ACTIVE MODE OBSERVABILITY OF SWITCHING LINEAR SYSTEMS ⋆

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Abstract

In this paper active mode observability is addressed for a class of discrete-time linear systems that may switch in an unknown and unpredictable way among different modes taken from a finite set. The active mode observation problem consists in determining control sequences (discerning control sequences) that allow to identify the switching sequence on the basis of the observations. The presence of unknown but bounded noises affecting both the system and measurement equations is taken into account. In particular, a general condition is derived that characterizes discerning controls in a finite-horizon setting. Such a result is extended to the infinite-horizon case in order to derive "persistently discerning" control sequences. Numerical examples are reported to clarify the approach.

Key words: active estimation, linear systems, mode observability, switching systems.

1 Introduction

Over the past decade, particular attention has been devoted to the study of *hybrid systems*. Such systems result from the interaction of continuous dynamics, discrete dynamics, and logic decisions, and can be used to describe a wide range of physical and engineering systems. For different classes of hybrid systems, specific definitions of observability can be given. For jump linear systems, i.e., systems in which the evolution of the discrete dynamics is governed by Markov processes, the reader is referred to [9] for a survey and to [6,5] for recent developments. For piece-wise affine systems, where the discrete state is a piece-wise function of the previous continuous state, the reader is referred to [3,4]. Finally, for systems where the control input is extended to the discrete state (the discrete state is a control variable) the reader is referred to [12].

In this paper, we focus on a particular class of hybrid systems: *switching discrete-time linear systems*, i.e., linear systems in which the matrices governing the system and measurement equations can take values in a finite set (the index denoting such values being called the "mode" or the "discrete state"). We assume the system matrices may switch at each time instant in an unknown and unpredictable way. For such a class of systems, we investigate the connection between the choice of the control sequence and the observability of the system mode. More specifically, the problem we address consists in looking for suitable control sequences such that the switching sequence can be reconstructed on the basis of the observations. We call this problem *active mode observation*, borrowing the term "active" from the literature on active estimation/identification (see [8,7,11]). It is important to remark that the knowledge of the past discrete states may be of interest in many practical situations. A first possible application is fault diagnosis, as the discrete state is often introduced to model possible faults in a plant. Moreover once the discrete state is known, standard filtering techniques can be efficiently applied to estimate the continuous state of the system.

For unforced noise-free switching systems (where the control is absent), the problem of determining the

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switching sequence on the basis of the observations was first addressed systematically in [13], under the assumption of a minimum dwell time between consecutive switches. More recent advances on this topic have been developed in [2], where arbitrary switching sequences were considered. In [1], such results have been extended to comply with the presence of bounded disturbances that corrupt the dynamics and the measures. It is important to point out that, for unforced systems, the possibility of exactly reconstructing the sequence of discrete states depends on the initial continuous state of the system. In fact, even in the best-case scenario, it is not possible to determine uniquely the switching sequence when the initial continuous state is null (or, in the noisy case, when it belongs to a non-empty neighborhood of the origin). In [2], it has been shown that, for noise-free systems, such a drawback may be overcome by means of a suitable choice of the controls, thus making it possible to determine uniquely the switching sequence over a finite horizon on the basis of the observations, regardless of the initial continuous state. Such controls are called *discerning controls*.

The original contributions of this paper can be summarized as follows: i) in the finite horizon setting, the results of [2] are extended to the noisy case and a characterization of discerning control sequence in the presence of unknown but bounded noises is derived; ii) it is proved that, when the mode observability property holds over a finite horizon, then it is possible to derive a *persistently discerning* control sequence over an infinite horizon that, at any time, allows one to reconstruct the discrete state up to the current time; iii) a sequential scheme is proposed that allows to derive a persistently discerning control sequence by choosing the control vectors on line, one at each time stage.

2 Mode observability over a finite horizon

Let us consider switching discrete-time linear systems described by

$$x_{t+1} = A(\lambda_t)\, x_t + B(\lambda_t)\, u_t + w_t \tag{1a}$$
$$y_t = C(\lambda_t)\, x_t + v_t \tag{1b}$$

where $t = 0, 1, \dots$ is the time instant, $x_t \in \mathbb{R}^n$ is the continuous state vector (the initial state x_0 is unknown), $\lambda_t \in \mathcal{L} \triangleq \{1, 2, \dots, L\}$ is the discrete state (or "mode" of the system), $u_t \in \mathbb{R}^k$ is the control vector, $w_t \in \mathcal{W} \subset \mathbb{R}^n$ is the system noise vector, $y_t \in \mathbb{R}^m$ is the vector of the measurements, and $v_t \in \mathcal{V} \subset \mathbb{R}^m$ is the measurement noise vector. $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$, $\lambda \in \mathcal{L}$, are $n \times n$, $n \times k$, and $m \times n$ matrices, respectively. We assume the statistics of the vectors x_0, w_t, and v_t as well as the law governing the evolution of the discrete state to be unknown.

In this section, we study the observability of the discrete state of system (1) over a finite horizon of length $N+1$. More specifically, we want to investigate whether it is possible to choose the control sequence [1] $\mathbf{u}_{0,N-1}$ in such a way that the switching sequence $\boldsymbol{\lambda}_{0,N}$ (or at least a portion of it) can be reconstructed on the basis of the observation sequence $\mathbf{y}_{0,N}$ and of the control sequence $\mathbf{u}_{0,N-1}$ for any possible initial continuous state and any noise sequence. Since the law governing the evolution of the discrete state is supposed to be completely unknown, the switching sequence $\boldsymbol{\lambda}_{0,N}$ may take on any value in the set $\mathcal{L}^{N+1}$. It it important to remark that the proposed approach is well-suited to take into account also more complicate frameworks (e.g., the case in which the evolution of the discrete state is governed by a hidden finite state machine). This would simply require to consider a restricted set $\mathcal{P}_N \subseteq \mathcal{L}^{N+1}$ of switching sequences.

