Control of constrained systems: closed-loop, open-loop, and hybrid solutions

Frédéric Grognard

Thèse présentée en vue de l'obtention du titre de docteur en Sciences Appliqués

Louvain-la-Neuve, October 15, 2001

Ce sont rarement les réponses qui apportent la vérité mais l'enchaînement des questions.

Daniel Pennac. La Fée Carabine.

La vérité, c'est qu'il n'y a pas de vérité.

Pablo Neruda. Fin du monde.

A mon Père

Remerciements

Je voudrais d'abord exprimer la chance que j'ai eue de pouvoir travailler pendant quatre ans dans un environnement scientifique stimulant. Cet environnement, je l'ai trouvé dans l'atmosphère détendue et novatrice du CESAME et ensuite dans la croissance du service Systèmes de l'Université de Liège. Cette chance m'a été donnée par le F.N.R.S.

Pour effectuer cette thèse, j'ai bénéficié de l'encadrement de deux co-promoteurs: les professeurs Rodolphe Sepulchre et Georges Bastin.
J'ai toujours apprécié de pouvoir collaborer étroitement avec Rodolphe. Son apport fut sans égal quand il a fallu progresser rapidement ou m'empêcher de reculer. Je suis souvent sorti épuisé de son bureau, fatigué mais enthousiaste. Je ne le remercierai jamais assez du temps qu'il a passé à lire mon écriture maladroite et à améliorer mes textes.
Le sens de la réalité et l'esprit critique de Jojo ont toujours été de précieux guides dans ma recherche. La justesse de son jugement n'a d'égale que son sens de l'humour.

Je dois ensuite remercier les professeurs Campion, Praly et Stoorvogel d'avoir accepté d'être membres de mon jury de thèse. La défense privée fut certes une épreuve difficile, mais je ne crois pas que j'aurai à nouveau la chance d'avoir une opinion extérieure si avisée et si détaillée sur mon travail.

Le professeur Michel Gevers m'a fait l'honneur de présider mon jury.

Durant quatre ans, j'ai cotoyé de nombreux chercheurs. Je tiens spécialement à remercier mes deux collègues de bureau: Benoît, dont le calme a toujours conféré à notre bureau une atmosphère studieuse, et Manu, dont la bonne humeur est rarement prise en défaut. Je veux également remercier Isabelle, Jose, Pierre-Antoine, Guy, Cristina, Benoît, Xavier, Radu, Amaury, et tous les autres chercheurs qui ont pu passer au CESAME et au service Systèmes pendant ces quatre ans. C'est également grâce à la disponibilité du personnel administratif et technique que j'ai pu mener cette thèse à terme: Lydia, Michou, Isabelle, Guido, Etienne, Yannick, Victor, Michel, Edouard et Madame De Baets.

Si j'en suis arrivé là, c'est avant tout grâce à ma Maman et à mes Grand-

Parents, qui m'ont aidé à déployer mes ailes. Il n'est pas de mots pour exprimer ma gratitude.

Et surtout, que serais-je sans Sylvie, qui m'a soutenu durant ces quatre années et avec qui j'ai découvert l'importance de la vie hors de l'université. Sylvie, avec qui j'ai pu réécrire les mots "Amour" et "bonheur" en dix lettres: Christophe.

Contents

Introduction

A fundamental problem of control theory consists in designing a control function $u(x,t)$ in order to steer a solution of the system of differential equations

$$\dot{x} = f(x,u) \tag{1}$$

from an initial condition $x(0) = x_0$ to the origin ($x \in I\!R^m$, $u \in I\!R$, f Lipschitz continuous). This problem is central to this thesis, in the sense that we will concentrate on the design of control functions that achieve this objective for some classes of systems in the form (1). Classically, it is solved through *open-loop* or *closed-loop* control. In this thesis, we will approach both families of control methods, as well as an intermediate, or hybrid, control method.

Open-loop control consists in designing a control function $u(t)$ as a function of time ($t \in [0,T]$ or $t \in I\!R^+$). This control law is built such that, if $x(0) = x_0$, the solution of system (1) with $u(t)$ as input satisfies $x(T) = 0$ or $\lim_{t\to+\infty} x(t) = 0$. Such a control function is usually computed as the solution of an optimal control problem, whose computation can be numerically demanding, or results from the inversion of the system dynamics, which requires a special structure for (1) (e.g. trajectory planning for flat systems). Those control methods can guarantee efficient control towards the origin, but, in any case, open-loop control can only be efficiently applied when the model of the system is very accurate and in the absence of disturbances, a situation rarely encountered in practice. Indeed, no action is taken to modify the control in case the solution does not follow the planned trajectory, so that the solution misses the target (the origin).

Closed-loop control consists in building a feedback control law $u(x)$ for the stabilization of the origin of system (1). This formulation of the control law expresses the fact that the control value at time t depends on the value $x(t)$ of the state at time t. Such a *feedback* law, as opposed to an open-loop control, guarantees the stabilization of the origin and the convergence is robust: if the control law asymptotically stabilizes the origin of system (1), it will also stabilize the origin of this system subject to small perturbations or model inaccuracies. Indeed, if the solution of the actual control system does not exactly follow the solution of (1), the control value is "adapted" so that the solution still converges to the origin. It is therefore more desirable to design a feedback control law than an open-loop control law.

Except for linear systems, for which linear feedback laws can be designed with some guarantee of performance and robustness, the design of stabilizing control laws is usually based on Lyapunov theory. Such methods guarantee stability, but may not lead to satisfying performance.

It is therefore desirable to combine the performance advantage of open-loop control with the stability and robustness property of closed-loop control: hybrid control and Model Predictive Control (MPC) schemes proposed in this thesis aim at combining these advantages.

A hybrid controller is a collection of continuous feedback controllers, along with some map for switching among them; inside this collection of controllers, some are designed to have stability in large regions and others are designed to have satisfying performance. The switching map is then designed such that the hybrid controller ensures stability in a large region without compromising performance.

Model Predictive Control (MPC) schemes are widely applied to industrial processes. Every τ units of time, they require the solution of an optimal control problem. The solution of this optimal control problem is an open-loop control law $u^*(t)$ ($t \in [0,T]$), which is applied to the system for the interval duration τ. The resulting MPC controller then ensures performance derived from the optimal control criterion that is solved. The recalculation of $u^*(t)$ every τ units of time then closes the loop, to recover the robustness property of feedback control.

We will be interested in solving the problem of steering $x(t)$ to the origin for "constrained" systems. The constraints that we consider appear in two families of systems. They prevent the inversion of the system dynamics, which makes the control design challenging. On the one hand, we have the systems in feedforward form, which can be seen as chains of integrators with feedforward connections. At the input of the integrator whose output is x_i, we have the output of the previous integrator x_{i+1} and a feedforward term. In the absence of feedforward term, x_{i+1} is free to steer x_i; the presence of the feedforward term adds a perturbation to this steering. The effect of this perturbation can be limited if x_{i+1} is small. In that sense, we say that the feedforward nonlinearities are limiting the amplitude (or the gain of a linear controller) at the input x_{i+1} of each integrator. On the other hand, we have affine constraints in the form $Fx + Gu \leq H$, that constrain the stabilization of linear systems (and in particular input magnitude and rate constraints). Those constraints prevent the inversion of the system, which would allow for the design of simple control laws, so that we have to use specific control methods. Specifically, the input magnitude constraint is also amplitude limiting, which translates in a gain limitation of linear controllers. In summary, in this thesis, we consider system with gain-limiting constraints.

In Chapter 2, we exhibit existing stabilizing designs for nonlinear systems, presenting a lower or upper triangular structure. Those control designs are globally or semiglobally stabilizing the origin (backstepping for systems in feedback form and forwarding for systems in feedforward form). However backstepping designs are not robust to the presence of feedforward terms and forwarding designs are not robust to the presence of feedback terms. In Chapter 2, we present a new family of feedforward systems with feedback interconnections, for which we design a globally stabilizing control law. This shows that stabilizability of some classes of feedforward systems is robust to some local feedback interconnections.

When system (1) is linear, but when the input is constrained in amplitude and/or in rate, simple linear feedback control laws $u = Kx$ do not suffice to solve the stabilization problem in large regions with satisfying performance. Proposed solutions to this problem include anti-windup schemes, low-gain designs, and Model Predictive Control. In this thesis, we expose contributions to the last two designs. Those contributions are aimed at improving the performance of existing designs.

Linear low-gain designs are known to allow for large regions of attraction at the expense of performance. Indeed, with such control laws, the convergence to the origin is slow, and the small control values that they produce are more likely to be corrupted by input noise than control laws which make better use of the available actuation. In Chapter 3, we present a hybrid scheme aimed at improving performance of existing low-gain schemes. This hybrid controller includes a stabilizing controller, a local controller, and a scheduling controller, respectively designed for stability in a large region of attraction, satisfying local performance, and a fast transfer from the stabilizing controller to the local controller. The scheduling controller is then presented in the case of linear systems subject to input magnitude and rate constraints and the case where the linear system is subject to affine constraints.

In order to apply Model Predictive Control, an optimal control problem must be recomputed every τ units of time, which imposes a heavy computation load to the control system. Therefore, if the process is fast and requires τ to be small, MPC cannot be applied. It is then crucial to develop algorithms that efficiently solve constrained optimal control problems. In Chapter 4, we show that the time-optimal control problem can be efficiently solved for linear systems with bounded input, due to the bang-bang property of the solution. We then present such an algorithm that solves the time-optimal control problem when the solution presents less than $n-1$ switchings.

