INTERLACED SYSTEMS AND RECURSIVE DESIGNS FOR GLOBAL STABILIZATION

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Abstract

Interlaced systems constitute a class of systems only characterized by the zero entries of their matrix configuration and a local stabilizability condition. All these systems are globally stabilizable by a recursive design procedure which combines steps of backstepping and forwarding. When a nonlinear system misses this structural characterization, other types of conditions (sign, growth) are needed to ensure global stabilization.

To obtain a systematic design, we rely on two recently developed recursive procedures which, starting from a stabilized system with a known stabilizing control law and a control Lyapunov function (CLF), achieve the same objective for an *augmented* system. The two recursive designs are dual to each other. *Top-down* augmentations are handled by *backstepping* (see [2] and the references therein) while *bottom-up* augmentations are handled by the more recent *forwarding* [1, 5, 4]. In a block-diagram representation of the system, one integrator is added at the input in the case of backstepping, while, in forwarding, the integrator is added to the state located at the other end of the chain.

1 Introduction

This paper addresses the problem of global stabilization (by state feedback) of single input nonlinear systems

$$\dot{x} = f(x, u), \quad u \in I\!R, \quad x \in I\!R^n \tag{1.1}$$

where f is smooth (that is, has continuous partial derivatives of any order) and $f(0,0) = 0$. Throughout this paper it is assumed that the Jacobian linearization of (1.1) at $(x,u) = (0,0)$ is stabilizable. Our goal is to characterize a set of conditions which guarantee, first, that the system is *globally* stabilizable, and, second, that the design of a globally stabilizing feedback can be achieved in a systematic way.

The systems which can be globally stabilized by a sequence of backstepping and/or forwarding steps are called *interlaced systems*. They extend the class of "strict-feedback" nonlinear systems, handled by backstepping, and the class of "strict-feedforward" nonlinear systems, handled by forwarding. We show that they can be characterized by a condition on the zero entries of their matrix configuration. This condition, which only restricts the allowed feedback and feedforward interconnections in the system, is sufficient to guarantee global stabilization and the construction of a CLF. When a system misses this structural condition, we show that the global stabilizability of the system is not ensured without extra conditions.

2 Backstepping and forwarding

Backstepping and forwarding start from a subsystem

$$\dot{z} = q(z, v), \quad v \in \mathbb{R} \tag{2.1}$$

where the control v is the true control variable u or a state of the entire system (1.1) which acts as the input of the subsystem (2.1). For the subsystem (2.1), we assume that a globally stabilizing control law and a control Lyapunov function (see [6]) have already been constructed.

Assumption 1 *A smooth control law $v = \alpha(z)$ and Lyapunov function $W(z)$ (positive definite and radially unbounded) are known such that the equilibrium $z = 0$ of $\dot{z} = q(z, \alpha(z))$ is globally asymptotically stable (GAS) and the time-derivative $\dot{W} = \frac{\partial W}{\partial z} q(z, \alpha(z))$ is negative definite.*

The fundamental backstepping result states that the control law $v = \alpha(z)$ can be "implemented" through a (nonlinear) integrator.

Proposition 2 *(Backstepping)*
Suppose that Assumption 1 holds. Then the augmented system

$$\begin{array}{rcl} \dot{z} & = & q(z, \zeta) \\ \dot{\zeta} & = & a(z, \zeta) + b(z, \zeta)u, \quad \zeta \in \mathbb{R} \end{array} \tag{2.2}$$

where $b(z, \zeta) \neq 0$ for all (z, ζ) is globally stabilizable. A control Lyapunov function for (2.2) is given by

$$V(z, \zeta) = W(z) + \frac{1}{2}(\zeta - \alpha(z))^2 \tag{2.3}$$

A globally stabilizing control law for the system (2.2) is easily obtained from the control Lyapunov function (2.3). Defining the new coordinate $\xi = \zeta - \alpha(z)$ and rewriting (2.2) as

