \quad (3)$$

Operators

An operator is a mapping $H : \mathcal{H} \to \mathcal{H}$. Two operators H_1, H_2 on $\mathcal{H}$ satisfy the following properties

(i) The composition $H_1 H_2$ is also an operator on $\mathcal{H}$ defined by $(H_1 H_2)(v) = H_1(H_2(v))$

(ii) The sum $\alpha H_1 + \beta H_2$ for any $\alpha, \beta \in \mathbf{R}$ is an operator on $\mathcal{H}$ defined by $(\alpha H_1 + \beta H_2)(v) = \alpha H_1(v) + \beta H_2(v)$

An operator is *linear* if

$$H(\alpha v + \beta w) = \alpha H(v) + \beta H(w)$$

Shorthand notation $G(v) = Gv$ for linear operators G.

- We always assume $H(0) = 0$
- An operator $H : \mathcal{H} \to \mathcal{H}$ is called *bounded* if its "gain" is finite

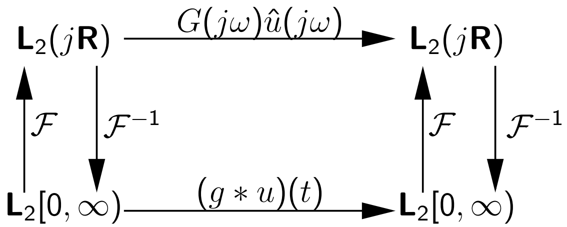
$$\|H\| = \sup_{\substack{v\in\mathcal{H} \\ v\neq 0}} \frac{\|H(v)\|}{\|v\|}$$

- The gain satisfies the submultiplicativity rule

$$\|H_1 H_2\| \leq \|H_1\| \cdot \|H_2\|$$

Examples of Operators

A stable transfer function $G(s) = C(sI - A)^{-1}B + D \in \mathbf{RH}_\infty$ defines an operator in both time and frequency domain



The convolution is defnend by

$$(g * u)(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

The gain is $\|G\|_{\mathbf{H}_\infty} = \sup_{\omega \in \mathbf{R}} \sigma_{\max}(G(j\omega))$

Consider a nonlinear function $\varphi : \mathbf{R} \to \mathbf{R}$ satisfying

$$|\varphi(x)| \le k|x|$$

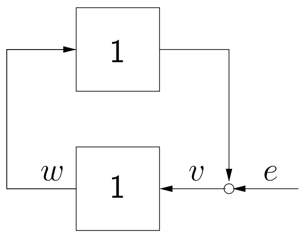
It defines a nonlinear operator $\Delta : \mathbf{L}_2[0,\infty) \to \mathbf{L}_2[0,\infty)$ as

$$(\Delta v)(t) = \varphi(v(t))$$

with gain

$$\|\Delta\| = \sup_{\substack{v \in \mathbf{L}_2[0,\infty) \\ v \neq 0}} \frac{\|\varphi(v(\cdot))\|}{\|v\|} = k$$

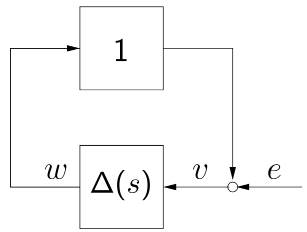
Ill-posed system



The system equation

$$v = e + v \quad \Rightarrow \quad e = 0$$

Not well defined for all inputs!!



If $\Delta(s) = I - e^{-sT}$ then the system equation

$$v(t) = e(t) + v(t) - v(t-T) \quad \Rightarrow \quad v(t) = e(t+T)$$

Not causal!!

Extended Spaces

- We do not know a priori if the system signals are bounded
- Truncate at arbtitrary finite time $\Rightarrow$ Extended space.

Definition 1. The truncation operator P_T is defined by

$$(P_T v)(t) = \begin{cases} v(t), & t \le T \quad (t, T \in \mathcal{T}) \\ 0, & t > T \end{cases}$$

for any signal $v : \mathcal{T} \to \mathcal{V}$

We will often use the notation $v_T = P_T v$.

Definition 2. The extended space $\mathcal{H}_e$ is then defined as

$$\mathcal{H}_e = \{v : \mathcal{T} \to \mathcal{V} : \|v_T\| < \infty,\ \forall T \ge 0\}$$

where $\|\cdot\|$ is the norm on $\mathcal{H}$. We will assume that the norm $\|\cdot\|$ is such that

- For every $v \in \mathcal{H}_e$ we have $\|v_{T_1}\| \le \|v_{T_2}\|$ for all $T_2 \ge T_1$.
- For all $v \in \mathcal{H}$ we have $\|v_T\| \to \|v\|$ as $T \to \infty$.

