Preprints of the 7th IFAC Symposium on Nonlinear Control Systems
21-24 August, 2007, Pretoria, South Africa

GROWTH RATE OF SWITCHED HOMOGENEOUS SYSTEM

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Abstract: We consider discrete-time homogeneous systems under arbitrary switching and study their growth rate, the analogue of joint spectral radius for switched linear systems. We show that a system is asymptotically stable if and only if its growth rate is less than unity. We also provide an approximation algorithm to compute growth rate to an arbitrary accuracy.

Keywords: homogeneity, switched systems, growth rate, joint spectral radius, approximation algorithm

1. INTRODUCTION

Stability analysis of systems under arbitrary switching has for a decade been a major direction of research in systems theory. The main reason, as indicated in an early survey (Liberzon and Morse, 1999), is that many recent engineering systems are of switched nature. The area is still in its infancy and even linear systems are yet not fully explored under switching. Hence much of the work is for now concentrated on switched linear systems, see the survey (Shorten *et al.*, 2007).

One way to determine the stability of a discrete-time switched linear system is to examine the *joint spectral radius* (JSR) of its system matrices: if JSR is less than unity then (and only then) the system is asymptotically stable. In fact, JSR is more than just a sharp indicator of stability; it also tells how fast the trajectories will converge to (or diverge from) the origin. As a consequence, JSR has been extensively studied (Gurvits, 1995) and an active research is going on to devise efficient algorithms approximating its value (Blondel and Nesterov, 2005).

In this paper we consider switched homogeneous systems, a superclass of switched linear systems, which have been the subject of (Filippov, 1980) and later (Holcman and Margaliot, 2003) and (Tuna and Teel, 2005). In particular, we consider a discrete-time homogeneous system under arbitrary switching and study its *growth rate*, the analogue of JSR for linear systems.

We first establish the result that the norm of the trajectories of a discrete-time switched homogeneous system can uniformly (in the norm of the initial condition) be upperbounded with an exponential envelope. The infimum of growth rates of such envelopes is the *growth rate* of the system. Then we show that stability of the system is equivalent to growth rate's being less than unity. The first contribution of the paper hence is the generalization of the result on JSR to the case of homogeneous systems. As a second contribution, we provide an algorithm to approximate growth rate to an arbitrary accuracy. This novel algorithm (cf. (Tuna, 2005)) may stand as an alternative for the known JSR approximation algorithms for certain applications.

2. DEFINITIONS AND NOTATION

Nonnegative integers are denoted by $\mathbb{N}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to *class*-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is zero at zero, continuous, and strictly increasing. It belongs to *class*-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it is also unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to *class-$\mathcal{KL}$* ($\beta \in \mathcal{KL}$) if it satisfies: for all $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s \to 0^+} \beta(s, t) = 0$, and for all $s \geq 0$, $\beta(s, \cdot)$ is nondecreasing and $\lim_{t \to \infty} \beta(s, t) = 0$. Notation $|\cdot|$ stands for some p-norm. Let u (and sometimes u_i) denote a unit vector in $\mathbb{R}^n$, i.e. $|u| = 1$.

An operator $f : \mathbb{R}^n \to \mathbb{R}^n$ is *homogeneous* (with respect to standard dilation) if for all $x \in \mathbb{R}^n$ and $\lambda \geq 0$ it satisfies $f(\lambda x) = \lambda f(x)$. The following map in $\mathbb{R}^2$, for instance,

$$f(x_1, x_2) = \begin{bmatrix} |x_1^2 - x_2^2|^{1/2} \\ |x_1 x_2|^{1/2} - x_1 \end{bmatrix}$$

is a homogeneous operator.

Consider the (discrete-time) *switched system*

$$x^+ = f_q(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the *state* and $q \in \{1, 2, \ldots, \bar{q}\}$ ($\bar{q} \in \mathbb{N}$) is the *index* that determines the right-hand side by selecting a transition map from a parametrized family of locally bounded operators $\{f_1, f_2, \ldots, f_{\bar{q}}\}$, and x^+ is the state at the next time instant. The *solution* of system (1) at time $k \in \mathbb{N}$, having started at the initial condition x, having evolved under the influence of an index sequence $\mathbf{q} := \{q_1, q_2, \ldots\}$ with $q_i \in \{1, 2, \ldots, \bar{q}\}$ is denoted by $\phi(k, x, \mathbf{q})$. For notational convenience we will sometimes write $f_q x$ instead of $f_q(x)$.