In order to address the active mode observation problem some preliminary definitions are useful. First note that the observation sequence $\mathbf{y}_{0,N}$ can be written as

$$\mathbf{y}_{0,N} = F(\boldsymbol{\lambda}_{0,N})x_0 + G(\boldsymbol{\lambda}_{0,N})\mathbf{u}_{0,N-1} + H(\boldsymbol{\lambda}_{0,N})\mathbf{w}_{0,N-1} + \mathbf{v}_{0,N},$$

where $F(\boldsymbol{\lambda}_{0,N})$ is the observability matrix and $G(\boldsymbol{\lambda}_{0,N})$, $H(\boldsymbol{\lambda}_{0,N})$ are suitable matrices (see [1]).

Furthermore, let us denote by $\mathcal{S}(\boldsymbol{\lambda}_{0,N}, \mathbf{u}_{0,N-1})$ the set of all the possible observation sequences associated with the switching sequence $\boldsymbol{\lambda}_{0,N}$ and the control sequence $\mathbf{u}_{0,N-1}$ for any possible initial continuous state and any possible noise sequence, i.e.,

$$\mathcal{S}(\boldsymbol{\lambda}_{0,N}, \mathbf{u}_{0,N-1}) \triangleq \Big\{ \mathbf{y} \in \mathbb{R}^{m(N+1)} : \mathbf{y} = F(\boldsymbol{\lambda}_{0,N})x + G(\boldsymbol{\lambda}_{0,N})\mathbf{u}_{0,N-1} + H(\boldsymbol{\lambda}_{0,N})\mathbf{w} + \mathbf{v}, \quad x \in \mathbb{R}^n, \mathbf{w} \in \mathcal{W}^N, \mathbf{v} \in \mathcal{V}^{N+1} \Big\}.$$

As the observation sequence $\mathbf{y}_{0,N}$ depends on the control sequence $\mathbf{u}_{0,N-1}$, one may think of suitably choosing $\mathbf{u}_{0,N-1}$ in order to make it possible to uniquely determine the switching sequence $\boldsymbol{\lambda}_{0,N}$ or, at least, a portion of it. More specifically, in the lines of [1], we shall look for two integers, α and ω, with $\alpha, \omega \geq 0$ and $\alpha + \omega \leq N$, such that it is possible to uniquely determine the discrete state λ_t in the restricted interval $[\alpha, N-\omega]$ on the basis of the observation sequence $\mathbf{y}_{0,N}$ and of the control sequence $\mathbf{u}_{0,N-1}$. Towards this end, given a switching sequence $\boldsymbol{\lambda}$ in the interval $[0, N]$, let

[1] Given a generic sequence $\mathbf{z}_{0,\infty} \triangleq \{z_t;\, t = 0, 1, \dots\}$ and two time instants $t_1 \leq t_2$, we define $\mathbf{z}_{t_1,t_2} \triangleq \mathrm{col}\,(z_{t_1}, z_{t_1+1}, \dots, z_{t_2})$.

us denote by $r^{\alpha,\omega}(\boldsymbol{\lambda})$ the restriction of $\boldsymbol{\lambda}$ to the interval $[\alpha, N-\omega]$. Of course, from a practical point of view, it would be preferable to have $\alpha = 0$ and $\omega = 0$; unfortunately, in some cases, reconstructing the entire sequence $\boldsymbol{\lambda}_{0,N}$ turns out to be an impossible task (see, for instance, the example system of Section 4). Note that a similar idea was proposed in [2], where only the case $\alpha = 0$ and $\omega \geq 0$ was considered. Here, the presence of α introduces a further degree of freedom in the mode observation scheme. We are now ready to give the following definition.

Definition 1 *System (1) is said to be* (N, α, ω)-mode observable *if there exists a control sequence* $\mathbf{u}_{0,N-1}$ *such that, for every pair of switching sequences* $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ *with* $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$, *we have*

$$\mathcal{S}(\boldsymbol{\lambda}, \mathbf{u}_{0,N-1}) \bigcap \mathcal{S}(\boldsymbol{\lambda}', \mathbf{u}_{0,N-1}) = \emptyset\,.$$

A control sequence $\mathbf{u}_{0,N-1}$ *that has such a property is called an* (N, α, ω)-discerning control sequence *for system (1).*

It is worth noting that such a definition extends to noisy switching systems the concept of *Strong Mode Observability* proposed in [2] in the noise-free case. According to Definition 1, if system (1) is (N, α, ω)-mode observable, then it is possible to find a control sequence $\mathbf{u}_{0,N-1}$ such that different switching sequences in the interval $[\alpha, N-\omega]$ generate different observation sequences in the interval $[0, N]$, regardless of the initial states and the noise sequences. Clearly, if $\mathbf{u}_{0,N-1}$ is an (N, α, ω)-discerning control sequence, then the switching sequence $\boldsymbol{\lambda}_{\alpha,N-\omega} = r^{\alpha,\omega}(\boldsymbol{\lambda}_{0,N})$ can be uniquely determined from the observation sequence $\mathbf{y}_{0,N}$. It is important to point out that such a goal can be accomplished only by means of a suitable choice of the control sequence. In fact, when the switching system is not controlled (i.e., the term $B(\lambda_t)u_t$ in equation (1a) is absent), it is not possible to make the sets $\mathcal{S}(\boldsymbol{\lambda}, 0)$ and $\mathcal{S}(\boldsymbol{\lambda}', 0)$ completely disjoint and so the switching sequence $\boldsymbol{\lambda}_{\alpha,N-\omega}$ can be uniquely determined only for certain initial continuous states (for a thorough discussion on this issue the interested reader is referred to [1]).

In [2], a necessary and sufficient condition was given for the (N, α, ω)-mode observability of system (1) in the noise-free case that is based on the concept of orthogonal projection. Furthermore, on the basis of such a condition, a characterization of all the discerning control sequences was given. The following theorem summarizes some of the results of [2]. [2]

[2] Even if the results in [2] are given for $\alpha = 0$, it is straightforward to verify that they hold also for $\alpha > 0$. We shall then refer to this (more general) case.