Definitions

Stability

The origin of system

$$\dot{x} = f(x) \tag{2}$$

where $x \in I\!R^n$ and $f(x)$ is Lipschitz continuous, is said to be

- stable if, for each $\epsilon > 0$, there exists $\delta > 0$ such that
$$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon, \forall t \geq 0$$
- attractive if there exists $\delta > 0$ such that
$$\|x(0)\| \leq \delta \Rightarrow \lim_{t\to\infty} x(t) = 0$$
- asymptotically stable (AS) if it is stable and attractive.
- globally asymptotically stable (GAS) if it is stable and
$$\forall\, x(0) \in I\!R^n,\ \lim_{t\to\infty} x(t) = 0$$
- locally exponentially stable (LES) if there exists $\delta, k, \beta > 0$ such that
$$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq k\|x(0)\|e^{-\beta t}$$

A sufficient condition for the origin of a nonlinear system to be stable is checked through Lyapunov's direct method:

Theorem 1 *Let $x = 0$ be an equilibrium of system (2). Let $V : I\!R^n \to I\!R^+$ be a $\mathcal{C}^1$ positive definite and radially unbounded function $V(x)$ such that*

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x}(x)f(x) \leq 0, \qquad \forall x \in I\!R^n$$

then $x = 0$ is stable, and all solutions of (2) are bounded and converge to the set where $\dot{V}(x) \equiv 0$. If $\dot{V}(x)$ is negative definite, the origin is globally asymptotically stable.

The function $V(x)$ is called a Lyapunov function for system (2).

Stabilization

Stabilization of the origin of the nonlinear system

$$\dot{x} = f(x, u) \tag{3}$$

where $x \in I\!R^n$, $u \in I\!R$, $f \in \mathcal{C}^1$ and $f(0,0) = 0$, is achieved through a static state feedback control law $\alpha(x)$, where α is a Lipschitz continuous function defined on the whole state space $I\!R^n$ and α takes values in $I\!R$. In this thesis, we will simply call α a control law.

The control law $\alpha(x)$ is said to be stabilizing if the equilibrium $x = 0$ of the controlled system

$$\dot{x} = f(x, \alpha(x))$$

is asymptotically stable with a region of attraction denoted D.

The control law $\alpha(x)$ is *globally* stabilizing when $D = I\!R^n$, the whole state space; it means that the controlled system is globally asymptotically stable.

When the dependence of (3) in u is affine, the system can be rewritten as:

$$\dot{x} = f(x) + g(x)u$$

and a classical control method consists in building a positive definite and radially unbounded function $V(x)$ such that we can find a control law $\alpha(x)$ that renders its derivative

$$\dot{V} = \frac{\partial V}{\partial x}(f(x) + g(x)\alpha(x)) = L_f V(x) + L_g V(x)\alpha(x)$$

strictly negative, which ensures global asymptotic stability of the closed loop system. Such a function is called a Control Lyapunov Function (CLF) for system (3):

Definition 1 (Sontag (1983)) *A smooth, positive definite, and radially unbounded function $V(x)$ is called a control Lyapunov function for the system (3) if, for all $x \neq 0$,*

$$L_g V(x) = 0 \Rightarrow L_f V(x) < 0$$

The construction of control Lyapunov functions is inherent to most nonlinear control designs; for example, it underlines most recursive control methods for systems in feedback or in feedforward form.

The more recent concept of semiglobal stabilization often allows for simpler control designs:

Definition 2 (Sussmann & Kokotović (1991)) *A system Σ is semiglobally stabilizable to the origin by the mean of a class $\mathcal{F}$ of feedback control laws if, for every bounded subset Ω of the state space, there exists a control law belonging to $\mathcal{F}$ that makes the origin an asymptotically stable equilibrium of the corresponding closed loop system, with a region of attraction containing Ω.*

Saturation

Several control methods described in this thesis use saturation functions:

Definition 3 *Given two constants $a \leq 0 \leq b$, with $a < b$, a function $sat_{[a,b]} : I\!R \to I\!R$ is said to be a saturation function when it satisfies, for $s \in I\!R$:*

- $sat_{[a,b]}(s) = a$ *when* $s \leq a$
- $sat_{[a,b]}(s) = s$ *when* $a \leq s \leq b$
- $sat_{[a,b]}(s) = b$ *when* $s \geq b$

Particular cases of this saturation function are

- $sat_{\epsilon}(s) = sat_{[-\epsilon,\epsilon]}(s)$, *for* $\epsilon > 0$
- $sat(s) = sat_1(s)$

Terminology

In Chapter 2, we will use functions which are at least quadratic near the origin. When using this terminology, we mean:

Definition 4 *A $\mathcal{C}^1$ function $h(x) : I\!R^n \to I\!R^p$ is said to be at least quadratic near the origin if $h(0) = 0$ and $Dh(0) = 0$.*

Chapter 1

Control design for systems with gain-limiting constraints

This chapter briefly introduces the families of systems which will be handled in this thesis and exposes the point of view that we have taken to design control laws for those systems. These families of systems are, on the one hand, nonlinear systems in feedforward form, and on the other hand, linear systems subject to input constraints. These two classes of systems impose a similar type of constraint on the design of control laws. Essentially, this constraint is a gain limitation, which is not active when the magnitude of the control input is small. Therefore, both classes of systems are classically stabilized through control laws that are termed *low-gain*:

Definition 5 *A control law $u = \alpha(x, \epsilon)$ with $\alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is said to be low-gain if, for all $x \in \mathbb{R}^n$:*

$$\lim_{\epsilon \to 0} \alpha(x, \epsilon) = 0$$

This chapter briefly describes feedforward systems and linear systems with input constraint, as well as the low-gain control designs that have been proposed in the literature and some of their limitations.

1.1 Systems in feedforward form

The systems in feedforward form are structured nonlinear systems, for which recursive methods for the construction of stabilizing feedback laws are known. Numerous mechanical systems fit in this structure, including the celebrated cart-pendulum system. The recursive control methods take advantage of the triangular structure of these systems:

$$\left\{\begin{array}{lcll} \dot{x}_1 & = & x_2+ & g_1(x_2, x_3, \cdots, x_n, u) \\ \dot{x}_2 & = & x_3+ & g_2(x_3, \cdots, x_n, u) \\ & \vdots & & \\ \dot{x}_{n-1} & = & x_n+ & g_{n-1}(x_n, u) \\ \dot{x}_n & = & u+ & g_n(u), \end{array}\right. \tag{1.1}$$

where x_i, $u \in I\!R$, the g_i nonlinearities are smooth, and the Jacobian linearization is controllable at the origin. System (1.1) is a chain of integrators with feedforward interconnections: g_i is called a feedforward nonlinearity because it depends only on the control law and the states which are "lower" in the system. A classical approach for the stabilization of the origin of such systems consists in first designing $u = \alpha_n(x_n)$ that stabilizes the origin of the $\dot{x}_n = u + g_n(u)$ subsystem; the next step of the procedure builds $\alpha_{n-1}(x_{n-1}, x_n)$ such that the control law $u = \alpha_n(x_n) + \alpha_{n-1}(x_{n-1}, x_n)$ stabilizes the origin of the (x_{n-1}, x_n) subsystem. This procedure can be continued until a control law $u = \alpha(x)$ is built for the stabilization of the whole system.

Several methods are known for the design of these recursive control laws. They can essentially be classified into two families: the Lyapunov forwarding procedures of Mazenc & Praly (1996) and Sepulchre et al. (1996), on the one hand, and the small gain designs of Teel (1992b), Teel (1996) and Grognard et al. (1998) on the other. We will be interested in the second family, which treats the feedforward nonlinearities as gain-limiting constraints at the input of each of the integrators. We will now illustrate this approach on two examples.

The following systems are in the feedforward form (1.1):

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 + x_2^2 \\ \dot{x}_2 & = & u \end{array}\right. \qquad \left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 + u^2 \\ \dot{x}_2 & = & u \end{array}\right. \tag{1.2}$$

A preliminary feedback $\alpha_2(x_2) = -x_2$ stabilizes the origin of the x_2 subsystems and systems (1.2) become:

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 + x_2^2 \\ \dot{x}_2 & = & -x_2 + v \end{array}\right. \qquad \left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 + (v - x_2)^2 \\ \dot{x}_2 & = & -x_2 + v \end{array}\right.$$

where v is a new control variable. The procedure of Teel (1992b) then uses the change of coordinates $(y_1, y_2) = (x_1 + x_2, x_2)$ to transform systems (1.2) into

$$\left\{\begin{array}{lcl} \dot{y}_1 & = & v + y_2^2 \\ \dot{y}_2 & = & -y_2 + v \end{array}\right. \qquad \left\{\begin{array}{lcl} \dot{y}_1 & = & v + (v - y_2)^2 \\ \dot{y}_2 & = & -y_2 + v \end{array}\right.$$

so that the control $v = \alpha_1(x_1, x_2) = -\ \mathrm{sat}_\epsilon y_1$ globally stabilizes the origin of the complete systems. Therefore, the original systems are globally stabilized by

$$u(x) = -x_2 - \ \mathrm{sat}_\epsilon(x_1 + x_2) \tag{1.3}$$

with ϵ small enough (see Teel (1992b)). This low-gain control method does not make any difference between the two feedforward nonlinearities x_2^2 and u^2: it ensures that both become small perturbations after a finite time. The parameter ϵ is forced to be small by the presence of the feedforward nonlinearities. Indeed, the control law (1.3) ensures that $|v(x)| \leq \epsilon$ and that $x_2 = y_2$ is $\mathcal{O}(\epsilon)$ after a finite time, so that, for both systems of (1.2), the y_1 differential equations become

$$\dot{y}_1 = -\epsilon y_1 + \mathcal{O}(\epsilon^2)$$

which ensures convergence of y_1 to the origin if ϵ is small enough. From this example, we see that the gain ϵ at the input v of the y_1 subsystem is limited by the feedforward nonlinearities x_2^2 or u^2. The feedforward structure of systems (1.2) ensures that the stabilization of the complete systems is possible because this amplitude limitation is not an obstacle to the stabilization of the x_1 subsystem.