System (1) is *asymptotically stable* if there exists $\beta \in \mathcal{KL}$ such that $|\phi(k, x, \mathbf{q})| \leq \beta(|x|, k)$ for all k, x, and $\mathbf{q}$. A continuous map $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *Lyapunov function* for system (1) if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{K}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\max_q V(f_q x) - V(x) \leq -\alpha_3(|x|)$$

for all x. We below make our first assumption which will henceforth hold.

Assumption 1. System (1) is homogeneous, i.e. for each index, f_q is homogeneous.

Observe that, due to homogeneity, the solution satisfies $\phi(k, \lambda x, \mathbf{q}) = \lambda \phi(k, x, \mathbf{q})$ for all k, λ, x, and $\mathbf{q}$. Given x, $\mathbf{q}$, and k let $\Phi_k(x) := \max_{\mathbf{q}} |\phi(k, x, \mathbf{q})|$. Then, we define the *(maximum) growth rate* of system (1) as

$$\sigma := \limsup_{k \to \infty} \left\{ \sup_{|x|=1} \Phi_k(x) \right\}^{1/k} . \tag{2}$$

Remark 2. The growth rate is independent of the choice of the norm for all p-norms are equivalent.

For an homogeneous operator $f : \mathbb{R}^n \to \mathbb{R}^n$ we define the *norm of f* as $|f| := \sup_u |fu|$. Finally, let $\mathbf{f} := \max_q |f_q|$ and $\mu := \inf_{u,\, q} |f_q u|$.

3. GROWTH RATE UNDER HOMOGENEITY

In this section we study the growth rate of switched homogeneous systems. The results we establish are the generalizations of what has been known for switched linear systems.

Lemma 3. Given homogeneous operators f, g, we have $|fg| \leq |f||g|$.

Claim 4. Growth rate σ is finite.

Lemma 5. In (2) the limit exists, i.e.

$$\lim_{k \to \infty} \left\{ \sup_{|x|=1} \Phi_k(x) \right\}^{1/k} = \sigma .$$

PROOF. For compactness let us adopt the shorthand notation Φ_k instead of $\sup_{|x|=1} \Phi_k(x)$. By Lemma 3 we can write

$$\begin{aligned} \Phi_k &= \max_{\mathbf{q}} |f_{q_k} f_{q_{k-1}} \cdots f_{q_1}| \\ &\leq \left\{ \max_{\{q_k,\, q_{k-1},\, \ldots,\, q_{m+1}\}} |f_{q_k} f_{q_{k-1}} \cdots f_{q_{m+1}}| \right\} \\ &\quad \times \left\{ \max_{\{q_m,\, q_{m-1},\, \ldots,\, q_1\}} |f_{q_m} f_{q_{m-1}} \cdots f_{q_1}| \right\} \\ &= \Phi_{k-m} \Phi_m . \end{aligned} \tag{3}$$

Let us first show that for each $m \in \mathbb{N}_{\geq 1}$ and real $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}_{\geq 1}$ such that for all $k \geq m_0$ we have $\Phi_k^{1/k} \leq \Phi_m^{1/m} + \varepsilon$.

The result follows trivially if $\Phi_1 = \mathbf{f} = 0$. Hence we suppose otherwise. Let us be given a pair (m, ε). First choose $\nu > 0$ such that $(1 + \varepsilon/\Phi_1)/(1 + \nu) > 1$. Then choose $l_0 \in \mathbb{N}_{\geq 1}$ large enough so that both $\Phi_1^{1/l_0} \leq (1 + \varepsilon/\Phi_1)/(1 + \nu)$ and $\Phi_m^{l_0/(m(l_0+1))} \leq \Phi_m^{1/m}(1 + \nu)$ hold. Let $m_0 := l_0 m$. Given $k \geq m_0$ there exist nonnegative integers $l \geq l_0$ and $d \leq m$ such that $k = lm + d$. By (3) we write