Theorem 1 *Suppose that* $\mathcal{V} = \{0\}$ *and* $\mathcal{W} = \{0\}$, *i.e.,* $w_t = 0$ *and* $v_t = 0$ *for* $t = 0, 1, \ldots$. *Then system (1) is* (N, α, ω)*-mode observable if and only if, for every* $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ *with* $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$, *we have*

$$[I - P(\boldsymbol{\lambda}, \boldsymbol{\lambda}')]\,[G(\boldsymbol{\lambda}) - G(\boldsymbol{\lambda}')] \neq 0 \tag{2}$$

where $P(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ *is the matrix of the orthogonal projection on* $\operatorname{span}([F(\boldsymbol{\lambda}) \mid F(\boldsymbol{\lambda}')])$. *Furthermore, the control sequence* $\mathbf{u}_{0,N-1}$ *is an* (N, α, ω)*-discerning control sequence if and only if*

$$[I - P(\boldsymbol{\lambda}, \boldsymbol{\lambda}')]\,[G(\boldsymbol{\lambda}) - G(\boldsymbol{\lambda}')]\,\mathbf{u}_{0,N-1} \neq 0 \tag{3}$$

for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ *with* $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$. □

In the following of the paper, for the sake of brevity, we shall use the definition

$$R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \triangleq [I - P(\boldsymbol{\lambda}, \boldsymbol{\lambda}')]\,[G(\boldsymbol{\lambda}) - G(\boldsymbol{\lambda}')]\,.$$

The set $\mathcal{K}_N^{\alpha,\omega}$ of all the *non-discerning* control sequences in the noise-free case can be defined as

$$\mathcal{K}_N^{\alpha,\omega} \triangleq \bigcup_{\substack{\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1} \\ r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')}} \ker\{R(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\}\,.$$

Provided that condition (2) is satisfied, then for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$, the set of control sequences $\mathbf{u}_{0,N-1}$ that does not satisfy condition (3) (i.e., the kernel of the matrix $R(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$) is a null set (i.e., a set with a null Lebesgue measure). As a consequence, also $\mathcal{K}_N^{\alpha,\omega}$ is a null set, since it is obtained as the union of a finite number of null sets.

In the following of this section, we shall show how, given a discerning control sequence for the noise-free system, it is possible to derive a discerning control sequence for system (1) in the presence of unknown but bounded noises. With this respect, according to Definition 1, we would like to find a control sequence $\mathbf{u}_{0,N-1}$ such that, for every pair of switching sequences $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$, we have

$$\begin{aligned} &F(\boldsymbol{\lambda})x + G(\boldsymbol{\lambda})\mathbf{u}_{0,N-1} + H(\boldsymbol{\lambda})\mathbf{w} + \mathbf{v} \\ &\neq F(\boldsymbol{\lambda}')x' + G(\boldsymbol{\lambda}')\mathbf{u}_{0,N-1} + H(\boldsymbol{\lambda}')\mathbf{w}' + \mathbf{v}' \end{aligned} \tag{4}$$

for every pair of initial states x, x', for every pair of system noise sequences $\mathbf{w}$, $\mathbf{w}' \in \mathcal{W}^N$, and for every pair of measurement noise sequences $\mathbf{v}$, $\mathbf{v}' \in \mathcal{V}^{N+1}$. Towards this end, let us rewrite condition (4) in the equivalent form

$$[-F(\boldsymbol{\lambda}) \mid F(\boldsymbol{\lambda}')] \begin{bmatrix} x \\ x' \end{bmatrix} \neq [G(\boldsymbol{\lambda}) - G(\boldsymbol{\lambda}')]\,\mathbf{u}_{0,N-1}$$

$$+H(\boldsymbol{\lambda})\mathbf{w} + \mathbf{v} - H(\boldsymbol{\lambda}')\mathbf{w}' - \mathbf{v}' . \quad (5)$$

Clearly, condition (5) is satisfied if and only if the right-hand side *is not* in the span of the columns of $[F(\boldsymbol{\lambda}) \mid F(\boldsymbol{\lambda}')]$, or, equivalently, if and only if the projection of the right-hand side on the subspace orthogonal to $\operatorname{span}([F(\boldsymbol{\lambda}) \mid F(\boldsymbol{\lambda}')])$ is not null, that is,

$$[I - P(\boldsymbol{\lambda}, \boldsymbol{\lambda}')] \Big\{ [G(\boldsymbol{\lambda}) - G(\boldsymbol{\lambda}')] \, \mathbf{u}_{0,N-1} + H(\boldsymbol{\lambda})\mathbf{w} + \mathbf{v} - H(\boldsymbol{\lambda}')\mathbf{w}' - \mathbf{v}' \Big\} \neq 0 . \quad (6)$$

If, for the sake of brevity, we define the following set

$$\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \triangleq \Big\{ z \in \mathbb{R}^{m(N+1)} : z = [P(\boldsymbol{\lambda}, \boldsymbol{\lambda}') - I] \times [H(\boldsymbol{\lambda})\mathbf{w} + \mathbf{v} - H(\boldsymbol{\lambda}')\mathbf{w}' - \mathbf{v}'] ; \; \mathbf{w}, \mathbf{w}' \in \mathcal{W}^N , \; \mathbf{v}, \mathbf{v}' \in \mathcal{V}^{N+1} \Big\} ,$$

then condition (6) turns out to be

$$R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \, \mathbf{u}_{0,N-1} \notin \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}') . \quad (7)$$