Adding a feedback term in those systems (1.2) has two possible consequences:

Possible loss of global stabilizability Systems presenting feedback and feedforward terms are not guaranteed to be globally stabilizable so that, if we wish to present a control law for systems in feedforward form presenting feedback connections, we will first have to define a class of feedforward and feedback connections, which are compatible with each other, that is which never prevent stabilizability. Let us examine a bounded feedback nonlinearity: $\frac{x_1}{1+x_1^2}$. The first system becomes:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & \frac{x_1}{1+x_1^2} + x_2 + x_2^2 \\ \dot{x}_2 & = & u \end{array} \right. \tag{1.4}$$

If the x_2 term is seen as the control that steers the $\dot{x}_1$ subsystem, x_2^2 is seen as a perturbation, and x_2 will have to be small, so that it dominates the perturbation. On the other hand, if x_2 is small, it cannot compensate for the feedback term, so that x_1 does not converge to the origin. This approach can therefore not yield a globally stabilizing control law. The whole expression $x_2 + x_2^2$ must then be used to compensate for $\frac{x_1}{1+x_1^2}$. However, $x_2 + x_2^2 \geq -\frac{1}{4}$, so that if $x_1(0) \geq 1$, then $x_1(t) \geq 1$ for all $t \geq 0$. Indeed, when $x_1(t) = 1$, we have $\dot{x}_1 = \frac{1}{2} + x_2 + x_2^2$. Therefore, $\dot{x}_1 \geq \frac{1}{4}$ for any choice of x_2, and, if $x_1(0) \geq 1$, we have $x_1(t) \geq 1$ for all $t \geq 0$. The global stabilizability of the first system of (1.2) is not robust to this feedback nonlinearity because the gain limiting feedforward nonlinearity x_2^2 prevents the compensation of $\frac{x_1}{1+x_1^2}$, which necessitates a higher gain for stabilization.

Possible failure of low-gain designs Low-gain designs are not a valid choice to globally stabilize the origin of systems in feedforward form presenting feedback connections. In some cases, they can be efficient, but we will show on a counter-example, that it does not always work, so that the introduction of a new control law is necessary.

If we now consider the same bounded feedback nonlinearity in the second system of (1.2), we have:

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& \frac{x_1}{1+x_1^2} + x_2 + u^2 \\ \dot{x}_2 &=& u \end{array} \right. \tag{1.5}$$

and the construction that we have used for the systems (1.2), applied to this system, yields:

$$\left\{ \begin{array}{rcl} \dot{y}_1 &=& \frac{y_1 - y_2}{1+(y_1-y_2)^2} + v + (v - y_2)^2 \\ \dot{y}_2 &=& -y_2 + v \end{array} \right.$$

A saturated control at the input v again yields $y_2 = \mathcal{O}(\epsilon)$ in finite time, and the y_1 subsystem becomes:

$$\dot{y}_1 = \frac{y_1}{1+y_1^2} + v + \mathcal{O}(\epsilon^2),\ v = \mathcal{O}(\epsilon)$$

The parameter ϵ must be small enough to dominate the $\mathcal{O}(\epsilon^2)$ term. On the other hand, ϵ has to be large so that the saturated control law $v = \mathcal{O}(\epsilon)$ can compensate for the feedback term $\frac{y_1}{1+y_1^2}$ (which can be as large as $\frac{1}{2}$). Those conflicting requirements on the value of ϵ result in an obstacle to the stabilization of system (1.5) through a classical low-gain approach. However, we show, in Chapter 2, that a control law can be designed to stabilize the origin of systems like (1.5): systems in feedforward form with bounded feedback nonlinearities and feedforward nonlinearities that do not yield strong gain limitations at the input of the integrators like in (1.4), but rather a limitation on the rate of the input of the integrators like in (1.5).

1.2 Linear systems with input constraints

The second class of problems that we consider in this thesis is the stabilization of the origin of linear systems with input magnitude limitation, that is

$$\dot{x} = Ax + bu \qquad |u| \leq 1 \tag{1.6}$$

Through the years, a lot of attention has been dedicated to the stabilization of such systems, because the input saturation is the most commonly encountered nonlinearity in applications, either because of physical limitation of the available actuation, or because of security considerations, which require the input to operate in a given safety region. If it is not accounted for in the control design, input saturation can lead to deterioration of the performance of the control system or even cause instability.

Two approaches are basically available for handling this input magnitude constraint: either the constraint is accounted for *a posteriori* or *a priori*.

Anti-windup The "a posteriori" approach consists in first designing a controller for the unconstrained system. In a second design phase, a supplementary control scheme is added to compensate for the destabilizing and performance degrading effects of saturation without modifying the local behavior of the control system. This method is called the *anti-windup compensation*. It usually leads to improved performance, but a rigorous stability and robustness analysis of the control scheme is rarely available. The conditioning technique as anti-windup is introduced in Hanus et al. (1987), and a general framework for the design and analysis of anti-windup schemes as well as references to the founding papers on anti-windup can be found in Kothare et al. (1994). Nonlinear stability analysis of anti-windup schemes can be found in Kapoor et al. (1996) and Teel (1999). Beyond the cited references, which propose systematic anti-windup schemes, the actual application of anti-windup is often ad-hoc, which complicates the stability analysis (see e.g. Grognard et al. (2001c) for an ad-hoc small gain analysis of anti-windup for semiglobal stabilization of a nonlinear cement mill model).

A canonical structure for anti-windup is illustrated on Figure 1.1 (see Teel (1998a)). The control system without anti-windup consists in the feedback loop containing the controller C, the nonlinearity "sat", and the plant P. In order to counter the adverse effects of saturation, the anti-windup scheme compares the computed control law u_c and the saturated control law u, and feeds the result back to the controller C through a compensator Σ. Note that, whenever the saturation is not active, $u = u_c$ is satisfied, so that the output of the anti-windup scheme is zero and the resulting control system behaves exactly as the original control system. If the saturation becomes active, classical anti-windup schemes prevents u_c from escaping to large values and attempts to keep u_c close to u.

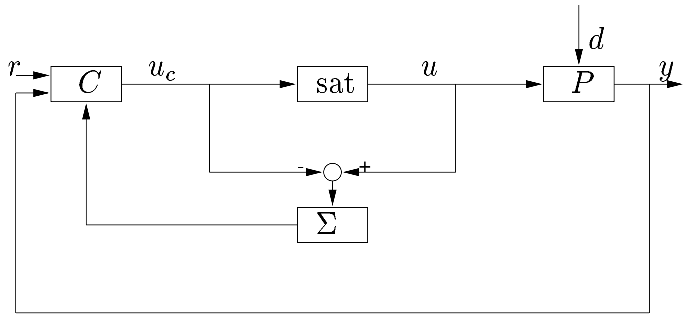

Figure 1.1: Canonical form of anti-windup schemes

The "a priori" approach consists in directly designing a controller taking the constraint into account. Low-gain designs and Model Predictive Control belong to this approach and are now described.

Low-gain designs Like for feedforward systems, *low-gain designs* can be applied to globally stabilizable linear systems with bounded input (which requires (A, b) to be stabilizable and the eigenvalues of A to be in the closed left half plane). Indeed those classes of systems are related because they are both subject to gain-limiting constraints. In Teel (1996), linear systems with bounded input are stabilized through the same nested saturation method as the systems in feedforward form: linear systems with a saturation nonlinearity at the input can be transformed into feedforward systems by putting them in Jordan form:

$$\begin{array}{lcl} \dot{x}_1 & = & A_1 x_1 + g_1(x_2) \\ \dot{x}_2 & = & A_2 x_2 + g_2(x_3) \\ & \vdots & \\ \dot{x}_{n-1} & = & A_{n-1} x_{n-1} + g_{n-1}(x_n) \\ \dot{x}_1 & = & A_n x_n + g_n(u) \end{array}$$

where the $g_1, \cdots, g_n$ functions are linear in x_i and in $\mathrm{sat}(u)$. The A_i matrices are Jordan blocks with eigenvalues in the closed left half plane. This structure is in feedforward form, except for the $A_i x_i$ terms. Because those terms are not exponentially destabilizing, the classical stabilization methods for feedforward systems can be applied.

Low-gain control laws guaranteeing stabilization in "large" regions of attraction have recently been proposed for (1.6) (Sontag & Sussmann 1990, Teel

1992a, Lin & Saberi 1993, Megretski 1996). Several of the proposed control laws rely on a one-parameter family of linear control laws $u = -K(\epsilon)x$. As the parameter $\epsilon \to 0$, the norm $\|K(\epsilon)\|$ decreases, so that the magnitude constraint $|u| \leq 1$ is satisfied in a large domain and semiglobal stabilization is achieved if A has no eigenvalue with strictly positive real part.

Let us now consider the double integrator:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & u \end{array} \right. \tag{1.7}$$

with $|u| \leq 1$. A Riccati based design (see Teel (1995) and Lin et al. (1996)) generates a classical low-gain control law for such a system (see Section 3.6 for the details of the calculations):

$$u = -\epsilon^2 x_1 - \sqrt{3}\epsilon x_2$$

With this control law the poles of the closed loop systems are:

$$\rho_{1,2} = \frac{-\sqrt{3} \pm i}{2}\epsilon$$

which means that the smaller ϵ is, the slower the rate of convergence to the origin is.

We only consider linear systems where A has no eigenvalue with strictly positive real part; a subclass of those systems is composed of the systems whose eigenvalues have strictly negative real part, that is asymptotically stable systems. For such systems, the application of a control law whose amplitude is $\mathcal{O}(\epsilon)$ hardly modifies the behavior of the solutions with respect to the solution of $\dot{x} = Ax$; this poses the question of the pertinence of applying such a control law for stable systems. Moreover, if there is a disturbance at the input, a control law which takes small control values is much more degraded than a control law which does a better use of the available actuation.

Classically, when stabilizing the origin of a system, two objectives are pursued: ensuring that a given x_0 belongs to the region of attraction, and ensuring satisfying performance. On this example, we have illustrated the conflict between those two requirements: on the one hand, we need ϵ to be small enough for stability, and on the other hand, ϵ has to be large enough for performance (convergence rate, robustness).

In Chapter 3, we design a family of low-gain control laws $u = -K(\epsilon)x$, parameterized with a parameter $\epsilon \in (0, 1]$ such that, when $\epsilon \to 0$, the region of attraction of the controller extends towards the whole state-space, and when $\epsilon = 1$, the regulator $u = -K(1)x$ ensures satisfying local performance. For x_0 outside the region of attraction $\mathcal{T}(1)$ of the regulator $u = -K(1)x$, we design a scheduling of the parameter that ensures that $x(t)$ enters $\mathcal{T}(1)$ as fast as possible, which ensures improved performance.