$$\begin{aligned} \Phi_k^{1/k} &\leq \{\Phi_{lm} \Phi_d\}^{1/k} = \Phi_{lm}^{1/(lm+d)} \Phi_d^{1/(lm+d)} \\ &\leq \Phi_m^{l/(lm+d)} \Phi_1^{d/(lm+d)} \\ &\leq \max \left\{ \Phi_m^{1/m},\, \Phi_m^{l_0/(m(l_0+1))} \right\} \\ &\qquad \times \max \left\{ \Phi_1^{1/l_0},\, 1 \right\} \\ &\leq \Phi_m^{1/m}(1 + \nu) \cdot (1 + \varepsilon/\Phi_1)/(1 + \nu) \\ &= \Phi_m^{1/m}(1 + \varepsilon/\Phi_1) = \Phi_m^{1/m} + \varepsilon(\Phi_m^{1/m}/\Phi_1) \\ &\leq \Phi_m^{1/m} + \varepsilon \end{aligned}$$

where we used the fact, an implication of (3), that $\Phi_k \leq \Phi_1^k$ for all k. We now move to the second part of the proof where we show that the (nonnegative) sequence $\{\pi_k\} := \{\Phi_k^{1/k}\}$ is a Cauchy sequence. Recall that a sequence $\{a_k\}$ is Cauchy if for each $\delta > 0$ there exist n_0 such that $|a_l - a_m| < \delta$ for all $l, m \geq n_0$. Suppose that $\{\pi_k\}$ is not Cauchy. Then there exists $\delta > 0$ such that for each n_0 there exist $l,\, m \geq n_0$ such that $|\pi_l - \pi_m| \geq \delta$. Suppose there exist integers $m,\, l_m \geq 1$ such that

$$\pi_{l_m} \leq \pi_1 - m\delta/2\,. \tag{4}$$

Due to what we have shown at the beginning of the proof, there exists m_0 such that for all $k \geq m_0$ we have

$$\begin{aligned}\pi_k &\leq \pi_{l_m} + \delta/2 \\ &\leq \pi_1 - (m-1)\delta/2\end{aligned}$$

from which one infers, due to our assumption that $\{\pi_k\}$ is not Cauchy, that there exists $l_{m+1} \geq m_0$ such that

$$\begin{aligned}\pi_{l_{m+1}} &\leq \pi_1 - (m-1)\delta/2 - \delta \\ &= \pi_1 - (m+1)\delta/2\,.\end{aligned}$$

Now recall that $\pi_k \leq \pi_1$ for all k. By assumption hence there must exist l_1 such that $\pi_{l_1} \leq \pi_1 - \delta$ which implies

$$\pi_{l_1} \leq \pi_1 - \delta/2\,.$$

Therefore by induction there exists an index sequence $\{l_1,\, l_2,\, \ldots\}$ such that (4) holds. However the sequence $\{\pi_k\}$ is nonnegative, i.e. bounded from below, which poses a contradiction to our assumption that it is not Cauchy. Hence $\{\pi_k\}$ *is* Cauchy. Recalling that every Cauchy sequence is convergent, the result follows. ■

Remark 6. Observe that for all k

$$\left\{\sup_{|x|=1} \Phi_k(x)\right\}^{1/k} \geq \sigma\,.$$

Lemma 7. For each $\omega > \sigma$ there exists $M \geq 1$ such that $|\phi(k,\, x,\, \mathbf{q})| \leq M\omega^k|x|$ for all $k,\, x,\, \mathbf{q}$.

4. STABILITY

For switched linear systems, the growth rate (known as the joint spectral radius) is a keen indicator of stability. A switched linear system is asymptotically stable if and only if its joint spectral radius is strictly less than unity. This result extends to the more general case of switched homogeneous systems, as indicated by the below result.

Theorem 8. If system (1) is asymptotically stable then and only then $\sigma < 1$.