$$\begin{array}{rcl} \dot{z} & = & q(z, \alpha(z)) + \psi(z, \xi)\xi \\ \dot{\xi} & = & a(z, \zeta) + b(z, \zeta)u - \frac{\partial \alpha}{\partial z}(q(z, \alpha(z)) + \psi(z, \xi)\xi), \end{array} \tag{2.4}$$

one can verify that the control law

$$\begin{array}{l} u = b^{-1}(z, \zeta)(-a(z, \zeta) + \dfrac{\partial \alpha}{\partial z} q(z, \alpha(z)) + \\ \qquad +\psi(z, \xi)\xi) - \dfrac{\partial W}{\partial z}\psi(z, \xi) - \xi) \end{array}$$

achieves $\dot{V} = \frac{\partial W}{\partial z} q(z, \alpha(z)) - \xi^2 < 0$, which implies global asymptotic stability of $(z, \xi) = (0, 0)$.

A recursive application of backstepping permits the global stabilization of any nonlinear system which can be represented as a scalar subsystem augmented by several top-down augmentations of the form (2.2). Such systems are said to be in *strict-feedback* form: in a block-diagram representation, all the nonlinearities are feedback connections.

In forwarding, the augmented system is obtained by adding an integrator not at the input but rather at the other end of the chain. The fundamental forwarding result states that, under a local stabilizability assumption, the control law $v = \alpha(z)$ which stabilizes the subsystem (2.1) can be augmented to achieve global stabilization of the augmented system.

Proposition 3 *(Forwarding)*
Suppose that Assumption 1 holds and that in addition $v = \alpha(z)$ achieves local exponential stability (LES) of $z = 0$. Then the augmented system

$$\begin{array}{rcl} \dot{\zeta} & = & \gamma(z, v), \quad \zeta \in \mathbb{R} \\ \dot{z} & = & q(z, v) \end{array} \tag{2.5}$$

is globally stabilizable if its Jacobian linearization at $(z, \zeta) = (0, 0)$ is stabilizable. A control Lyapunov function for (2.5) is given by

$$V(z, \zeta) = W(z) + \frac{1}{2}(\zeta + \int_0^\infty \gamma(\tilde{z}(s), \alpha(\tilde{z}(s))ds)^2 \tag{2.6}$$

where $\tilde{z}(s)$ is the solution of $\dot{z} = q(z, \alpha(z))$ at time s for the initial condition $\tilde{z}(0) = z$.

The proof of this result, see [1, 5] is based on the fact that, with $v = \alpha(z)$, the equilibrium $(\zeta, z) = (0, 0)$ of (2.5) is globally stable (but not asymptotically stable), that is, the Lyapunov function (2.6) is positive definite, radially unbounded, and nonincreasing along the solutions. Global *asymptotic* stability of $(z, \xi) = (0, 0)$ is then achieved with a control law $v = \alpha(z) + \tilde{v}(\zeta, z)$ where $\tilde{v}(\zeta, z)$ is chosen to enhance the negativity of $\dot{V}$. For an affine system $\dot{x} = f(x) + g(x)\tilde{v}$, a standard choice is the control law $\tilde{v} = -\frac{\partial V}{\partial x} g(x)$. An extension to the non-affine case can be found in [3].

Forwarding allows the stabilization of any nonlinear system which can be represented as a scalar stabilizable subsystem augmented by several bottom-up augmentations of the form (2.5). Such systems are said to be in *strict-feedforward* form: in a block-diagram representation, all the nonlinearities are feedforward connections. More general forwarding results allow an additional feedback connection around each integrator, that is, $\dot{\zeta} = \gamma(\zeta, z, v)$ in (2.5). However, these feedback connections are restricted by stability requirements and growth conditions, see [1, 4] for details. The characterizations discussed in the present paper are only in terms of allowed interconnections, with no further sign or growth restriction.

3 Interlaced systems

We call a nonlinear system *interlaced* if it can be represented as a scalar subsystem augmented by several top-down or/and bottom-up augmentations. Hence interlaced systems extend the class of strict-feedback and strict-feedforward systems. We will characterize interlaced systems by certain properties of the configuration matrix

$$P(x, u) = \frac{\partial f}{\partial(x, u)} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial u} \end{pmatrix} \tag{3.1}$$

The zero entries of $P(x,u)$ determine the feedback and feedforward connections which are absent from a block-diagram representation of the system. A same sequence of backstepping and forwarding steps will be applicable to different systems if their configuration matrices have the same zero entries.