Note that the boundedness of the sets $\mathcal{W}$ and $\mathcal{V}$ implies the boundedness of the set $\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$. Let us now suppose that $\mathbf{u}'_{0,N-1}$ is an (N, α, ω)-discerning control sequence for the noise-free system; then $R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \, \mathbf{u}'_{0,N-1} \neq 0$ (see Theorem 1). As a consequence, it is always possible to "stretch" the vector $R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \, \mathbf{u}'_{0,N-1}$ so that it results outside the bounded set $\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$. Formally, there always exists a suitable positive scalar γ such that $\gamma R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \, \mathbf{u}'_{0,N-1} \notin \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$. By defining the scalar $\bar{\gamma}(\mathbf{u}'_{0,N-1}, \boldsymbol{\lambda}, \boldsymbol{\lambda}')$ as

$$\bar{\gamma}(\mathbf{u}'_{0,N-1}, \boldsymbol{\lambda}, \, \boldsymbol{\lambda}') \triangleq \sup_{\gamma > 0 : \; \gamma R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \, \mathbf{u}'_{0,N-1} \in \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')} \gamma , \quad (8)$$

it is immediate to see that $R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \, \mathbf{u}_{0,N-1} \notin \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ for all $\mathbf{u}_{0,N-1} = \gamma \, \mathbf{u}'_{0,N-1}$ with $\gamma > \bar{\gamma}(\mathbf{u}'_{0,N-1}, \boldsymbol{\lambda}, \boldsymbol{\lambda}')$ (for the reader's convenience, an example of such quantities is shown in Fig. 1). Then, in order to satisfy condition (7) (and consequently condition (4)) for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$, it is sufficient to choose the scalar parameter γ so that

$$\gamma > \bar{\gamma}(\mathbf{u}'_{0,N-1}) \triangleq \max_{\substack{\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1} \\ r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')}} \bar{\gamma}(\mathbf{u}'_{0,N-1}, \boldsymbol{\lambda}, \boldsymbol{\lambda}') .$$

The foregoing can be summarized in the following theorem.

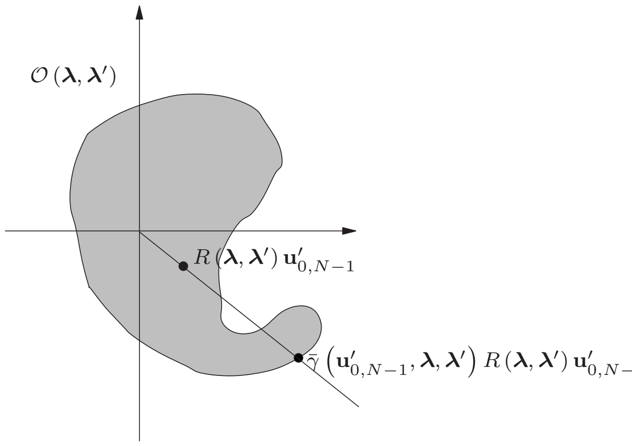

Fig. 1. Graphical interpretation of the condition on the control sequence in the presence of bounded noises

Theorem 2 *Suppose that the sets* $\mathcal{W}$ *and* $\mathcal{V}$ *are bounded. Furthermore, suppose that the noise-free system is* (N, α, ω)*-mode observable and let* $\mathbf{u}'_{0,N-1}$ *be an* (N, α, ω)*-discerning control sequence for the noise-free system. Then there exists a positive scalar* $\bar{\gamma}(\mathbf{u}'_{0,N-1})$ *such that, for every* $\gamma > \bar{\gamma}(\mathbf{u}'_{0,N-1})$ *, the control sequence* $\mathbf{u}_{0,N-1} = \gamma \, \mathbf{u}'_{0,N-1}$ *is an* (N, α, ω)*-discerning control sequence for system (1).* □

In general, Theorem 2 provides only a sufficient condition for the derivation of discerning control sequences for system (1). However, the following considerations can be made.

i) Let us suppose that $0 \in \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ (e.g., this happens when $0 \in \mathcal{W}$). In this case, a control sequence $\mathbf{u}_{0,N-1}$ may be (N, α, ω)-discerning for system (1) only if $R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \neq 0$ for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$, i.e., only if it is an (N, α, ω)-discerning control sequence for the noise-free system.
ii) If no further assumptions are made on the sets $\mathcal{W}$ and $\mathcal{V}$, then there may exist some $\gamma < \bar{\gamma}(\mathbf{u}'_{0,N-1})$ such that $\mathbf{u}_{0,N-1} = \gamma \, \mathbf{u}'_{0,N-1}$ is an (N, α, ω)-discerning control sequence for system (1) (see, for instance, the situation depicted in Fig. 1). However, if the sets $\mathcal{W}$ and $\mathcal{V}$ are convex and if $0 \in \mathcal{W}$, then a control sequence $\mathbf{u}_{0,N-1} = \gamma \, \mathbf{u}'_{0,N-1}$ is (N, α, ω)-discerning for system (1) if and only if $\gamma > \bar{\gamma}(\mathbf{u}'_{0,N-1})$. Thus, in this case, Theorem 2 provides a necessary and sufficient condition.

It is important to remark that, in general, determining exactly the positive scalar $\bar{\gamma}(\mathbf{u}'_{0,N-1})$ for a given $\mathbf{u}'_{0,N-1}$ might be a difficult task. However, in the special case where the sets $\mathcal{W}$ and $\mathcal{V}$ are polytopes, such a task becomes quite simple, since also each set $\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ turns out to be a polytope. Therefore, it is possible to find a suitable matrix $\Psi(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ and a suitable vector

$\mu(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ such that

$$\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}') = \left\{ z \in \mathbb{R}^{m(N+1)} : \Psi(\boldsymbol{\lambda}, \boldsymbol{\lambda}') z \leq \mu(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \right\} .$$

Then, each scalar $\bar{\gamma}(\mathbf{u}'_{0,N-1}, \boldsymbol{\lambda}, \boldsymbol{\lambda}')$ can be found as the maximum $\gamma \geq 0$ such that

$$\gamma \Psi(\boldsymbol{\lambda}, \boldsymbol{\lambda}') R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \mathbf{u}'_{0,N-1} \leq \mu(\boldsymbol{\lambda}, \boldsymbol{\lambda}') .$$

A similar consideration holds also for the procedure to determine the switching sequence $\boldsymbol{\lambda}_{\alpha,N-\omega}$ on the basis of the observation sequence $\mathbf{y}_{0,N}$. Indeed, if the sets $\mathcal{W}$ and $\mathcal{V}$ are polytopes, then each set $\mathcal{S}(\boldsymbol{\lambda}, \mathbf{u}_{0,N-1})$ turns out to be a polyhedron (see [1]). Therefore, it is possible to find a suitable matrix $\Phi(\boldsymbol{\lambda})$ and a suitable vector $\nu(\boldsymbol{\lambda})$ such that

$$\mathcal{S}(\boldsymbol{\lambda}) = \left\{ \mathbf{y} \in \mathbb{R}^{m(N+1)} : \Phi(\boldsymbol{\lambda})\mathbf{y} \leq \nu(\boldsymbol{\lambda}) \right\} .$$

In this case, given an observation sequence $\mathbf{y}_{0,N}$, in order to determine the switching sequence $\boldsymbol{\lambda}_{\alpha,N-\omega}$, it is sufficient to find the set of switching sequences $\boldsymbol{\lambda} \in \mathcal{L}^{N+1}$ such that $\Phi(\boldsymbol{\lambda})\mathbf{y}_{0,N} \leq \nu(\boldsymbol{\lambda})$.