PROOF. Suppose $\sigma < 1$. Then by Lemma 7 there exist $M \geq 1$ and $\omega \in (\sigma,\, 1)$ such that $|\phi(k,\, x,\, \mathbf{q})| \leq M\omega^k|x|$ for all k, x, and $\mathbf{q}$. Define $\beta_0(s,\, t) := M\omega^t s$. Note that $\beta_0 \in \mathcal{KL}$. Hence the asymptotic stability.

Let us now show the other direction. Suppose that there exists $\beta \in \mathcal{KL}$ such that $|\phi(k,\, x,\, \mathbf{q})| \leq \beta(|x|,\, k)$ for all k, x and $\mathbf{q}$. Let the integer $k_0 \geq 1$ be such that $\beta(1,\, k_0) \leq 1/2$. Let $\sigma_0 := (1/2)^{1/k_0}$. Note that $\sigma_0 < 1$. Then

$$\begin{aligned}\sup_{|x|=1} \Phi_{k_0}(x)^{1/k_0} &\leq \beta(1,\, k_0)^{1/k_0} \\ &\leq \sigma_0\,.\end{aligned} \tag{5}$$

The result follows once we combine (5) and Remark 6. ■

We will need the following classic result for later use.

Theorem 9. System (1) is asymptotically stable if there exists a Lyapunov function for it.

5. APPROXIMATING GROWTH RATE

Due to its importance in determining the stability of a system, it may be essential to compute σ for certain applications. For that reason, we provide an algorithm to calculate the growth rate to an arbitrary accuracy. We begin the section with the following assumptions on system (1) which will henceforth hold.

Assumption 10. For each index, f_q is continuous.

Assumption 11. We have that $\mu > 0$.

Define $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as

$$\rho(s) := \sup_{u_1,\, u_2,\, q} |f_q(u_1 + su_2) - f_q u_1|\,.$$

Claim 12. Function ρ is zero at zero, continuous, and monotonic.

5.1 Gridding unit sphere

Let us be given $\mathcal{G} := \{x^i\}$, a finite *grid* with $i \in \{1,\, 2,\, \ldots,\, \bar{\imath}\}$ ($\bar{\imath} \in \mathbb{N}$) and $|x^i| = 1$ for all i. Then let

$$h := \sup_u \min_i |u - x^i|\,.$$

We define

$$\lambda_{iq} := |f_q x^i| \,.$$

Also, for each i let

$$\mathcal{L}_{iq} := \left\{ j : |f_q x^i - \lambda_{iq} x^j| \le \mathbf{f} h + (2+h)\rho(h) \right\}$$

which is never empty. Then define for $k \in \mathbb{N}$

$$\Psi_{i,k+1} := \max_q \max_{j \in \mathcal{L}_{iq}} \lambda_{iq} \Psi_{j,k} \tag{6}$$

with $\Psi_{i,0} = 1$ for all i.

Lemma 13. Given i, j, q, and u, if $|u - x^i| \le h$ and $||f_q u|^{-1} f_q u - x^j| \le h$ then $j \in \mathcal{L}_{iq}$.

PROOF. Let us be given i, j, q, and u satisfying $|u - x^i| \le h$ and $||f_q u|^{-1} f_q u - x^j| \le h$. Note that there exist u_1 and $\varepsilon \in [0,\, h]$ such that $u = x^i + \varepsilon u_1$. Also note that there exist $\delta_1 \in [0,\, \rho(\varepsilon)]$ and u_2 such that $f_q x^i = f_q u + \delta_1 u_2$. It follows that $|f_q u| - \lambda_{iq} \le \delta_1$. Let u_3 and $\delta_2 \in [0,\, h]$ be such that $|f_q u|^{-1} f_q u = x^j + \delta_2 u_3$. We can now write

$$\begin{aligned}
&f_q x^i - \lambda_{iq} x^j \\
&= f_q u + \delta_1 u_2 - \lambda_{iq} x^j \\
&= |f_q u|(x^j + \delta_2 u_3) + \delta_1 u_2 - \lambda_{iq} x^j \\
&= (|f_q u| - \lambda_{iq} + \lambda_{iq})(x^j + \delta_2 u_3) + \delta_1 u_2 - \lambda_{iq} x^j \\
&= (|f_q u| - \lambda_{iq})(x^j + \delta_2 u_3) + \lambda_{iq} \delta_2 u_3 + \delta_1 u_2
\end{aligned}$$