With one step of backstepping or forwarding, the configuration matrix is augmented by one additional row and one additional column. It undergoes a *top-down augmentation* of the form

$$P_{bst} = \begin{pmatrix} & & & 0 \\ & P & & \vdots \\ & & & 0 \\ \tilde{p}_1 & \dots & \tilde{p}_{n+1} & b \end{pmatrix} \tag{3.2}$$

The zeros in the last column are necessary to apply one step of backstepping. They guarantee the absence of *feedforward connections* from u to the subsystem $\dot{z} = q(z,\zeta)$. In a dual manner, with a step of forwarding, the configuration matrix P undergoes a *bottom-up augmentation* to

$$P_{fwd} = \begin{pmatrix} 0 & \tilde{p}_1 & \dots & \tilde{p}_{n+1} \\ 0 & & & \\ \vdots & & P & \\ 0 & & & \end{pmatrix} \tag{3.3}$$

where the zero entries in the first column are necessary to apply one step of forwarding. They guarantee the absence of *feedback connections* from the state ζ to the subsystem $\dot{z} = q(z,u)$.

Backstepping imposes restrictions on some nonzero entries of the configuration matrix. For the system (2.2), we must have $b(z,\zeta) \neq 0$ for all $(z,\zeta) \in I\!R^{n+1}$. This implies that, if the Jacobian linearization of (2.1) is stabilizable, so is the Jacobian linearization of the augmented system (2.2). Forwarding requires stabilizability of the Jacobian linearization.

Examining all the configuration matrices which can be generated by repeated top-down augmentations of the type (3.2) or bottom-up augmentations of the type (3.3), we arrive at the following characterization of interlaced systems.

Definition 4 *(Interlaced systems)*
A system (1.1) is called interlaced if its Jacobian linearization is stabilizable and its configuration matrix $P(x,u)$ satisfies the following requirements:

(i) If $j > i+1$ and $p_{ij} \not\equiv 0$, then $p_{kl}(x) \equiv 0$ for all $k \geq l$, $k \leq j-1$, and $l \leq i$.

(ii) If $p_{ij} \not\equiv 0$ for some $j \leq i$, then $p_{i\,i+1}$ is independent of x_{i+1} and $p_{i\,i+1}(x) \neq 0$ for all x.

□

Definition 4 characterizes interlacing by (i) and excludes degenerate situations in which the lack of stabilizability occurs in the Jacobian linearization or in which backstepping cannot be applied because of a nonglobal relative degree (condition (ii)).

Example 5 *(Three-dimensional interlaced systems)*
For third-order systems, the four different types of configuration matrix which satisfy the requirement (i) of Definition 4 are listed below with the two-step sequences of backstepping (bst) and forwarding (fwd):

$$bst\ +\ bst: \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}$$

$$fwd\ +\ fwd: \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

$$bst\ +\ fwd: \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & 0 \\ 0 & * & * & * \end{pmatrix}$$

$$fwd\ +\ bst: \begin{pmatrix} 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & * & * & * \end{pmatrix}$$

Only these four configuration matrices can be generated with two top-down and/or bottom-up augmentations. In each case, a sequence of backstepping and forwarding steps to be followed can be determined from the matrix configuration. □

Theorem 6 *(Stabilization of interlaced systems)*
Every interlaced system is globally stabilizable by a sequence of scalar steps of backstepping and/or forwarding. The design simultaneously yields the construction of a globally stabilizing feedback and of a CLF. □