This method is then extended to the control of linear systems with amplitude and rate limited inputs. This problem is not trivial: the combination of input magnitude with rate constraints has been identified as the main contributing factor to the crashes of YF-22 (Dornheim (1992)) and Gripen prototypes (Lenorowitz (1990)); the rate limit introduces what can be seen as a delay in the system. And this delay can have a destabilizing effect in itself, or can lead to oscillations due to the destabilizing actions of the pilot. The problem is then absolutely relevant and requires a solution taking the rate saturation into account.

Low-gain control laws solving that problem are directly derived from the ones solving the problem without rate saturation. It suffices to take the gain ϵ small enough to force both the amplitude and the rate of the control law $u = -K(\epsilon)x$ to be small (see Lin (1997) and Stoorvogel & Saberi (1999). Like in the case of magnitude constraints, large regions of attraction require ϵ to be small, which results in poor performance.

Finally, our scheduling is exposed for linear systems subject to constraints that are more general than constraints on the input: affine constraints $Fx + Gu \leq H$.

Model Predictive Control The solution of an optimal control problem often produces better performance than low-gain designs. This optimal control problem consists in designing a control law that minimizes a prespecified performance index:

$$\begin{aligned} &\min V(x,u) \\ &\quad \text{s.t.} \quad \dot{x}(t) = Ax(t) + Bu(t) \\ &\qquad\quad\; |u(t)| \leq 1 \\ &\qquad\quad\; x(0) = x_0 \end{aligned} \tag{1.8}$$

where $V(x,u)$ is often taken to be a quadratic cost:

$$V(x,u) = \frac{1}{2}\int_0^{+\infty} x^T(s)Qx(s) + u^T(s)Ru(s)ds$$

with $Q, R > 0$. However, the problem of computing the closed-loop controller $u^*(x)$ that ensures the minimization of this index for any initial condition is usually untractable, and only open-loop solutions $u^*(t)$ of this problem for given initial conditions can be computed numerically. The cost is therefore approximated by a finite-horizon cost in the form

$$V(x,u) = \frac{1}{2}\int_0^{T} x^T(s)Qx(s) + u^T(s)Ru(s)ds \tag{1.9}$$

We have seen that open-loop control lacked robustness. Embedding this open-loop control in a MPC scheme will then guarantee robustness by closing the loop. MPC schemes act as follows: every τ units of time, a finite-horizon optimal control law $u^*(t)$ is numerically computed online with the current $x(k\tau)$ as initial condition, and this control law is applied during τ units of time; at time $(k+1)\tau$, the same control problem is recomputed,... In order to guarantee stability of the resulting MPC scheme, the cost (1.9) is usually not sufficient (Mayne et al. (2000)) unless a suitable terminal cost penalty $x(T)^T Px(T)$ is added to the cost (1.9). This approach yields stability and optimality results (Mayne et al. (2000)) and an explicit formulation of the controller resulting from a discretized MPC scheme is given in Bemporad et al. (2001) and De Doná (2000). Another solution to the stability problem consists in imposing the terminal constraint $x(T) = 0$ in the problem formulation (1.8) (see Mayne & Michalska (1990)).

However, when the resulting controller cannot be explicitly computed offline, the online computation burden associated with this method prevents its application to systems with fast dynamics. It is therefore important to design algorithms that can rapidly solve online the finite-horizon optimal control problem that is posed every τ units of time. In Chapter 4, instead of the quadratic cost (1.9), we choose the total transfer time from $x = x_0$ to $x = 0$ as performance index to be minimized, and we propose an algorithm that efficiently computes the *time-optimal control* $u^*(t)$ for any given $x(0)$ when this solution presents at most $n-1$ switchings.

1.3 Summary

In this chapter, we have exposed two families of nonlinearities for which we have given the following information

Feedforward nonlinearities	Bounds on the input
$\begin{cases} \dot{x}_1 &= x_2 + u^2 \\ \dot{x}_2 &= u \end{cases}$	$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \quad \|u\| \leq 1 \end{cases}$
Property	Property
Gain limiting	Gain limiting
Control methods	Control methods
Low-gain designs Forwarding	Low-gain designs MPC Anti-windup
Drawbacks of low-gain	Drawbacks of low-gain
Poor performance Lack of robustness to feedback connections	Poor performance

From this table, the link between the families of systems presenting those nonlinearities is evident: both systems in feedforward form and linear systems with a constrained input present gain-limiting nonlinearities, so that their stabilization can be achieved through low-gain designs, which have, as main drawback, that they yield a slow behavior of the solutions. In the case of feedforward systems, we will not consider the performance aspect, which is largely addressed by forwarding, but we will rather address the problem of lack of robustness of the existing designs to the presence of destabilizing feedback terms. As for the systems with bounded input, we will consider the problem of performance, by introducing a scheduling of low-gain designs and giving an algorithm producing open-loop time-optimal controls that can be used in an MPC scheme.

Chapter 2

Stabilization of nonlinear systems with feedback and feedforward interconnections

The global stabilization of nonlinear systems has been the subject of an important literature over the last decade and significant progresses have been made towards the development of systematic design methods and a better understanding of structural limitations to large regions of attractions (see Freeman & Kokotović (1996), Isidori (1995), Krstić et al. (1995), Sepulchre et al. (1996), Sepulchre (2000), Sussmann & Kokotović (1991), Coron et al. (1995) and references therein). It is fair to say that most existing results have been obtained by exploiting certain triangularity properties of the differential equations of the considered system.

These structural properties can be classified in two categories and it is sufficient for our purpose to illustrate them on the system

$$\left\{ \begin{array}{lllll} \dot{x}_1 & = & f_1(x_1) & +x_2+ & g_1(x_2, x_3, \cdots, x_n, u) \\ \dot{x}_2 & = & f_2(x_1, x_2) & +x_3+ & g_2(x_3, \cdots, x_n, u) \\ & \vdots & & & \\ \dot{x}_{n-1} & = & f_{n-1}(x_1, \ldots, x_{n-1}) & +x_n+ & g_{n-1}(x_n, u) \\ \dot{x}_n & = & f_n(x_1, \ldots, x_n) & +u+ & g_n(u), \end{array} \right. \tag{2.1}$$

where $x = (x_1, \ldots x_n) \in \mathbb{R}^n$ and $u \in \mathbb{R}$, with f_i and g_i smooth functions on $\mathbb{R}^i$ and $\mathbb{R}^{n-i+1}$ such that $f_i(0, \cdots, 0) = g_i(0, \cdots, 0) = 0$. We view (2.1) as a chain of integrators perturbed by feedback connections (f_i functions) and feedforward connections (g_i functions). The two possible forms of triangularity are obtained by setting to zero either the feedforward terms or the feedback terms:

Setting $g_i \equiv 0$ yields a lower triangular structure to the system, which is termed in "strict feedback form". This structure ensures that a recursive approach can be employed to globally stabilize the origin of the system: the control law is "backstepped" from the first equation to the last one, so that this method is called backstepping. In principle it acts as follows: the first equation is:

$$\dot{x}_1 = f_1(x_1) + x_2 \tag{2.2}$$

and a globally stabilizing virtual control law $x_2 = x_2^c(x_1)$ can be built. Indeed, there is no constraint on x_2 so that the control law x_2^c always has enough power to compensate the possibly destabilizing $f_1(x_1)$ term. Considering the equation

$$\dot{x}_2 = f_2(x_1, x_2) + x_3$$

a virtual control law $x_3^c(x_1, x_2)$ can then be designed so that x_2 follows $x_2^c(x_1)$. This procedure is repeated until the last equation, where u is designed so that x_n follows the virtual control law $x_n^c(x_1, \cdots, x_{n-1})$. This results in globally or semiglobally stabilizing control laws, even in the presence of strongly destabilizing nonlinearities, which can be compensated for with sufficiently high gain feedback (see Freeman & Kokotović (1996), Isidori (1995), Krstić et al. (1995), Saberi et al (1990), Sepulchre et al. (1996)). This design is possible because of the absence of constraints on the virtual controls. Indeed, in the presence of an explicit state constraint in the form $x_2 \leq C$, the origin of equation (2.2) would not be globally stabilizable if, for some x_1, there was $f_1(x_1) > C$. In some cases, the presence of a feedforward term $g_1(x_2)$ can also act as a constraint and prevent global stabilization (e.g., if $x_1(0) > -0.25$ and $\dot{x}_1 = x_1 + x_2 + x_2^2$, no virtual control can be built in order to steer x_1 to the origin because of the presence of the x_2^2 feedforward term).

Setting $f_i \equiv 0$ yields a system in "strict feedforward form", with an upper triangular structure. In this case, the absence of feedback connections limits the instability of the open-loop system (in particular the Jacobian linearization cannot be exponentially unstable). Thanks to this property, if the Jacobian linearization at the origin is controllable, a low gain at the input of each integrator is sufficient to stabilize the system, and systems in strict feedforward form are globally stabilizable without further restrictions of the feedforward connections. Globally and semiglobally stabilizing recursive control laws are known (see Jankovic et al. (1996),Mazenc & Praly (1996),Teel (1992b)); they start by stabilizing the last equation, and progressively stabilize the whole system going from the bottom up. Even if the feedforward nonlinearities impose severe limitations on the available gain, stabilization is achieved with sufficiently low gain feedback. This is particularly well studied in the case where the feedforward terms are at least quadratic near the origin. The last equation becomes

$$\dot{x}_n = u + \mathcal{O}(u^2)$$

and the feedforward term $\mathcal{O}(u^2)$ is dominated by u if the amplitude of the control law is sufficiently small. Therefore, low-gain and saturation designs have been very successful for the stabilization of such feedforward systems. However, the presence of a feedback term $f_n(x_n)$ that would need to be compensated for can ruin the stabilization; the $\mathcal{O}(u^2)$ term limits the amplitude of the input while $f_n(x_n)$ might require much control power for its compensation.