whence, since $|f_q u| - \lambda_{iq} \le \delta_1$,

$$\begin{aligned}
|f_q x^i - \lambda_{iq} x^j| &\le \delta_1 + \delta_1 \delta_2 + \lambda_{iq} \delta_2 + \delta_1 \\
&= \lambda_{iq} \delta_2 + (2 + \delta_2)\delta_1 \\
&\le \mathbf{f} h + (2+h)\rho(h)
\end{aligned}$$

which implies $j \in \mathcal{L}_{iq}$. ■

Lemma 14. For all u and i, satisfying $|u - x^i| \le h$, and k we have

$$\Phi_k(u) \le (1 + \rho(h)/\mu)^k \, \Psi_{i,k} \,. \tag{7}$$

PROOF. Suppose that there exists k such that for all u_1 and m satisfying $|u_1 - x^m| \le h$ we have

$$\Phi_k(u_1) \le (1 + \rho(h)/\mu)^k \, \Psi_{m,k} \,. \tag{8}$$

Now let us be given some u_2. Let i, δ_1, and u_3 be such that $x^i + \delta_1 u_3 = u_2$ and $\delta_1 \in [0,\, h]$. Let q be such that $\Phi_{k+1}(u_2) = \Phi_k(f_q u_2)$. There exist $\delta_2 \in [0,\, h]$, j, and u_4 such that

$$f_q u_2 = |f_q u_2|(x^j + \delta_2 u_4) \,.$$

By Lemma 13 we have that $j \in \mathcal{L}_{iq}$. Therefore $\lambda_{iq} \Psi_{j,k} \le \Psi_{i,k+1}$. We can write

$$\begin{aligned}
\Phi_{k+1}(u_2) &= |f_q(x^i + \delta_1 u_3)| \Phi_k(x^j + \delta_2 u_4) \\
&\le (|f_q x^i| + \rho(\delta_1)) \Phi_k(x^j + \delta_2 u_4) \\
&\le (\lambda_{iq} + \rho(h)) \Phi_k(x^j + \delta_2 u_4) \\
&\le \lambda_{iq} \left(1 + \rho(h)/\mu\right) \Phi_k(x^j + \delta_2 u_4) \\
&\le (1 + \rho(h)/\mu)^{k+1} \, \lambda_{iq} \Psi_{j,k} \\
&\le (1 + \rho(h)/\mu)^{k+1} \, \Psi_{i,k+1} \,.
\end{aligned}$$

Eq. (8) trivially holds for $k = 0$. Hence the result by induction. ■

5.2 Approximate growth rate

Let us now define the *approximate growth rate* γ, which depends on system (1) and our choice of grid $\mathcal{G}$, as

$$\gamma := \limsup_{k \to \infty} \left\{ \max_i \Psi_{i,k} \right\}^{1/k} \,. \tag{9}$$

Claim 15. The approximate growth rate is finite.

Lemma 16. In (9) the limit exists, i.e.

$$\lim_{k \to \infty} \left\{ \max_i \Psi_{i,k} \right\}^{1/k} = \gamma \,.$$

Below result is a direct implication of Lemma 14.

Theorem 17. We have that $\sigma \le (1 + \rho(h)/\mu)\gamma$.

Theorem 17 is practically important. It implies that if the computed approximate growth rate satisfies $\gamma < (1 + \rho(h)/\mu)^{-1}$ then one can be sure, by Theorem 8, that system (1) is asymptotically stable. Of course, it would not be of much worth if γ did not converge to σ as the grid gets finer. Although we have not been able to come up with an explicit lower bound on σ in terms of γ and h, it is still the case that γ converges to σ as the grid fineness h gets smaller. Below result formalizes that observation.

Theorem 18. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $h \le \delta$ then $|\sigma - \gamma| \le \varepsilon$.