Proof (sketch): for notational convenience, let $x_{n+1} := u$. By definition, the configuration matrix $P(x,u)$ of an interlaced system is made of a core submatrix $[*\,*]$ or $[0\,*]$ augmented by a sequence of top-down or bottom-up augmentations which determine the sequence of backstepping forwarding steps for the design of a stabilizing feedback and the construction of a Control Lyapunov Function. The submatrix $[*\,*]$ corresponds to a scalar subsystem

$$\dot{x}_k = f_k(x_k, x_{k+1}) = a_k(x_k) + b_k(x_k)x_{k+1} \tag{3.4}$$

for some $k \in \{1,\dots,n\}$ and a function $b(x_k) \neq 0$, see (ii). Treating x_{k+1} as the control variable, a Control Lyapunov Function is x_k^2 and a stabilizing feedback is $x_{k+1} = \alpha(x_k)$ such that $(a_k(x_k) + b_k(x_k)\alpha(x_k))x_k < 0$ for all $x_k \neq 0$. Choosing $\alpha'(0) \neq 0$ ensures local exponential stability such that Assumption 1 is then satisfied for the subsystem (3.4).

Alternatively, the submatrix $[0\ *]$ corresponds to a scalar subsystem

$$\dot{x}_k = f_k(x_k, x_{k+1}) = h_k(x_{k+1}) \tag{3.5}$$

for some $k \in \{1, \ldots, n\}$ and $h_k'(0) \neq 0$, since the Jacobian linearization is stabilizable. Treating x_{k+1} as the control variable, a Control Lyapunov Function is x_k^2 and a stabilizing feedback is $x_{k+1} = \alpha(x_k)$ such that $h_k(\alpha(x_k))x_k < 0$ for all $x_k \neq 0$. Choosing $\alpha'(0) \neq 0$ ensures local exponential stability such that Assumption 1 is then satisfied for the subsystem (3.4).

Having a scalar subsystem which satisfies Assumption 1, the next step is either a top-down augmentation, which yields the subsystem

$$\begin{aligned} \dot{x}_k &= f_k(x_k, x_{k+1}) \\ \dot{x}_{k+1} &= f_{k+1}(x_k, x_{k+1}, x_{k+2}) \\ &= a_{k+1}(x_k, x_{k+1}) + b_{k+1}(x_k, x_{k+1})x_{k+2} \end{aligned}$$

or a bottom-up augmnentation, which yields the subsystem

$$\begin{aligned} \dot{x}_{k-1} &= f_{k-1}(x_{k-1}, x_k, x_{k+1}) = h_{k-1}(x_k, x_{k+1}) \\ \dot{x}_k &= f_k(x_k, x_{k+1}) \end{aligned}$$

In the first case, Proposition 2 can be applied with x_{k+2} treated as the control variable. In the second case, Proposition 3 can be applied with x_{k+1} treated as the control variable. After n steps of backstepping or/and forwarding, the entire matrix configuration is recovered, the control variable is the true control u, and a Control Lyapunov Function is obtained for the entire system. □

Definition 4 guarantees that global stabilization can be achieved without any restriction on the sign or the growth of the nonlinearities. Global stabilizability of systems which are not interlaced cannot be guaranteed.

Theorem 7 *(Loss of stabilizability in noninterlaced systems)*
Let $P(x,u)$ be a configuration matrix for a system whose Jacobian linearization is a chain of integrators. If $P(x,u)$ does not satisfy the condition (i) of Definition 4, then there exists a system, with the configuration matrix which contains all the zeros of $P(x,u)$, and satisfies the other requirements of Definition 4, which is not globally stabilizable.

Proof: Let $p_{ij}(x)$ and $p_{kl}(x)$ be two nonzero entries of $P(x,u)$ such that $j > i+1$, $k \geq l$, $k \leq j-1$, and $l \leq i$. We consider a system consisting of a chain of integrators $\dot{x}_1 = x_2$, $\ldots$, $\dot{x}_n = u$, except for its i-th and k-th equations, which are

$$\begin{aligned} \dot{x}_i &= x_{i+1} + x_j^2 \\ \dot{x}_k &= x_{k+1} + x_l^2 \quad if\ i \neq k \end{aligned} \tag{3.6}$$

and

$$\dot{x}_i = x_{i+1} + x_j^2 + x_i^2 \quad if\ i = k (= l) \tag{3.7}$$

In the configuration matrix $P(x,u)$ of this system the only nonzero entries are the off-diagonal entries $p_{m\,m+1} \equiv 1$, $m \in \{1, \ldots, n\}$, and the two entries $p_{ij}(x) = 2x_j$, $p_{kl}(x) = 2x_l$. Therefore, it satisfies the conditions (i) and (ii) of Definition 4 and contains all the zeros of $P(x,u)$.