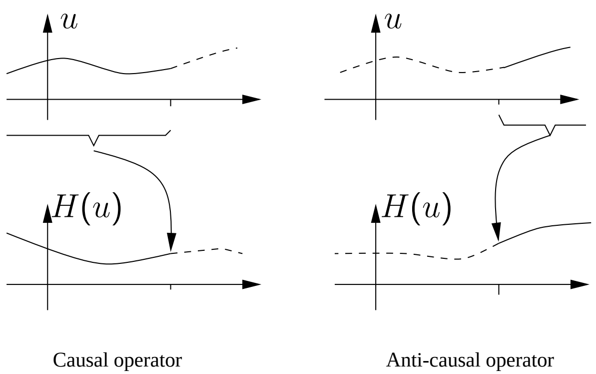
Example 1.

1. $\sin(t) \in \mathbf{L}_{2e}[0, \infty)$
2. $e^t \in \mathbf{L}_{2e}[0, \infty)$

Causality of operators on extended spaces

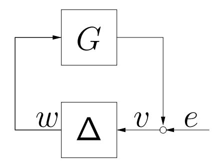
An operator $H : \mathcal{H}_e \to \mathcal{H}_e$ (or $H : \mathcal{H} \to \mathcal{H}$) is said to be causal if

$$P_T H P_T = P_T H, \quad \text{for all } T \in \mathcal{T}.$$



H is causal and bounded on $\mathcal{H}_e \Leftrightarrow H$ is causal and bounded on $\mathcal{H}$

The gain is the same on both spaces



Definition 3 *(Well-posedness).* Suppose G, Δ are bounded and causal on $\mathcal{H}_e$. The system is well-posed if for any $e \in \mathcal{H}_e$ there exist a solution $w, v \in \mathcal{H}_e$ and if furthermore, the loop signals v and w depend causally on e.

Definition 4 *(Stability).* The system is stable if it is well-posed and if there are positive constants c such that

$$\|v_T\| + \|w_T\| \le c\|e_T\| \tag{4}$$

for all $T \in \mathcal{T}$ and all $e \in \mathcal{H}_e$.

Conditions for well-posedness can be found in

1 J. C. Willems. The Analysis of Feedback Systems

2 Desoer and Vidyasagar. Feedback Systems: Input-Output Properties

Adjoint Operators and Quadratic Forms

Definition 5. Let $H : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Then the Hilbert adjoint H^* of H is the operator $H^* : \mathcal{H} \to \mathcal{H}$ such that

$$\langle Hv, w\rangle = \langle v, H^* w\rangle \quad \forall v, w \in \mathcal{H}$$

Example 2. Let $G = C(sI - A)^{-1}B + D \in \mathbf{RL}_\infty^{m\times m}$ be an operator on $\mathbf{L}_2^m(-\infty,\infty)$ (or $\mathbf{L}_2[0,\infty)$). Then

$$G^*(s) = G(-s)^T = -B^T(sI + A^T)^{-1}C^T + D^T$$

Definition 6. A bounded linear operator $H : \mathcal{H} \to \mathcal{H}$ is self-adjoint if $H^* = H$. A self-adjoint operator is

Positive semi-definite, denoted $H \geq 0$ if and only if

$$\langle Hv, v\rangle \geq 0, \qquad \forall v \in \mathcal{H}.$$

Positive definite, denoted $H > 0$, iff there exists $\varepsilon > 0$ such that

$$\langle Hv, v\rangle \geq \varepsilon\|v\|^2, \quad \forall v \in \mathcal{H}.$$

- H negative semi-definite if $-H$ is positive semi-definite.
- H is negative definite if $-H$ is positive definite

Definition 7. A bounded self-adjoint operator $\Phi = \Phi^* : \mathcal{H} \to \mathcal{H}$ defines a (bounded) quadratic form $\sigma : \mathcal{H} \to \mathbf{R}$ as

$$\sigma(v) = \langle \Phi v, v\rangle .$$

The quadratic form is positive definite if there exists $\varepsilon > 0$ such that

$$\sigma(v) \geq \varepsilon\|v\|^2, \qquad \forall v \in \mathcal{H}$$

Proposition 1. *Let $\Phi = \Phi^* \in \mathbf{RL}_\infty^{m\times m}$ and define the quadratic form $\sigma(v) = \langle \Phi v, v\rangle$ on $\mathbf{L}_2[0,\infty)$. Then the following are equivalent*

(i) $\sigma(v) \geq 0$ *for all* $v \in \mathbf{L}_2[0,\infty)$

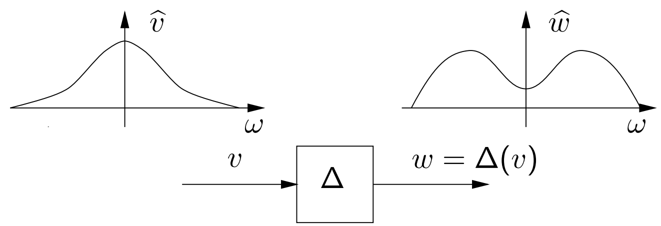
(ii) $\Phi(j\omega) \geq 0$ *for all* $\omega \geq 0$.

and similarly for strict inequalities.