PROOF. Let us be given $\varepsilon > 0$. We let $r := \sigma + \varepsilon$ and define

$$V(x) := \sup_{\mathbf{q}} \sum_{k=0}^{\infty} r^{-k} |\phi(k,\, x,\, \mathbf{q})| \,.$$

Due to homogeneity, V satisfies $V(\lambda x) = \lambda V(x)$. Observe that $V(x) \ge |x|$. By Lemma 7 there exists $M \ge 1$ such that $|\phi(k,\, x,\, \mathbf{q})| \le M(\sigma + \varepsilon/2)^k |x|$ for all k, x, and $\mathbf{q}$. If we let $c := (\sigma + \varepsilon/2)/(\sigma + \varepsilon) < 1$, then it follows that $V(x) \leq M(1-c)^{-1}|x|$. We can write therefore

$$|x| \leq V(x) \leq M(1-c)^{-1}|x|\,.$$

Then we observe that V satisfies

$$\max_q V(r^{-1}f_q x) - V(x) \leq -|x|\,.$$

Finally, V is continuous due to Assumption 10. As a result, V is a Lyapunov function for the system

$$x^+ = r^{-1}f_q x\,.$$

Let α be a class-$\mathcal{K}$ function satisfying

$$\alpha(s) \geq \sup_{|x|\leq \mathbf{f},\,u} |V(x+su) - V(x)|\,.$$

Such α exists since V is continuous. Now let us pick $\delta > 0$ such that the following inequalities hold.

$$r^{-1}\alpha(\mathbf{f}\delta + (2+\delta)\rho(\delta)) + \alpha(\delta) \leq 1/2 \qquad (10)$$

$$\mathbf{f}\rho(\delta)/\mu \leq \varepsilon\,. \qquad (11)$$

We are able to choose such δ for ρ is monotonic by Claim 12, continuous, and zero at zero. Now let grid $\mathcal{G}$ be such that $h \leq \delta$. Let us define an auxiliary system

$$\xi^+ = g_t \xi \qquad (12)$$

for $t \in \{1, 2, \ldots, \bar{t}\}$ such that (i) system (12) is homogeneous and (ii) for each $|\xi| = 1$ we have $\{g_t\xi : t \in \{1, 2, \ldots, \bar{t}\}\} = \{\lambda_{iq}x^j : q \in \{1, 2, \ldots, \bar{q}\},\ |\xi - x^i| \leq h,\ j \in \mathcal{L}_{iq}\}$. Observe that $|g_t| \leq \mathbf{f}$ for all t. Hence the locally boundedness. Our second observation is that the growth rate of system (12) equals γ.

Given ξ, satisfying $|\xi| = 1$, let i be such that $|\xi - x^i| \leq h$. Then let j and q_0 be such that $j \in \mathcal{L}_{iq_0}$ and $\max_t V(r^{-1}g_t\xi) = r^{-1}V(\lambda_{iq_0}x^j)$. Since $j \in \mathcal{L}_{iq_0}$ there exists $\omega_1 \leq \mathbf{f}h + (2+h)\rho(h)$ and u_1 such that $\lambda_{iq_0}x^j = f_{q_0}x^i + \omega_1 u_1$. Moreover, there exist $\omega_2 \leq h$ and u_2 be such that $\xi = x^i + \omega_2 u_2$. We can write by (10)

$$\begin{aligned}
&\max_t V(r^{-1}g_t\xi) - V(\xi)\\
&= r^{-1}V(\lambda_{iq_0}x^j) - V(\xi)\\
&= r^{-1}V(\Lambda_{q_0}x^i + \omega_1 u_1) - V(x^i + \omega_2 u_2)\\
&\leq r^{-1}\left(V(\Lambda_{q_0}x^i) + \alpha(\omega_1)\right) - V(x^i) + \alpha(\omega_2)\\
&\leq \max_q V(r^{-1}f_q x^i) - V(x^i) + r^{-1}\alpha(\omega_1) + \alpha(\omega_2)\\
&\leq \max_q V(r^{-1}f_q x^i) - V(x^i) + 1/2\\
&= -1/2\,. \qquad (13)
\end{aligned}$$