We will now show that this system is not globally stabilizable. Using the fact that $l \leq k, i \leq j-1$, we define the new state $\eta = x_l + \ldots + x_{j-1}$ which satisfies

$$\begin{aligned} \dot{\eta} &= (x_{l+1} + \ldots + x_j) + x_l^2 + x_j^2 & (3.8) \\ &= \eta - x_l + x_l^2 + x_j + x_j^2 \geq \eta - 1 & (3.9) \end{aligned}$$

This proves that initial conditions $\eta(0) > 1$ cannot be driven to zero, irrespective of the choice of the control. □

4 Wei's configuration for linear uncertain systems

Our characterization of interlaced systems is reminiscent of the results of Wei [7, 8] for the following robust stabilization problem: given a linear system (A, b) where the elements a_{ij} of the matrix A and b_j of the vector b are only known to belong to given compact sets $a_{ij}^- \leq a_{ij} \leq a_{ij}^+$, $b_j^- \leq b_j \leq b_j^+$, what are the conditions on (A, b) which guarantee the existence of a robust linear stabilizing feedback $u = k^T x$?

A structural characterization of the pairs (A, b) which are robustly stabilizable employs the following definition.

Definition 8 *(Weis' configuration)*
A linear uncertain system (A, b) has a generalized anti-symmetric stepwise configuration (or Wei's configuration) if its configuration matrix satisfies the following requirements:

(i) If $j > i+1$ and $p_{ij} \not\equiv 0$, then $p_{kl} \equiv 0$ for all $k \geq l$, $k \leq j-1$, and $l \leq i$.

(ii) Each superdiagonal element $p_{i\,i+1}$ is sign definite (that is, $a_{i\,i+1}^- a_{i\,i+1}^+ > 0$ and $b_n^- b_n^+ > 0$).

(iii) If P^r denotes the $n \times n$ matrix obtained by removing the first column of the configuration matrix P, then $\det(P^r) = \pi_{i=1}^n p_{i\,i+1}$.

□

For linear systems, the requirements (i) and (ii) of Definition 4 coincide with the requirements (i) and (ii) of Definition 8. Condition (iii) replaces the requirement that the Jacobian linearization is stabilizable.

Example 9 *(Three-dimensional Wei's configurations)*
For third-order systems, the five different types of configuration matrix which satisfy the requirements of Definition 8 are listed below. The symbol θ is used to denote the additional zeros required by (iii) to exclude the loss of controllability for particular values of the coefficients.

$$bst\ +\ bst: \begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}$$

$$fwd\ +\ fwd: \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & \theta & * \end{pmatrix} \quad \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & \theta \\ 0 & 0 & * & * \end{pmatrix}$$

$$bst\ +\ fwd: \begin{pmatrix} 0 & * & \theta & * \\ 0 & * & * & 0 \\ 0 & \theta & \theta & * \end{pmatrix}$$

$$fwd\ +\ bst: \begin{pmatrix} 0 & * & * & 0 \\ 0 & \theta & * & 0 \\ * & * & * & * \end{pmatrix}$$

As in the case of interlaced systems, these five configuration matrices can be generated with two top-down and/or bottom-up augmentations of a scalar subsystem. In each case, a sequence of backstepping and forwarding steps to be followed can be determined from the matrix configuration. □

It is shown in [7, 8] that Wei's configuration is sufficient for the design of a robust linear stabilizing feedback. As in the case of interlaced systems, the design is recursive. For systems satisfying the condition (ii), the two other requirements (i) and (iii) are also shown to be necessary for stabilization in the absence of further sign restrictions.

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