Integral Quadratic Constraints

1. Definition of IQC and some examples of IQC
2. Basic stability theorem
3. How to use IQCs in practice

Integral Quadratic Constraint



- Π bounded self adjoint linear operator
- $\Delta \in \text{IQC}(\Pi)$ if

$$\sigma_\Pi(v, \Delta(v)) = \left\langle \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}, \Pi \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} \right\rangle \geq 0, \qquad \forall v \in \mathcal{H}$$

- Describes spectral distribution

If $\mathcal{H} = \mathbf{L}_2[0,\infty)$ then $\Pi(j\omega) = \Pi(j\omega)^*$ (self adjoint transfer function) and

$$\sigma_\Pi(v, \Delta(v)) = \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix} d\omega$$

If $\mathcal{H} = l_2(Z_+)$ then $\Pi(e^{j\omega}) = \Pi(e^{j\omega})^*$ (self adjoint transfer function) and

$$\sigma_\Pi(v, \Delta(v)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \begin{bmatrix} \hat{v}(e^{j\omega}) \\ \widehat{\Delta(v)}(e^{j\omega}) \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} \hat{v}(e^{j\omega}) \\ \widehat{\Delta(v)}(e^{j\omega}) \end{bmatrix} d\omega$$

Examples

- Unstructured uncertainty $\|\Delta\|_{\mathbf{H}_\infty} \leq 1$

$$\Pi_1(j\omega) = \begin{bmatrix} x(j\omega)I & 0 \\ 0 & -x(j\omega)I \end{bmatrix}, \quad x(j\omega) \geq 0$$

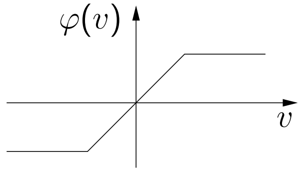
- Parametric uncertainty $\delta(t)I$, $\delta(t) \in [-1, 1]I$

$$\Pi_2(j\omega) = \begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}, \quad \begin{matrix} X = X^T \geq 0 \\ Y = -Y^T \end{matrix}$$

Proof:

$$\begin{bmatrix} I \\ \delta(t)I \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix} \begin{bmatrix} I \\ \delta(t)I \end{bmatrix} = X - \delta(t)^2 X \geq 0$$

- Saturation function



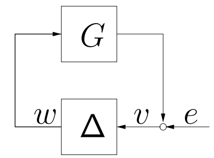
$$\Pi_3(j\omega) = \begin{bmatrix} 0 & 1 + H(j\omega) \\ 1 + H(j\omega)^* & -2(1 + \text{Re}\, H(j\omega)) \end{bmatrix}$$

where $\|H\|_1 = \int_{-\infty}^{\infty} |h(t)|dt \le 1$.

$$\begin{aligned} \text{Proof : } & [v(t) - \varphi(v(t))] \cdot [\varphi(v(t)) + (h * \varphi(v))(t)] \\ & \ge [v(t) - \varphi(v(t))] \cdot [\varphi(v(t)) - \text{sign}(v(t)) \sup_{v \in \mathbf{R}} |\varphi(v)| \cdot \|H\|_1] \\ & \ge [v(t) - \varphi(v(t))] \cdot [\varphi(v(t)) - \text{sign}(v(t))] = 0 \end{aligned}$$

Integration and use of Parsevals theorem gives the result.

Basic Stability Theorem



$$\begin{aligned} v &= Gw + e \\ w &= \Delta(v) \end{aligned} \qquad (5)$$

- G bounded and causal linear operator on $\mathcal{H}_e$
- Δ bounded and causal on $\mathcal{H}_e$
- The system (5) is stable if there exists $c > 0$ such that

$$\|v_T\| + \|w_T\| \le c\|e_T\|$$

for all $T \in \mathcal{T}$ and $e \in \mathcal{H}_e$.