When feedback and feedforward terms are present at the same time, classical control methods for systems in strict feedback or feedforward forms can be inefficient to achieve global asymptotic stability. Therefore, the present chapter aims at presenting a control law that stabilizes the origin of a class of feedforward systems and allows for the presence of (destabilizing) feedback connections. As seen in Chapter 1, the simultaneous presence of feedback and feedforward terms can prevent global stabilizability so that we have to specify a class of such systems which is still globally stabilizable (see Section 2.1.4: the feedback connections are bounded, with bounded derivatives, and the feedforward connections are not gain-limiting, but only rate-limiting). Also, we have shown that low-gain was not always sufficient to achieve stability so that we introduce a new control law.

The main contributions of this chapter are the introduction of a globally stabilizable class of feedforward systems presenting feedback connections, the construction of a control law and the proof that it globally stabilizes the system.

This chapter is structured as follows: classical recursive control methods for systems in feedforward form are presented in Section 2.1, as well as the conflict between the gain limitations induced at the input of each integrator by the feedforward nonlinearities, and the necessity of having a large available gain for compensation of feedback terms. Section 2.2 contains the main result of this chapter: the control law that ensures the stability of a certain class of feedforward systems. Examples of application of this control law are given in Section 2.3. The limitations of its application are then exposed in Section 2.4, while its use for robust global stabilization of some class of feedforward systems is given in Section 2.5, before some conclusions in Section 2.6.

2.1 Systems in feedforward form

In this chapter, we consider the design of a control law for systems in feedforward form perturbed by local feedback connections, so that we will start by introducing classical control methods for systems in feedforward form, and show that they can be inefficient for the stabilization of systems presenting feedback connections. In (2.1), they are characterized by $f_i = 0$, which results in the lower triangular structure:

$$\begin{cases} \dot{x}_1 & = & x_2 + & g_1(x_2, x_3, \cdots, x_n, u) \\ \dot{x}_2 & = & x_3 + & g_2(x_3, \cdots, x_n, u) \\ & \vdots & & \\ \dot{x}_{n-1} & = & x_n + & g_{n-1}(x_n, u) \\ \dot{x}_n & = & u + & g_n(u), \end{cases} \tag{2.3}$$

with $x_i, u \in I\!R$, g_i continuous and the Jacobian linearization controllable at the origin. This is a chain of integrators perturbed by feedforward interconnections. Since the introduction of this form, several control laws have been presented to stabilize such systems, most notably forwarding (Mazenc & Praly (1996) and Sepulchre et al. (1996)) and the low-gain saturated control (Teel (1992b) and Teel (1996)), which all lead to globally stabilizing control laws. Forwarding generates a CLF for the complete feedforward system, while the original formulation of saturated control considered the feedforward nonlinearities as higher order perturbations that constrained the available gain at the input.

2.1.1 Saturated controls

Nested saturation methods for global stabilization of feedforward systems can be found in Teel (1992b) and Teel (1996). The latter is more general and is based on the nonlinear small gain theorem, but the first version more directly exemplifies the fact that the feedforward connections impose gain restrictions.

The feedforward systems considered in Teel (1992b) are in the form (2.3) where the g_i functions are continuous and are required to be at least quadratic in (x, u).

Theorem 2 (Teel (1992b)) *There exist linear functions $T_i : I\!R^n \to I\!R$ for $i \in \{1, \cdots, n\}$ and scalar saturation bounds $\epsilon_n > \epsilon_{n-1} > \cdots > \epsilon_1$ such that the control*

$$u = -\ sat_{\epsilon_n}\big(T_n(x) +\ sat_{\epsilon_{n-1}}(T_{n-1}(x) + \cdots +\ sat_{\epsilon_1}(T_1(x)) \cdots)\big)$$

globally asymptotically stabilizes the origin of (2.3).

The proof that can be found in Teel (1992b) is constructive. A transformation $y = Tx$ is exhibited, so that the system becomes

$$\left\{\begin{array}{lll} \dot{y}_1 & = & y_2 + \cdots + y_n + \quad u \quad + \phi_1(y_2, y_3, \cdots, y_n, u) \\ \dot{y}_2 & = & y_3 + \cdots + y_n + \quad u \quad + \phi_2(y_3, \cdots, y_n, u) \\ & \vdots & \\ \dot{y}_{n-1} & = & \qquad\qquad y_n + \quad u \quad + \phi_{n-1}(y_n, u) \\ \dot{y}_n & = & \qquad\qquad\qquad\quad u \quad + \phi_n(u), \end{array}\right. \tag{2.4}$$

where the ϕ_i functions are quadratic in (y, u). The analysis goes from the bottom up. A quadratic Lyapunov function $V_n = y_n^2$ is first considered for the last equation and its derivative is

$$\dot{V}_n = -2y_n[\,\mathrm{sat}_{\epsilon_n}(y_n + \,\mathrm{sat}_{\epsilon_{n-1}}(\cdots)) - \phi_n(u)]$$

We have $|u| \leq \epsilon_n$ and, since ϕ_n is at least quadratic, there exists C_n such that $|\phi_n(u)| \leq C_n\epsilon_n^2$. If we take ϵ_n small enough such that

$$\frac{\epsilon_n}{2} > C_n\epsilon_n^2 \tag{2.5}$$

and ϵ_{n-1} such that

$$\epsilon_{n-1} \leq \frac{\epsilon_n}{4} \tag{2.6}$$

we can see that $\dot{V}_n < 0$ for all $|y_n| > \frac{3\epsilon_n}{4}$, which forces $|y_n| \leq \frac{3\epsilon_n}{4}$ to be achieved in finite time, and to stay satisfied afterwards. Moreover, the saturation $\mathrm{sat}_{\epsilon_n}$ is not active once $|y_n| \leq \frac{3\epsilon_n}{4}$, so that the saturation $\mathrm{sat}_{\epsilon_n}$ stops being active in finite and never becomes active afterwards. The y_{n-1} equation then becomes:

$$\dot{y}_{n-1} = -\,\mathrm{sat}_{\epsilon_{n-1}}(y_{n-1} + \,\mathrm{sat}_{\epsilon_{n-2}}(\cdots)) + \phi_{n-1}(y_n, u)$$

which is in the appropriate form to use the same argument as for y_n, so that the same argument can be used up to y_1. Therefore, in finite time, the solution $y(t)$ reaches the region where none of the nested saturations is active. Local stability of the closed-loop system in that region completes the proof.

This proof again illustrates that the feedforward connections are considered as a limitation of the available gain. Indeed, the bound (2.5) forces the amplitude of the control law to be small; this bound comes from the feedforward nonlinearity $\phi_n(u) = g_n(u)$. The next step imposes y_n to be small while stabilizing the y_{n-1} equation. The bound (2.6) shows that the available gain is reduced at the input of each integrator from the bottom up.

Example 1 *The following system is in feedforward form:*

$$\left\{\begin{array}{lcl}\dot{x}_1 & = & x_2 + x_2^2 \\ \dot{x}_2 & = & u\end{array}\right. \tag{2.7}$$

The change of variables $(y_1, y_2) = (x_1 + x_2, x_2)$ *puts the system in the form (2.4):*

$$\left\{\begin{array}{lclll}\dot{y}_1 & = & y_2+ & u & +y_2^2 \\ \dot{y}_2 & = & & u & \end{array}\right.$$

and the control law is:

$$u = -\ sat_{\epsilon_2}(x_2 +\ sat_{\epsilon_1}(x_1 + x_2))$$

The absence of ϕ_2 *term keeps* ϵ_2 *free, while (2.5) forces*

$$\epsilon_1 < \frac{1}{2}$$

which shows that the x_2^2 *term imposes a bound on the available amplitude at the input of the* y_1 *subsystem.*

2.1.2 Semiglobal forwarding

A semiglobally stabilizing linear controller is available for systems satisfying the structure (2.3), with g_i affine in u. For readability's sake, we will write $x_2 + g_i(x_{i+1}, \cdots, x_n, u) = h_i(x_{i+1}, \cdots, x_n) + \psi_i(x_{i+1}, \cdots, x_n)u$, and system (2.3) rewrites as:

$$\left\{\begin{array}{lcl}\dot{x}_1 & = & h_1(x_2, \cdots, x_n) + \psi_1(x_2, \cdots, x_n)u \\ \dot{x}_2 & = & h_2(x_3, \cdots, x_n) + \psi_2(x_3, \cdots, x_n)u \\ & \vdots & \\ \dot{x}_{n-1} & = & h_{n-1}(x_n) + \psi_{n-1}(x_n)u \\ \dot{x}_n & = & u\end{array}\right. \tag{2.8}$$

with the Jacobian linearization at the origin controllable, $h_i(0) = 0$, and h_i and ψ_i are $\mathcal{C}^1$ functions on $I\!R^{n-i}$. The philosophy of this linear forwarding is low-gain and recursive starting from the last equation. Before step $i+1$ of the design, a control law $u(x_{n-i+1}, \cdots, x_n) + u^{n-i}$ has been built to stabilize the subsystem composed of the last $i-1$ equations of (2.8). At step $i+1$, u^{n-i} is used to stabilize the x_{n-i} equation. Defining $\xi = x_{n-i}$ and $z = (x_{n-i+1}, \cdots, x_n)$, a forwarding step works on system

$$\begin{array}{lcl}\dot{\xi} & = & h(z) + \psi(z)u \\ \dot{z} & = & f(z) + g(z)u\end{array} \tag{2.9}$$

where the control $u(x_{n-i+1}, \cdots, x_n)$ is integrated into $f(z)$ and $h(z)$, and u^{n-i} is renamed u. Because of $u(x_{n-i+1}, \cdots, x_n)$, the system $\dot{z} = f(z)$ is GAS.

The linear change of coordinates

$$y = \xi + q^T z, \; q^T = -Dh(0)Df(0)^{-1}$$

transforms the first equation of (2.9) into

$$\dot{y} = (\psi(0) + q^T g(0))u + h.o.t.$$

where $h.o.t.$ denotes higher-order terms. If the Jacobian linearization of (2.9) is stabilizable, then $\psi(0) + q^T g(0) \neq 0$. Up to the multiplication of y by a constant, we assume without loss of generality that $\psi(0)+q^T g(0) = 1$. System (2.9) becomes

$$\begin{array}{rcl} \dot{y} &=& \tilde{h}(z) + (1 + \tilde{\psi}(z))u, \\ \dot{z} &=& f(z) + g(z)u \end{array} \tag{2.10}$$

where

$$\tilde{h}(z) := h(z) - Dh(0)z + q^T(f(z) - Df(0)z), \;\; \tilde{h}(0) = 0, D\tilde{h}(0) = 0$$

and

$$\tilde{\psi}(z) := \psi(z) - \psi(0) + q^T(g(z) - g(0)) \quad \tilde{\psi}(0) = 0$$

Then we have the following result.