Now let us be given an arbitrary ξ. If $|\xi| = 0$ then we can trivially write $\max_t V(r^{-1}g_t\xi) - V(\xi) \leq -|\xi|/2$ since each term in the inequality is zero. If $|\xi| > 0$ then we can write by homogeneity and (13) that

$$\begin{aligned}
&\max_t V(r^{-1}g_t\xi) - V(\xi)\\
&= |\xi|\left(\max_t V\left(r^{-1}g_t\frac{\xi}{|\xi|}\right) - V\left(\frac{\xi}{|\xi|}\right)\right)\\
&\leq -|\xi|/2
\end{aligned}$$

since $\xi/|\xi|$ is a unit vector. Therefore V is a Lyapunov function for the system

$$\xi^+ = r^{-1}g_t\xi \qquad (14)$$

and by Theorem 9 system (14) is stable. Note that since the growth rate of system (12) is γ, the growth rate of system (14) has to be γ/r. Stability, by Theorem 8, implies that $\gamma/r < 1$. Therefore

$$\begin{aligned}
\gamma &\leq r\\
&= \sigma + \varepsilon\,. \qquad (15)
\end{aligned}$$

Recall Lemma 14. Combining (7) with Lemma 16 and that $\gamma \leq \mathbf{f}$ one obtains

$$\begin{aligned}
\sigma &\leq (1 + \rho(h)/\mu)\gamma\\
&\leq \gamma + \mathbf{f}\rho(\delta)/\mu\\
&\leq \gamma + \varepsilon \qquad (16)
\end{aligned}$$

thanks to (11). All that is left is to put together (15) and (16). ■

Remark 19. Assumption 11 can be removed but is preferred to be had for the ease of analysis. To be precise, Theorem 18 would still hold without it. However, with it the proof is simpler. Moreover, the upperbound in Lemma 14 can be bettered in a way that obviates Assumption 11. It may also be of worth to point out that if f_q are obtained via sample and hold from continuous-time homogeneous (with degree zero) systems, then the assumption comes for free.

The following remark is for the case when ρ is not known exactly but an upperbound is known instead.

Remark 20. Let α be any class-$\mathcal{K}$ function satisfying $\alpha(s) \geq \mathbf{f}s + (2+s)\rho(s)$ and

$$\mathcal{L}^*_{iq} := \left\{j : |f_q x^i - \lambda_{iq}x^j| \leq \alpha(h)\right\}\,.$$

Then Lemma 14 and Theorem 18 would still hold if $\mathcal{L}_{iq}$ were replaced by $\mathcal{L}^*_{iq}$. However, then, a finer grid (i.e. a smaller h) would possibly be required to attain a desired closeness of the approximate growth rate γ to the actual growth rate σ.

Remark 21. For $k \in \mathbb{N}$ consider the recursion

$$\gamma_{i,k+1} = \max_{q} \max_{j \in \mathcal{L}_{iq}} \lambda_{iq}^{\frac{1}{k+1}} \gamma_{j,k}^{\frac{k}{k+1}} \tag{17}$$

with $\gamma_{i,0} = 1$ for all i. Note that $\gamma_{i,k}^k = \Psi_{i,k}$ and therefore $\lim_{k\to\infty} \max_i \gamma_{i,k} = \gamma$. Also note that $\mu \leq \gamma_{i,k} \leq \mathbf{f}$ for all i, k whereas $\Psi_{i,k}$ for some i may either blow up or vanish as $k \to \infty$. Hence for numerical calculations, (17) is more preferable than (6).

6. HOMOGENEITY CONSIDERED IN GENERAL

In this section we consider homogeneity in a more general sense and extend our analysis on growth rate. First, we need the following definition. A *dilation* Δ is such that for each λ (nonnegative) $\Delta_\lambda = \operatorname{diag}(\lambda^{r_1}, \lambda^{r_2}, \ldots, \lambda^{r_n})$ with fixed $r_i > 0$.