Theorem 1. *Suppose*

(i) *The system (5) is well-posed*

(ii) $\Delta \in \text{IQC}(\Pi)$

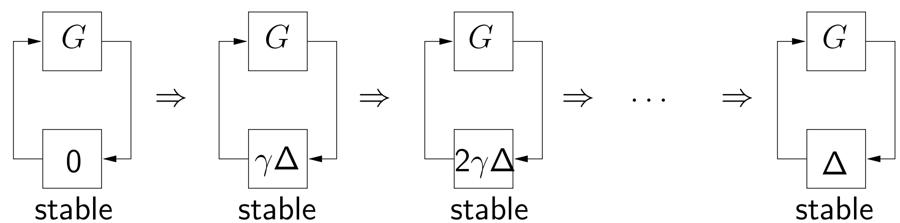
(iii) $\Pi_{11} \ge 0$ *and* $\Pi_{22} \le 0$

(iv) $\begin{bmatrix} G \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I \end{bmatrix} < 0,$

then system (5) is stable.

Sketch of proof:

1. Use $(ii) - (iv)$ to show that there exists $c_0 > 0$, which is independent of $\tau \in [0, 1]$, such that $\|v\| \le c_0\|(I - \tau G\Delta)(v)\|, \ \forall v \in \mathcal{H}$
2. Hence if the feedback interconnection $(G, \tau\Delta)$ is stable then $\|(I - \tau G\Delta)^{-1}\| \le c_0$.
3. Use "continuity" in τ to prove the following implications



for some $\gamma > 0$.

Practical Issues

If $\Delta = \text{diag}(\Delta_1, \Delta_2)$ where $\Delta_k \in \text{IQC}(\Pi_k)$, $k = 1, 2$, then $\Delta \in \text{IQC}(\Pi)$

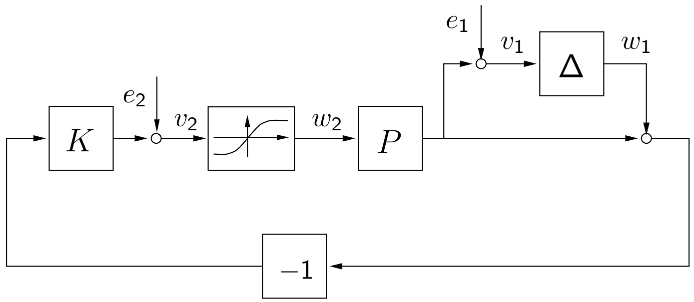
$$\text{where } \Pi = \begin{bmatrix} \Pi_{1(11)} & 0 & \Pi_{1(12)} & 0 \\ 0 & \Pi_{2(11)} & 0 & \Pi_{2(12)} \\ \Pi_{1(12)}^* & 0 & \Pi_{1(22)} & 0 \\ 0 & \Pi_{2(12)}^* & 0 & \Pi_{2(22)} \end{bmatrix}$$

Example 3. For mixed uncertainty $\Delta = \begin{bmatrix} \Delta_1(s) & 0 \\ 0 & \Delta_2(t) \end{bmatrix}$ we use

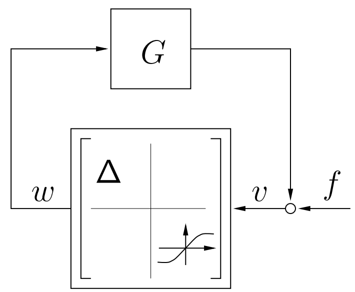
$$\Pi_1(j\omega) = \begin{bmatrix} X(j\omega) & 0 \\ 0 & -X(j\omega) \end{bmatrix} \qquad \Pi_2 = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$$

which gives the stability criterion (1)

Example 4 (Robust Stability).



- The multiplicative uncertainty $\Delta \in \text{IQC}(\Pi_1)$
- The saturation nonlinearity $\varphi \in \text{IQC}(\Pi_3)$



$$G = \begin{bmatrix} 0 & P \\ -K & -KP \end{bmatrix}$$

Stability criterion

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty]$$

where

$$\Pi(j\omega) = \begin{bmatrix} x(j\omega)I & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 + H(j\omega) \\ 0 & 0 & -x(j\omega)I & 0 \\ 0 & 1 + H(j\omega)^* & 0 & -2(1 + \text{Re}\, H(j\omega)) \end{bmatrix}$$

Concluding Remarks

1. IQCs provide a unifying framework for robust stability analysis
2. Sufficient conditions for stability
3. The next lectures
   - (a) The S-procedure losslessness can for special cases prove that the stability conditions are necessary.
   - (b) Robust performance analysis
   - (c) How to verify the stability criterion using convex optimization
   - (d) Toolbox for IQC analysis