Theorem 3 (Grognard et al. (1998)) *Let $\Omega = \Omega_\xi \times \Omega_z \subset I\!R \times I\!R^i$ be any compact set. Then there exists $\bar{\epsilon} > 0$ such that, for all $0 < \epsilon \leq \bar{\epsilon}$, the equilibrium $(\xi, z) = (0, 0)$ of (2.9) is locally exponentially stable with the control law $u = -\epsilon y$ and the region of attraction includes Ω.*

which allows for a recursive semiglobal stabilization of system (2.8).

This control law ensures that, if ϵ is small enough, z converges to an ϵ neighborhood of the origin, for initial conditions (ξ, z) in a large region of the state-space. The first equation of (2.10) then becomes

$$\dot{y} = -\epsilon y + \mathcal{O}(\epsilon^2)$$

which forces convergence to the origin.

It directly appears that such a method could not work with a destabilizing feedback term $k(\xi)$ in the first equation of (2.9). Indeed, after the convergence of z to an ϵ-neighborhood of the origin, the first equation of (2.10) becomes

$$\dot{y} \approx k(y) + u + \mathcal{O}(\epsilon^2)$$

and u has to be taken small, so that $z = \mathcal{O}(\epsilon)$ stays valid in order to keep the feedforward term $\tilde{h}(z) + \tilde{\psi}(z)u$ negligeable. This prevents the compensation of $k(y)$. Those feedforward terms are therefore gain-limiting.

Example 2 *The previous example (2.7), with the preliminary feedback* $u = -x_2$*, results in the form (2.11), which is suitable for the application of the semiglobally stabilizing linear control law. Indeed, the change of variable* $y_1 = x_1 + x_2$ *results in*

$$\begin{cases} \dot{y}_1 &= x_2^2 + v \\ \dot{x}_2 &= -x_2 + v \end{cases}$$

and the original system is semiglobally stabilized by

$$u = -x_2 - \epsilon(x_1 + x_2)$$

which, when (x_1, x_2) *belongs to the region of attraction of the origin, first forces* x_2 *to be* $\mathcal{O}(\epsilon)$ *and then*

$$\dot{y}_1 = -\epsilon y_1 + \mathcal{O}(\epsilon^2)$$

which leads to slow convergence of x_1 *to the origin.*

2.1.3 Global forwarding

Forwarding, as exposed in Sepulchre et al. (1996), can handle feedforward systems in the form (2.3) with g_i affine in u, which results in the form (2.8). This is not the only structure to which forwarding can be applied: note, for example, that local feedback terms $f_i(x_i)$ are tolerated at each integrator as long as the origin of system $\dot{x}_i = f_i(x_i)$ is stable, but we restricted to (2.8) for simplicity.

We now briefly illustrate forwarding, a recursive procedure to globally stabilize (2.8). Therefore, like for semiglobal forwarding, we consider the building block (2.9) of system (2.8):

$$\begin{aligned} \dot{\xi} &= h(z) + \psi(z)u \\ \dot{z} &= f(z) + g(z)u \end{aligned}$$

where $\xi = x_i$, $z = (x_{i+1}, \cdots, x_n, z)$, the core system $\dot{z} = f(z)$ is globally asymptotically stable, $h(0) = 0$ and the Jacobian linearization at the origin stabilizable. Global stabilization of system (2.9) allows for the global stabilization of the whole system (2.8) through a recursive use of this procedure. It works as follows:

We first have to find a Lyapunov function corresponding to the ξ system with $u = 0$ and $z = 0$. This system is $\dot{\xi} = 0$ and we can then take the quadratic function:

$$W(\xi) = \frac{\xi^2}{2}$$

The necessity of building such a Lyapunov functions prevents the presence of destabilizing feedback terms in the $\dot{\xi}$ equation.

We can then find a locally quadratic and radially unbounded Lyapunov function $U(z)$ for the $\dot{z} = f(z)$ subsystem; classically, this Lyapunov function is given by the previous step of forwarding. We define an output of the z system:

$$y_0 = L_g U(z)$$

In order to build a control law for system (2.9), a control Lyapunov function for the whole system, based on U and W, is generated:

$$V(\xi, z) = W(\xi) + \Psi(\xi, z) + U(z)$$

where Ψ is called a cross-term: it is designed to cancel non negative definite terms in $\dot{W}$. We must use this cross-term in order to render the derivative of V non positive (when $u = -y_0$). Forwarding builds this cross term through an integration:

$$\begin{aligned} \Psi(\xi, z) &= \int_0^\infty \frac{\partial W}{\partial \xi}(\tilde{\xi}(s))(h(\tilde{z}(s)) - \psi(\tilde{z}(s))y_0)ds \\ &= \int_0^\infty \tilde{\xi}(s)(h(\tilde{z}(s)) - \psi(\tilde{z}(s))y_0)ds \end{aligned}$$

where, at time s, $(\tilde{\xi}(s), \tilde{z}(s)) = (\tilde{\xi}(s; (\xi, z)), \tilde{z}(s; z))$, the states of the system controlled by $u = -y_0$ and with initial conditions $(\tilde{\xi}(0), \tilde{z}(0)) = (\xi, z)$. This integral is well-defined (see Theorem 5.8 of Sepulchre et al. (1996)).

We see that Ψ can be evaluated differently:

$$\Psi(\xi, z) = \lim_{s \to \infty} \frac{\tilde{\xi}^2(s)}{2} - \frac{\xi^2}{2}$$

because the argument of the integral is the time derivative of $\frac{\tilde{\xi}^2(s)}{2}$.

We define a new output, which is the term multiplying u in the derivative of the new Lyapunov function $V(\xi, z)$:

$$y_1 = \frac{\partial V}{\partial \xi}\psi(z) + \frac{\partial V}{\partial z}g(z)$$

Theorem 4 (Sepulchre et al. (1996)) *The origin of the system (2.9) is globally stabilized by the control law:*

$$u = -y_1$$

and has $V(\xi, z)$ as Lyapunov function.

Forwarding produces a control law and a Lyapunov function to be used at the next step of the recursion. When the procedure is complete for the whole system, it produces a globally stabilizing control law and a global Lyapunov function.

We can better see its application on a simple example:

Example 3 *In (2.7), x_1 is taken as the ξ variable and x_2 as the z variable of the general form (2.9). This system is not exactly in the form of (2.9) because the z subsystem is not GAS; therefore, we apply a preliminary feedback $u = -x_2$, which transforms the system into*

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 + x_2^2 \\ \dot{x}_2 & = & -x_2 + v \end{array} \right. \tag{2.11}$$

The Lyapunov functions corresponding to the first and second equations are, respectively, $W(x_1) = \frac{x_1^2}{2}$ and $U(x_2) = \frac{x_2^2}{2}$. We are left with the construction of the cross-term $\Psi(x_1, x_2)$, which is:

$$\Psi(x_1, x_2) = \lim_{s \to \infty} \frac{\tilde{x}_1(s)^2}{2} - \frac{x_1^2}{2}$$

with $v = -L_g U(\tilde{x}_2) = -\tilde{x}_2(s)$ as input, and $(\tilde{x}_1(0), \tilde{x}_2(0)) = (x_1, x_2)$. This results in

$$\left\{ \begin{array}{rcl} \tilde{x}_2(s) & = & x_2 e^{-2s} \\ \tilde{x}_1(s) & = & x_1 + (1 - e^{-2s})\frac{x_2}{2} + (1 - e^{-4s})\frac{x_2^2}{4} \end{array} \right.$$

and the cross-term is

$$\Psi(x_1, x_2) = \frac{(x_1 + \frac{x_2}{2} + \frac{x_2^2}{4})^2}{2} - \frac{x_1^2}{2}$$

which results in the Lyapunov function

$$V(x_1, x_2) = \frac{(x_1 + \frac{x_2}{2} + \frac{x_2^2}{4})^2}{2} + \frac{x_2^2}{2}$$

whose derivative with the control $u = -x_2 - L_g U(x_2) + w = -2x_2 + w$ is

$$\dot{V} = (x_1 + \frac{x_2}{2} + \frac{x_2^2}{4})\frac{1 + x_2}{2} w - 2x_2^2 + x_2 w = -2x_2^2 + y_1 w$$

and the globally asymptotically stabilizing control law is

$$u = -2x_2 - y_1 = -3x_2 - (x_1 + \frac{x_2}{2} + \frac{x_2^2}{4})\frac{1 + x_2}{2}$$

Like in (1.4), we have $x_2 + x_2^2 > -\frac{1}{4}$. In this case, this does not prevent global stabilization because no feedback term needs to be compensated.

2.1.4 Gain and rate limitations

The previous subsections have shown that general gain-limiting feedforward connections are not compatible with destabilizing feedback connections. The objective of this chapter being the introduction of some feedback connections inside the structure of feedforward systems, the considered feedforward and feedback connections will have to be compatible with each other.

Our result is best illustrated with the help of the simple benchmark system

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1) + x_2 + g_1(x_2, x_3) \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & u \end{array}\right. \tag{2.12}$$

As shown earlier, existing control methods for feedforward systems require the feedback connection $f_1(x_1)$ to be missing or "stabilizing", that is $f_1(s)s \leq 0$; at least in the linear forwarding and saturated control methods, convergence is first enforced towards a manifold where $x_2 = \mathcal{O}(\epsilon)$ and $x_3 = \mathcal{O}(\epsilon)$ (so that g_1 is negligeable), before steering x_1 to 0 (which is impossible with $x_2 = \mathcal{O}(\epsilon)$ in the presence of a destabilizing feedback term).