3 Mode observability over an infinite horizon

In this section, the active mode observation problem is addressed over an infinite horizon. Towards this end, the following definition can be introduced.

Definition 2 *System (1) is said to be* uniformly (N, α, ω)-mode observable *over an infinite horizon if there exists a control sequence* $\mathbf{u}_{0,\infty}$ *such that, for every* $t = N, N+1, \ldots,$ *the restriction* $\mathbf{u}_{t-N,t-1}$ *is an* (N, α, ω)*-discerning control sequence for system (1) in the interval* $[t-N, t]$. *A control sequence* $\mathbf{u}_{0,\infty}$ *that has such a property is called an* (N, α, ω)-persistently discerning control sequence *for system (1).*

By definition, if the control sequence $\mathbf{u}_{0,\infty}$ is (N, α, ω)-persistently discerning, then, for every time $t = N, N+1, \ldots,$ the switching sequence $\boldsymbol{\lambda}_{t-N+\alpha,t-\omega}$ in the restricted interval $[t-N+\alpha, t-\omega]$ can be uniquely determined from the observation sequence $\mathbf{y}_{t-N,t}$ in the interval $[t-N, t]$. Thus, such an input sequence makes it possible, at each time t greater than or equal to N, to reconstruct the discrete state up to time $t-\omega$ (with the exception of the first α time instants) on the basis of the information available up to time t.

In the light of the results of Section 2, in order to be (N, α, ω)-persistently discerning a generic control sequence $\mathbf{u}_{0,\infty}$ has to satisfy the observability conditions uniformly over time, i.e., it must be

$$R(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}_{t-N,t-1} \notin \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \qquad (9)$$

for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ such that $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$ and for every time $t = N, N+1, \ldots$. Therefore, an infinite number of conditions have to be met.

Actually, it turns out that satisfying all the observability conditions uniformly over time is not more difficult than satisfying them only in the first interval. More specifically, the following theorem holds.

Theorem 3 *The following facts are equivalent:*

(i) system (1) is (N, α, ω)*-mode observable in the interval* $[0, N]$ *(according to Definition 1);*
(ii) system (1) is uniformly (N, α, ω)*-mode observable over an infinite horizon (according to Definition 2).*

Proof: $(ii) \Rightarrow (i)$ It follows directly from the definitions.

$(i) \Rightarrow (ii)$ This can be proved by showing that if system (1) is (N, α, ω)-mode observable then it is possible to construct a periodic control sequence $\tilde{\mathbf{u}}_{0,\infty}$ with period N that is (N, α, ω)-persistently discerning. For such a sequence, each restriction $\tilde{\mathbf{u}}_{t-N,t-1}$ can be obtained as a cyclic permutation of $\tilde{\mathbf{u}}_{0,N-1}$, i.e.,

$$\tilde{\mathbf{u}}_{t-N,t-1} = \Omega^{t-N} \tilde{\mathbf{u}}_{0,N-1}$$

where Ω is a $Nk \times Nk$ cyclic permutation matrix that cyclically permutes the vectors composing $\tilde{\mathbf{u}}_{0,N-1}$ one position upward, that is, $\Omega \mathbf{u}_{0,N-1} = \mathrm{col}(u_1, u_2, \ldots, u_{N-1}, u_0)$. Since, by construction, Ω^{t-N} is periodic with period N, in this particular case conditions (9) turn out to be equal to

$$R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \Omega^i \tilde{\mathbf{u}}_{0,N-1} \notin \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}'), \quad i = 0, 1, \ldots, N-1,$$

for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ such that $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$. Hence, when periodic control sequences are considered, only a *finite* number of conditions have to be satisfied.

It is immediate to see that fact (i) ensures the existence of a control sequence $\tilde{\mathbf{u}}'_{0,N-1}$ such that

$$R(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \Omega^i \tilde{\mathbf{u}}'_{0,N-1} \neq 0, \qquad i = 0, 1, \ldots, N-1 \qquad (10)$$

for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ such that $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$. With this respect, recall that for a generic matrix M, we have $\ker(M\,\Omega^i) = \Omega^{-i} \ker(M)$. Then, the set $\tilde{\mathcal{K}}_N^{\alpha,\omega}$ of all the control sequences $\tilde{\mathbf{u}}_{0,N-1}$ such that at least one of conditions (10) is not satisfied turns out to be equal to

$$\tilde{\mathcal{K}}_N^{\alpha,\omega} \triangleq \bigcup_{i=0}^{N-1} \Omega^{-i} \mathcal{K}_N^{\alpha,\omega} .$$

As pointed out in Section 2, when the (N, α, ω)-mode observability property holds, then each $\mathcal{K}_N^{\alpha,\omega}$ is a null set. As a consequence, the set $\tilde{\mathcal{K}}_N^{\alpha,\omega}$, obtained as a finite union of null sets, is also a null set. Then, almost every choice of the control sequence $\tilde{\mathbf{u}}'_{0,N-1}$ satisfies conditions (10).