Consider the following system in $\mathbb{R}^n$

$$\xi^+ = \Gamma_q(\xi) \tag{18}$$

that is homogeneous with respect to Δ, i.e. $\Gamma_q(\Delta_\lambda x) = \Delta_\lambda \Gamma_q(x)$ for all q, λ, and ξ. Assume that Γ_q is a locally bounded operator for each q. Let ψ denote the solution of system (18). The homogeneous (w.r.t. Δ) norm is defined as

$$\|\xi\| := \left(|\xi_1|^{p/r_1} + |\xi_2|^{p/r_2} + \ldots + |\xi_n|^{p/r_n}\right)^{1/p}$$

for some $p \geq 1$. Let us define for $k \in \mathbb{N}$, ${}^{\Delta}\Phi_k(\xi) := \max_{\mathbf{q}} \|\psi(k, \xi, \mathbf{q})\|$. We then define the growth rate of system (18) as

$${}^{\Delta}\sigma := \limsup_{k\to\infty} \left\{ \sup_{\|\xi\|=1} {}^{\Delta}\Phi_k(\xi) \right\}^{1/k}.$$

Remark 22. Growth rate ${}^{\Delta}\sigma$ is norm-independent.

For $b \in \mathbb{R}$ and $r > 0$, let $b^{[r]} := \operatorname{sgn}(b)|b|^r$. Now let us define a transformation $H : \mathbb{R}^n \to \mathbb{R}^n$ as

$$Hx := \left(x_1^{[r_1]}, x_2^{[r_2]}, \ldots, x_n^{[r_n]}\right).$$

Note that H^{-1} exists and is

$$H^{-1}\xi := \left(\xi_1^{[r_1^{-1}]}, \xi_2^{[r_2^{-1}]}, \ldots, \xi_n^{[r_n^{-1}]}\right).$$

We observe that $H\lambda = \Delta_\lambda H$ and $H^{-1}\Delta_\lambda = \lambda H^{-1}$. We also observe that $\{H^{-1}\Gamma_q H\}\lambda = \lambda\{H^{-1}\Gamma_q H\}$ and $\{H^{-1}\Gamma_q H\}$ is locally bounded for each q. Therefore we may without loss of generality assume that system (18) and system (1) satisfy the following relation $H^{-1}\Gamma_q H = f_q$. The following result then accrues.

Theorem 23. We have that ${}^{\Delta}\sigma = \sigma$.

Remark 24. The practical importance of Theorem 23 is in that the growth rate of an arbitrary homogeneous system $\xi^+ = \Gamma_q \xi$ can be computed by the approximation algorithm presented in Section 5 applied to the auxiliary system $x^+ = f_q x$ which is only homogeneous with respect to standard dilation.

We end the section with the following results.

Lemma 25. For each $\omega > {}^{\Delta}\sigma$ there exists $M \geq 1$ such that $\|\psi(k, \xi, \mathbf{q})\| \leq M\omega^k \|\xi\|$ for all k, ξ, $\mathbf{q}$.

Theorem 26. If system (18) is asymptotically stable then and only then ${}^{\Delta}\sigma < 1$.

7. CONCLUSION

We define the growth rate for a homogeneous system under arbitrary switching. We then show that the system is stable if and only if its growth rate is less than unity. We also provide an approximation algorithm to compute the growth rate to an arbitrary accuracy.

REFERENCES

Blondel, V.D. and Y. Nesterov (2005). Computationally efficient approximations of the joint spectral radius. *SIAM Journal of Matrix Analysis* **27(1)**, 256–272.

Filippov, A.F. (1980). Stability conditions in homogeneous systems with arbitrary regime switching. *Automation and Remote Control* **41**, 1078–1085.

Gurvits, L. (1995). Stability of discrete linear inclusion. *Linear Algebra Appl.* **231**, 47–85.

Holcman, D. and M. Margaliot (2003). Stability analysis of second-order switched homogeneous systems. *SIAM Journal on Control and Optimization* **41(5)**, 1609–1625.

Liberzon, D. and A.S. Morse (1999). Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine* **19(5)**, 59–70.

Shorten, R., F. Wirth, O. Mason, K. Wulff and C. King (2007). Stability criteria for switched and hybrid systems. (to appear). *SIAM Review.*

Tuna, S.E. (2005). Optimal regulation of homogeneous systems. *Automatica* **41(11)**, 1879–1890.

Tuna, S.E. and A.R. Teel (2005). Regulating discrete-time homogeneous systems under arbitrary switching. In: *Proc. 44th Conference on Decision and Control.* pp. 2586–2591.