On the other hand, existing methods for strict feedback systems require $g_1(x_2, x_3) \equiv 0$ because backstepping forces x_2 to $-f_1(x_1) - k(x_1)$ (where $sk(s) < 0$ when $s \neq 0$), which renders the perturbation coming from x_2 in the feedforward connection large. We will show in this chapter that we will be able to tolerate x_3 terms in the feedforward connections.

Our result will show that a bounded destabilizing nonlinearity $f_1(x_1)$ can be tolerated provided that the feedforward term $g_1(x_2, x_3)$ is not "gain limiting" but only "rate limiting", that is, $g_1(x_2, x_3) = g_1(x_3) = g_2(\dot{x}_2)$, and is at least quadratic near the origin. Our design will enforce the convergence of solutions towards a region of the state space where $\dot{x}_2(t)$ is kept small enough, but not necessarily $x_2(t)$, thereby guaranteeing enough gain to compensate for the destabilizing connection $f_1(x_1)$. This control law can be viewed as a *slow control design* in contrast to the low-gain schemes previously considered for feedforward systems.

Expanding on this idea, we achieve global stabilization of systems which can be written in the form

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1) + x_2 + g_1(\dot{x}_2, \dot{x}_3, \cdots, \dot{x}_n) \\ \dot{x}_2 & = & f_2(x_2) + x_3 + g_2(\dot{x}_3, \cdots, \dot{x}_n) \\ & \vdots & \\ \dot{x}_{n-1} & = & f_{n-1}(x_{n-1}) + x_n + g_{n-1}(\dot{x}_n) \\ \dot{x}_n & = & f_n(x_n) + u \end{array}\right. \tag{2.13}$$

with $x \in I\!R^n$, $u \in I\!R$. Substituting $\dot{x}_i$ by its expression in the g_i nonlinearity, we see that this system can directly be put in the classical state-space form (2.1), and that this representation is a chain of integrators with feedforward and local feedback nonlinearities:

- Feedback connections $f_i(x_i)$ which are depending only on the local integrator state. They are bounded with bounded derivatives for all $x_i \in I\!R$;
- Rate limiting feedforward connections g_i with an upper triangular structure. These functions are at least quadratic near the origin.

Our design will enforce convergence of the solutions towards a sequence of nested manifolds where an increasing number of state derivatives $\dot{x}_i$ are kept small (hence we term our control "slow control"). Near-invariance of these manifolds will be guaranteed despite of the destabilizing feedback connections by allowing for enough gain in a restricted neighborhood of the manifolds.

2.2 Global stabilization of rate limited feedforward systems

The next theorem is the main result of this chapter. We present a recursive design to achieve global asymptotic and local exponential stability of the origin of system (2.13) using saturation functions. It successively force $\dot{x}_n, \dot{x}_{n-1}, \cdots, \dot{x}_2$ to small values so that the feedforward terms are negligeable. As opposed to the step by step construction of control laws through recursive methods for systems in feedback or feedforward form, we have to analyze the whole system at once because the feedback terms force an approach starting from the first equation and the feedforward terms from the last equation.

Theorem 5 *Consider the $\mathcal{C}^1$ system:*

$$\left\{\begin{array}{lll} \dot{x}_1 & = & f_1(x_1) + x_2 + g_1(\dot{x}_2, \dot{x}_3, \cdots, \dot{x}_n) \\ \dot{x}_2 & = & f_2(x_2) + x_3 + g_2(\dot{x}_3, \cdots, \dot{x}_n) \\ & \vdots & \\ \dot{x}_{n-1} & = & f_{n-1}(x_{n-1}) + x_n + g_{n-1}(\dot{x}_n) \\ \dot{x}_n & = & f_n(x_n) + u \end{array}\right.$$

with x_i, $u \in I\!R$. Suppose that the feedback interconnections are bounded, with bounded derivatives, i.e. $\exists\, M_i, D_i > 0\, \forall\, x_i : |f_i(x_i)| < M_i$, $|f_i'(x_i)| < D_i$ and the g_i functions are at least quadratic near the origin.

Then there exists positive constants $\epsilon_i > 0$ (sufficiently small) and $K_i > 0$ (sufficiently large) such that the origin of the system is globally asymptotically and locally exponentially stabilized by

$$u = -f_n(x_n) - \ sat_{\epsilon_n}(K_n e_n) \tag{2.14}$$

where

$$\begin{array}{rcl} e_1 & = & x_1 \\ e_i & = & x_i + f_{i-1}(x_{i-1}) + \ sat_{\epsilon_{i-1}}(K_{i-1}e_{i-1}) \quad for\ i \in \{2, \cdots, n\} \end{array}$$

for a suitable choice of ϵ_i and K_i. One such choice is:

$$\begin{array}{ll} K_1 = K & K_{i+1} = 16(K_i + D_i) \\ \epsilon_1 = \epsilon & \frac{\epsilon \prod_{j=2}^{i} K_j}{2^{i-1}} \leq \epsilon_i \leq \frac{3}{2} \frac{\epsilon \prod_{j=2}^{i} K_j}{2^{i-1}} \end{array} \tag{2.15}$$

Proof The control law (2.14) cannot generate finite escape time. Indeed, the $\dot{x}_i$ equation is:

$$\dot{x}_i = f_i(x_i) + x_{i+1} + g_i(\dot{x}_{i+1}, \cdots, \dot{x}_n)$$

which can be rewritten as

$$\dot{x}_i = f_i(x_i) + x_{i+1} + h_i(x_{i+1}, \cdots, x_n, - \ \mathrm{sat}_{\epsilon_n}(K_n e_n))$$

Finite escape time can only arise through a x_i term in the right-hand side of this equation. Those x_i terms are in $f_i(x_i)$ and in $\mathrm{sat}_{\epsilon_n}(K_n e_n)$, which both are bounded functions, and cannot generate finite escape time.

Consider the *nested* sequence of subsets $\Omega_1 \subset \cdots \subset \Omega_n \subset I\!R^n$ defined by

$$\Omega_i = \{x \in I\!R^n |\ K_j|e_j| \leq \epsilon_j, j \geq i\}$$

whose choice is suggested by the form of the control law:

We will show that the choice (2.15) for the parameters K_i and ϵ_i ensures the following properties:

(i) each Ω_i is invariant

(ii) each solution in $I\!R^n$ reaches Ω_n in finite time, and each solution in Ω_{i+1} reaches Ω_i in finite time, so that each solution reaches $\Omega = \Omega_1$ in finite time

(iii) each solution in Ω converges exponentially to the equilibrium $x = 0$

We first show that the Jacobian linearization of the closed-loop system is Hurwitz when the parameters K_i are chosen accordingly to (2.15), with K large enough. The Jacobian linearization of the system in x coordinates is:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & F_1 x_1 + x_2 \\ & \vdots & \\ \dot{x}_i & = & F_i x_i + x_{i+1} \\ & \vdots & \\ \dot{x}_n & = & -\sum_{k=1}^{n} \left(\prod_{j=k+1}^{n} K_j\right)(F_k + K_k) x_k \end{array}\right.$$

where we define $F_i = f_i'(0)$ (with $F_n = 0$ because $f_n(x)$ is compensated by u). Introducing the change of coordinates $y_i = \frac{x_i}{K^i}$ and the new time scale $s = Kt$, the matrix of this system can be decomposed into two parts:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_2 & -a_1 \end{pmatrix}$$

$$+ \begin{pmatrix} \times & 0 & 0 & \cdots & 0 & 0 \\ 0 & \times & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & \times & 0 \\ \times & \times & \times & & \times & 0 \end{pmatrix} = T + U$$

where $a_i = \lim_{K\to\infty} \frac{K_n \cdots K_{n-i+1}}{K^i}$. The choice (2.15) ensures that the matrix T is Hurwitz and that the matrix U converges to 0 as K tends to ∞. Therefore, the matrix A is Hurwitz for K large enough. We have thus fixed the gains K_i such that our controller ensures local exponential stability of the system. The original system controlled without saturation has a region of attraction Λ (independent of ϵ). The rest of the proof will show that any solution of the controlled system reaches the set Ω and that we can design this set to be included inside Λ by taking ϵ small enough.

To this end, we use the following lemma, proven in Appendix, showing crucial properties that are verified when the parameters satisfy inequalities that are verified by the choice of parameters (2.15)

Lemma 1 Suppose that we have parameters such that:

$$\begin{array}{lll} \epsilon_1 & > & \frac{\epsilon_2}{K_2} \\ \epsilon_i & > & \frac{\epsilon_{i+1}}{K_{i+1}} + 2(D_{i-1} + K_{i-1})\epsilon_{i-1} \quad i \in \{2, \cdots, n-1\} \\ \epsilon_n & > & 2(D_{n-1} + K_{n-1})\epsilon_{n-1} \end{array} \tag{2.16}$$

then (2.16) is still satisfied after a $\mu - scaling$ of the ϵ_i parameters, that is if we replace ϵ_i by $\mu\epsilon_i$, and, for μ small enough, we have: $\forall e \in \Omega_i$:

a $|\dot{x}_i| < 2\epsilon_i$

b $|e_i| = \frac{\epsilon_i}{K_i} \Rightarrow e_i\dot{e}_i < 0$

c $|\dot{e}_i| < 2\epsilon_i$

It is an easy calculation to verify that (2.16) is satisfied with the choice (2.15) and that a μ-scaling is achieved by taking ϵ small enough.

Part **(i)** of the proof of Theorem 1 is a consequence of Lemma 1 (b). When a solution is on $\partial\Omega_i$, it satisfies $|e_j| = \frac{\epsilon_j}{K_j}$ for some $j \geq i$. Due to the fact that $\Omega_i \subset \Omega_j$ for $j \geq i$, Lemma 1 (b) implies $e_j\dot{e}_j < 0$. Therefore, the solution stays inside Ω_i.

In order to prove **(ii)**, we will now show that any solution starting in $I\!R^n \setminus \Omega_n$ reaches Ω_n in finite time and that any solution in Ω_{i+1} reaches Ω_i in finite time. By the invariance property of Ω_i, this means that any solution reaches $\Omega_1 = \Omega$ in finite time.