Thus, following the lines of Section 2, in order to find a periodic (N, α, ω)-persistently discerning control sequence, it is sufficient to set $\tilde{\mathbf{u}}_{0,N-1} = \gamma\, \tilde{\mathbf{u}}'_{0,N-1}$, where the scalar parameter γ is such that

$$\gamma > \tilde{\gamma}(\tilde{\mathbf{u}}'_{0,N-1}) \triangleq \max_{i,\boldsymbol{\lambda},\boldsymbol{\lambda}'} \bar{\gamma}(\Omega^i\, \tilde{\mathbf{u}}'_{0,N-1}, \boldsymbol{\lambda}, \boldsymbol{\lambda}')$$

where the maximum is extended to every $i = 0, 1, \ldots, N-1$ and to every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ such that $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$. The scalars $\bar{\gamma}(\Omega^i\, \tilde{\mathbf{u}}'_{0,N-1}, \boldsymbol{\lambda}, \boldsymbol{\lambda}')$ are defined as in Section 2 (see (8)). ■

The proof of Theorem 3 is constructive in that it provides a simple method to derive a persistently discerning control sequence. It is important to point out that such a sequence is chosen with a very specific structure, i.e., it is a periodic sequence with period N. Such a result may be useful when the observation of the discrete state is the unique control objective. However, in many applications (e.g., when the input has to be chosen on line according to some feedback strategy in order to meet some optimal control performance) assigning a priori a fixed structure to the whole control sequence $\mathbf{u}_{0,\infty}$ is not possible. With this respect, in the following, we shall develop a sequential observation scheme that allows one to choose the control vectors on line, one at each time stage, without such a restrictive requirement.

3.1 Sequential observation scheme

Suppose that, at a generic time t, only the control u_t has to be chosen. Such a control influences the observability of the discrete state in all the intervals of length $N+1$ containing the instant t, i.e., in all the intervals

$$[t-N+1+i,\, t+1+i], \quad i = 0, 1, \ldots, N-1$$

when $t \geq N-1$, and

$$[j,\, j+N], \quad j = 0, 1, \ldots, t$$

when $t < N-1$. Such intervals can be equivalently denoted for any $t = 0, 1, \ldots$ by

$$[t-N+1+i, t+1+i]\,, \quad i = i_{0|t}, i_{0|t}+1, \ldots, N-1,$$

where $i_{0|t} \triangleq \max\{0, N-1-t\}$. The definition of $i_{0|t}$ will be useful in the following to make the formulas more compact. By using such a definition, in order to have Definition 2 fulfilled, u_t has to be chosen so that

$$R(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}_{t-N+1+i,t+i} \notin \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\,, \quad i = i_{0|t}, i_{0|t}+1, \ldots, N-1\,, \tag{11}$$

for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ such that $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$. Note that such conditions depend on the last $N-1$ controls $\mathbf{u}_{t-N+1,t-1}$ (that have already been applied at time t) and on the control sequence $\mathbf{u}_{t,t+N-1}$ (which is not yet fixed at time t), composed by the control u_t (to be chosen at the current time) and by the future $N-1$ controls $\mathbf{u}_{t+1,t+N-1}$. With this respect, suitable matrices $R_i^-(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ and $R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ can be defined such that conditions (11) can be rewritten as

$$R_i^-(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}_{t-N+1,t-1} + R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}_{t,t+N-1} \notin \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\,, \quad i = i_{0|t}, i_{0|t}+1, \ldots, N-1\,.$$

More specifically, each matrix $R_i^-(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ is obtained as the juxtaposition of a $(N+1)m \times ik$ matrix of zeros with the first $(N-1-i)k$ columns of $R(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$. Similarly, each matrix $R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ is obtained as the juxtaposition of the last $(i+1)k$ columns of $R(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ with an $(N+1)m \times (N-1-i)k$ matrix of zeros.

As to the control sequence $\mathbf{u}_{t+1,t+N-1}$ (to be fixed and applied in the future), one may adopt the following strategy: first, determine a control input sequence $\mathbf{u}_{t,t+N-1|t} = \mathrm{col}\big(u_{t|t}, u_{t+1|t}, \ldots, u_{t+N-1|t}\big)$ of length N that satisfies all the mode observability constraints; then, actually apply only the first control of such a sequence. The same mechanism can be applied stage after stage. Then the proposed strategy gives rise to the following iterative procedure.

Procedure 1 *For every $t = 0, 1, \ldots$:*

(1) *Find a control sequence $\mathbf{u}_{t,t+N-1|t}$ such that*

$$R_i^-(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}_{t-N+1,t-1} + R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}_{t,t+N-1|t} \notin \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\,, \quad i = i_{0|t}, i_{0|t}+1, \ldots, N-1\,, \tag{12}$$

for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ such that $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$.

(2) *Apply the first control of $\mathbf{u}_{t,t+N-1|t}$, i.e., set $u_t = u_{t|t}$.*

In the following, we shall call *feasible* the control sequences $\mathbf{u}_{t,t+N-1|t}$ that satisfy all the conditions in step 2 of Procedure 1. It is worth noting that Procedure 1 does not lead to the choice of a *unique* control sequence $\mathbf{u}_{t,t+N-1|t}$; on the contrary it just gives a certain number of conditions that the control sequence $\mathbf{u}_{t,t+N-1|t}$ has to satisfy in order to ensure the observability of the discrete state. As a consequence, such a procedure is well-suited to take into account other possible control objectives.

For example, the proposed iterative scheme may be used in connection with a receding-horizon model predictive control scheme, where each control action is generated by solving an open-loop optimal control problem over a finite horizon (see, for an introduction, [10]). In this case, conditions (12) have to be seen as constraints on the control sequence to be applied whenever the observability of the discrete state is required.

The following theorem ensures the solvability of step 2 of Procedure 1 at every time stage and then the possibility of applying it to obtain an (N, α, ω)-persistently discerning control sequence.

Theorem 4 *Suppose that the sets $\mathcal{W}$ and $\mathcal{V}$ are bounded. Furthermore, suppose that system (1) is (N, α, ω)-mode observable. If Procedure 1 is applied iteratively to generate the control vectors u_t, $t = 0, 1, \ldots$, then*

i) at every time $t = 0, 1, \ldots$, feasible control sequences $\mathbf{u}_{t,t+N-1|t}$ exist in almost all directions;
ii) there exists a finite scalar M (independent of time) such that, at every time $t = 0, 1, \ldots$, one can always find a feasible control sequence $\mathbf{u}_{t,t+N-1|t}$ with $\|\mathbf{u}_{t,t+N-1|t}\| \leq M$.

Proof: The proof can be given by induction. First note that, at the initial time $t = 0$, since system (1) is supposed (N, α, ω)-mode observable, it is possible to find a control sequence $\mathbf{u}_{0,N-1|0}$ satisfying all the observability conditions (12).

Suppose now that, at time $t-1$ (with $t \geq 1$), there exists a control sequence $\mathbf{u}_{t-1,t+N-2|t-1}$ that satisfies all the conditions of step 2 of Procedure 1. Moreover, suppose that u_{t-1} is chosen as the first control of such a sequence. Then, in order to prove Theorem 4, it is sufficient to show that, also at time t, there exists a control sequence $\mathbf{u}_{t,t+N-1|t}$ satisfying all the observability conditions (12). With this respect, for each condition in (12), two possibilities have to be considered:

a) The matrix $R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ is equal to 0 (note that this may happen only for $i < N-1$, in that for $i = N-1$ we have $R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}') = R(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ which is different from 0 by Theorem 1). Clearly, in this case, the corresponding condition *does not* depend on the control sequence $\mathbf{u}_{t,t+N-1|t}$. As a consequence, by the induction hypotheses, the past controls $\mathbf{u}_{t-N+1,t-1}$ have certainly been chosen in such a way that $R_i^-(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}_{t-N+1,t-1} \notin \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ (otherwise $\mathbf{u}_{t-1,t+N-2|t-1}$ would have not been feasible).

b) The matrix $R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ is different from 0. In this case, the corresponding condition depends on the future controls, and the sequence $\mathbf{u}_{t,t+N-1|t}$ has to be chosen so that

$$R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}_{t,t+N-1|t} \notin \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}') - R_i^-(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}_{t-N+1,t-1}\,. \tag{13}$$

In order to find a feasible control sequence, let us choose a control sequence $\mathbf{u}'_{t,t+N-1|t}$ such that

$$R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}'_{t,t+N-1|t} \neq 0 \tag{14}$$

for every $i = i_{0|t}, i_{0|t}+1, \ldots, N-1$ and for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$ and $R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \neq 0$ (note that such a vector always exists since each of the kernels of the matrices $R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ is a null set). Without loss of generality, let us suppose $\|\mathbf{u}'_{t,t+N-1|t}\| = 1$. Since the set $\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ is bounded, it is always possible to find a suitable scalar γ such that the vector $\mathbf{u}_{t,t+N-1|t} = \gamma\, \mathbf{u}'_{t,t+N-1|t}$ satisfies condition (13) (note that $\|\mathbf{u}_{t,t+N-1|t}\| = |\gamma|$). Furthermore, let us denote by $\delta(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ the diameter of the set $\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$, i.e.,

$$\delta(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \triangleq \sup_{z,z' \in \mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')} \|z - z'\|\,.$$

The boundedness of $\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ ensures that either condition (13) is satisfied for every γ or there exist two values $\underline{\gamma}_i(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathbf{u}'_{t,t+N-1|t}) \leq \bar{\gamma}_i(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathbf{u}'_{t,t+N-1|t})$ such that for any $\gamma \notin \left[\underline{\gamma}_i(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathbf{u}'_{t,t+N-1|t}), \bar{\gamma}_i(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathbf{u}'_{t,t+N-1|t})\right]$ condition (13) is satisfied. Then, the set of all the γ such that the vector $\mathbf{u}_{t,t+N-1|t} = \gamma\, \mathbf{u}'_{t,t+N-1|t}$ *does not* satisfy condition (13) has a Lebesgue measure $m_i(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathbf{u}'_{t,t+N-1|t})$ less than or equal to $\left|\bar{\gamma}_i(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathbf{u}'_{t,t+N-1|t}) - \underline{\gamma}_i(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathbf{u}'_{t,t+N-1|t})\right| \leq \delta(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \,/\, \left\|R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}'_{t,t+N-1|t}\right\|$. As a consequence the set of all the γ such that the vector $\mathbf{u}_{t,t+N-1|t} = \gamma\, \mathbf{u}'_{t,t+N-1|t}$ *does not* satisfy condition (13) for at least one i and at least one pair of switching sequences has a Lebesgue measure less or equal to

$$\begin{aligned} m\left(\mathbf{u}'_{t,t+N-1|t}\right) &= \sum_{i;\,\boldsymbol{\lambda};\,\boldsymbol{\lambda}'} m_i(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \mathbf{u}'_{t,t+N-1|t}) \\ &\leq \sum_{i;\,\boldsymbol{\lambda};\,\boldsymbol{\lambda}'} \delta(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \,/\, \left\|R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\mathbf{u}'_{t,t+N-1|t}\right\| \end{aligned}$$

where the summation is extended to every $i = i_{0|t}, i_{0|t}+1, \ldots, N-1$ and to every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$ and $R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \neq 0$. As a consequence, there exist infinite many choices of γ such that the vector $\mathbf{u}_{t,t+N-1|t} = \gamma\, \mathbf{u}'_{t,t+N-1|t}$ satisfies all the mode-observability conditions in step 2 of Procedure 1. Moreover, when $R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \neq 0$, each set $\ker\left\{ R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}')\right\}$

is a null set, then almost every choice of the versor $\mathbf{u}'_{t,t+N-1|t}$ satisfies condition (14) for every $i = i_{0|t}, i_{0|t}+1, \ldots, N-1$ and for every $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$ and $R_i^+(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \neq 0$. Hence the proposed construction can be applied for almost every choice of the versor $\mathbf{u}'_{t,t+N-1|t}$; thus proving fact i).

As to fact ii), note that each scalar $m\left(\mathbf{u}'_{t,t+N-1|t}\right)$ depends only on versor $\mathbf{u}'_{t,t+N-1|t}$ and does not depend on the time t and on the past controls $\mathbf{u}_{0,t-1}$. Then, at every time $t = 0, 1, \ldots$, it is possible to find feasible control sequences of the form $\mathbf{u}_{t,t+N-1|t} = \gamma\, \mathbf{u}'_{t,t+N-1|t}$ with norm γ bounded by a scalar M, provided that $M > m\left(\mathbf{u}'_{t,t+N-1|t}\right)$. □

A last remark is important. Since Procedure 1 has been constructed on the basis of Definition 2, it leads to a control sequence that satisfies the observability conditions *uniformly* over time. Actually such a strong requirement is not necessary, in that, at the generic time t (with $t \geq N+1$), the whole sequence $\boldsymbol{\lambda}_{\alpha,t-\omega-1}$ may have been already reconstructed. As a consequence, when addressing the observation of the switching sequence $\boldsymbol{\lambda}_{t-N+\alpha,t-\omega}$, such information would allow one to consider only a subset of the pairs of switching sequence $\boldsymbol{\lambda} \in \mathcal{L}^{N+1}$ and $\boldsymbol{\lambda}' \in \mathcal{L}^{N+1}$ with $r^{\alpha,\omega}(\boldsymbol{\lambda}) \neq r^{\alpha,\omega}(\boldsymbol{\lambda}')$. It is worth noting that Procedure 1 could be easily modified to take into account all the available information about the past discrete states. This would simply lead to a reduction of the number of conditions to be considered, thus making the choice of a feasible control sequence $\mathbf{u}_{t,t+N-1|t}$ less restrictive. Such a reduction may be important whenever some optimal control performance has to be met by the controller.

4 Numerical example

Let us consider a switching system described by means of equations (1) with

$$A(1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 3 \end{bmatrix},$$

$$B(1) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad B(2) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

$$C(1) = C(2) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

It is immediate to verify that, in this case, the observability matrix $F(\boldsymbol{\lambda})$ does not depend on the switching sequence; hence, in the absence of control, it would not be possible to distinguish between the two discrete states. On the other hand, by choosing $\alpha = 0$, $\omega = 1$, and $N = 2$, one can satisfy conditions (2), hence such a system turns out to be $(2, 0, 1)$-mode observable.

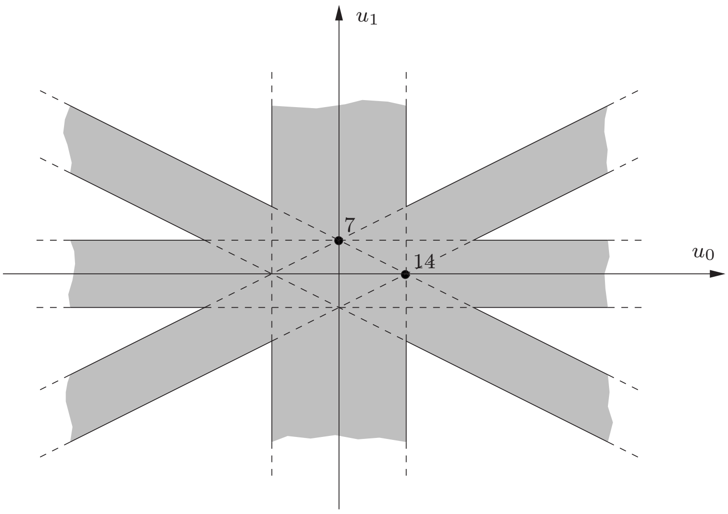

Fig. 2. Graphical representation of the conditions on the control sequence $\mathbf{u}_{0,1}$ that ensure the observability of the discrete state for the considered system with $N = 2$, $\alpha = 0$, and $\omega = 1$. The grey region corresponds to the set of the *non-discerning* control sequences.

As to the determination of the discerning control sequences in the absence of noises, it is immediate to see that in this case the conditions of Theorem 1 (see (3)) simply turn out to be

$$u_0 \neq 0, \quad u_1 \neq 0, \quad u_0 - 2u_1 \neq 0, \quad u_0 + 2u_1 \neq 0.$$

Let us now consider the noisy case and suppose that w_t and v_t, $t = 0, 1, \ldots$ belong to the polytopic compact sets $\mathcal{W} = [-1, 1]^3$ and $\mathcal{V} = [-1, 1]$, respectively. Then the sets $\mathcal{O}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ turn out to be independent from the switching sequences and equal to a line segment with extreme points $(7/3, -14/3, 7/3)$ and $(-7/3, 14/3, -7/3)$. As a consequence, one can see that, in order for a control sequence $\mathbf{u}_{0,1}$ to be discerning, it must satisfy the following conditions (see (7)):

$$|u_0| > 14, \ |u_1| > 7, \ |u_0 - 2u_1| > 14, \ |u_0 + 2u_1| > 14.$$

For the reader's convenience a graphical representation of such conditions is provided in Fig. 2.

As to the sequential scheme for the infinite horizon, at every time $t = 1, 2, \ldots$, the observability conditions in step 2 of Procedure 1 turn out to be

$$|u_{t-1}| > 14, \quad |u_{t|t}| > 7,$$
$$|u_{t-1} - 2u_{t|t}| > 14, \quad |u_{t-1} + 2u_{t|t}| > 14$$

for $i = 0$, and

$$\begin{aligned} & |u_{t|t}| > 14\,, \quad |u_{t+1|t}| > 7\,, \\ & |u_{t|t} - 2u_{t+1|t}| > 14\,, \quad |u_{t|t} + 2u_{t+1|t}| > 14\,. \end{aligned}$$

for $i = 1$.

5 Conclusions

In this paper, the connection between the choice of the control sequence and the observability of the system mode has been investigated for a class of switching discrete-time linear systems. In particular, suitable conditions have been derived in order to characterize the discerning control sequences (i.e., the control sequences that allow the reconstruction of the switching sequence on the basis of the observations). With respect to previous results, the presence of unknown but bounded noises has been explicitly taken into account. Moreover it has been shown that mode observability on a finite horizon is a necessary and sufficient condition for mode observability on an infinite horizon. In order to derive persistently discerning control sequences over an infinite horizon, a procedure has been proposed that is based on the on-line satisfaction of a set of conditions that have been proved to be feasible under suitable assumptions.

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