Let us define

$$\phi_i(x_i, e_i) = f_i(x_i) + \text{ sat}_{\epsilon_i}(K_i e_i)$$

Let $x(0) \in I\!R^n \setminus \Omega_n$. As long as $x(t) \notin \Omega_n$, the solution satisfies:

$$\dot{x}_n = -\epsilon_n \text{ sign}(x_n + \phi_{n-1}(x_{n-1}, e_{n-1}))$$

which means that x_n converges towards $-\phi_{n-1}(x_{n-1}, e_{n-1})$. Because ϕ_{n-1} is bounded, $x_n(t)$ eventually enters an interval where the saturation is no longer active. This happens when $|x_n + \phi_{n-1}(x_{n-1}, e_{n-1})| = |e_n| = \frac{\epsilon_n}{K_n}$: the solution enters Ω_n in finite time.

Next, we show that any solution in Ω_{i+1} reaches Ω_i in finite time. The solution satisfies

$$\dot{x}_i = e_{i+1} - \text{ sat}_{\epsilon_i}(K_i(x_i + \phi_{i-1}(x_{i-1}, e_{i-1}))) + g_i(\dot{x}_{i+1}, \cdots, \dot{x}_n) \tag{2.17}$$

and as long as $x_i(t) \in \Omega_{i+1} \setminus \Omega_i$, (2.17) can be rewritten using Lemma 1 (b) as:

$$\dot{x}_i = e_{i+1} - \epsilon_i \text{ sign}(x_i + \phi_{i-1}(x_{i-1}, e_{i-1})) + \mathcal{O}(\epsilon_{i+1}^2, \cdots, \epsilon_n^2)$$

Because $|e_{i+1}| \leq \frac{\epsilon_{i+1}}{K_{i+1}} < \epsilon_i$, the second term dominates the others while $x_i(t) \in \Omega_{i+1} \setminus \Omega_i$. Hence $x_i(t)$ converges towards ϕ_{i-1} until the saturation stops being active, i.e. the solution has reached Ω_i.

Repeating the argument for each i, the solution reaches Ω in finite time. Observe that a prescaling of the parameters ϵ_i can be used to include Ω in an arbitrarily small compact set containing the origin (this means that practical stability is achieved for any choice of the K_i satisfying (2.16)).

Part **(iii)** of the proof is direct from the fact that A is Hurwitz. It is sufficient to take Ω to be included inside Λ, the region of attraction of the controlled system without saturation (because Ω is an invariant set for the system controlled without saturation). In order to do so, we take ϵ small enough.

□

The proof of the previous theorem shows why we may use the term slow control: the derivatives of the variables are successively brought to small values in order to be able to neglect the values of the g_i functions.

When $n = 2$, condition (2.15) on the parameters reduces to $K_2 > 16(K_1 + D_1)$ (in this case, it could be weakened to $K_2 > 2(K_1 + D_1)$). We observe that the latter condition is stronger than the condition imposed for the local asymptotic stability, which only requires $K_2 > D_1$. This shows that the K_i constants must be chosen large enough not only to ensure local asymptotic stability, but also to render the designed manifolds near-invariant.

2.3 Examples

Example 4 *We will now illustrate our control design on the second example of (1.2), which contains the bounded feedback term of our running examples and a rate-limiting feedforward nonlinearity:*

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & \frac{x_1}{1+x_1^2} + x_2 + u^2 \\ \dot{x}_2 & = & u \end{array} \right. \tag{2.18}$$

This system is in the form (2.13) because $u^2 = \dot{x}_2^2$, so that we can calculate a globally stabilizing controller. We first calculate the e coordinates:

$$e_1 = x_1 \qquad e_2 = \frac{x_1}{1+x_1^2} + x_2 + \ sat_{\epsilon_1}(K_1 x_1)$$

and the control law is:

$$u = -\ sat_{\epsilon_2}\left(\frac{x_1}{1+x_1^2} + x_2 + \ sat_{\epsilon_1}(K_1 x_1)\right)$$

with K_1, K_2, ϵ_1, ϵ_2 satisfying the constraint (2.15). Noting that $M_1 = 1$ and $D_1 = 1$, we take

$$\begin{array}{ll} K_1 = K & K_2 = 16(K+1) \\ \epsilon_1 = \epsilon & \epsilon = 10(K+1)\epsilon \end{array}$$

The phase plane of the evolution of the resulting control system is illustrated on Figure 2.1 for the particular choice $K_1 = K = 1$, $K_2 = 32$, $\epsilon_1 = \epsilon = 0.05$ and $\epsilon_2 = 1$. The solution is simulated for two different initial conditions: $x_0 = (-5\ 1)^T$ and $x_0 = (5\ 0)^T$. Both those solutions first enter an interval around the manifold $x_2 + \frac{x_1}{1+x_1^2} = 0$. In that interval, e_2 is small, which means that u is small and the u^2 term does not perturb the x_1 equation. In fact, we have $e_2 \approx 0$, which means that $\dot{x}_1 \approx -\ sat_{\epsilon_1}(K_1 x_1)$. As long as x_1 is far from 0, this result in $\dot{x}_1 = -\epsilon_1\ sign(x_1)$ and x_1 slowly converges to the origin. Therefore, the convergence of the complete solution along the manifold is very slow. Finally, when both x_1 and x_2 are close to the origin, the convergence is exponential.

Example 5 *The calculation of the parameters is now illustrated on the benchmark system (2.12) with f a bounded function such that $\|f'(x_1)\|_\infty < D$ and $g_1(x_2, x_3) = x_3^2 = \dot{x}_2^2$:*

$$\left\{ \begin{array}{lcl} \dot{x}_1 &=& f_1(x_1) + x_2 + x_3^2 \\ \dot{x}_2 &=& x_3 \\ \dot{x}_3 &=& u \end{array} \right. \tag{2.19}$$

The set of constraints (2.16) is:

$$\begin{array}{lcl} \epsilon_1 &>& \frac{\epsilon_2}{K_2} \\ \epsilon_2 &>& \frac{\epsilon_3}{K_3} + 2(D_1 + K_1)\epsilon_1 \\ \epsilon_3 &>& 2K_2\epsilon_2 \end{array} \tag{2.20}$$

which could easily be solved for such a low-order system but becomes more intricate for larger dimensions. Therefore, we rather use the explicit parameter values that we gave in Theorem 5.

Let us take $\epsilon_1 = \epsilon$ small enough and $K_1 = K$ large enough. We choose the parameters according to (2.15) with $K_{i+1} = 16(D_i + K_i)$ and ϵ_i in the middle of the intervals:

$$\begin{array}{lll} K_1 = K & K_2 = 16(D+K) & K_3 = 256(D+K) \\ \epsilon_1 = \epsilon & \epsilon_2 = 10\epsilon(D+K) & \epsilon_3 = 1280\epsilon(D+K)^2 \end{array}$$

With such parameters, the control law is:

$$\begin{array}{lcl} u &=& -\ sat_{1280\epsilon(D+K)^2}[256(D+K)(x_3 +\ sat_{10\epsilon(D+K)}[16(D+K)(x_2 \\ && +f_1(x_1) +\ sat_\epsilon[Kx_1])])] \end{array}$$

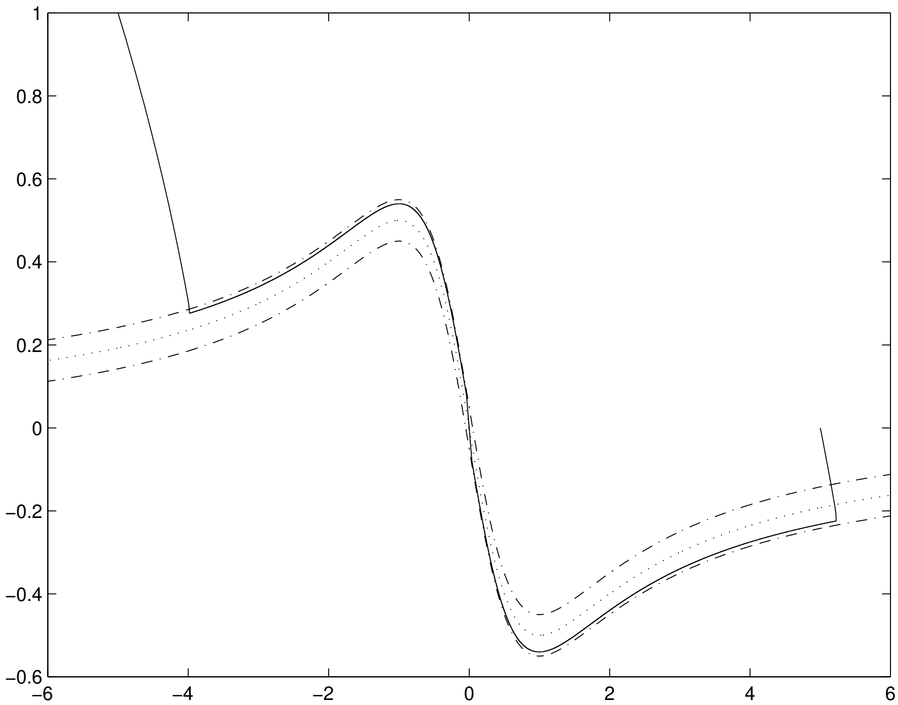

Figure 2.1: Phase plane of the slow control of example (2.18) with $x_0 = (-5\ 1)^T$ and $x_0 = (5\ 0)^T$ as initial conditions (solid line). The manifold $x_2 + \frac{x_1}{1+x_1^2} = 0$ is illustrated (dotted line), as well as the invariant interval around it (between the dash-dotted lines)

We see that both K_i and ϵ_i parameters are increasing. Such a controller can be designed to satisfy any rate constraint on the control and on the states. K must be taken sufficiently large in order to get asymptotic stability. ϵ must be sufficiently small to be able to neglect the ϵ_2^2 term that will arise because of the $x_3^2 = \dot{x}_2^2$ term.

We can justify on this example the boundedness condition of Theorem 5 imposed on the f_i functions. Indeed, when $f_1 : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$ function such that $\lim_{s\to+\infty} f(s) = +\infty$ and $\exists\epsilon > 0 : \forall s \in \mathbb{R} : f_1'(s) > \epsilon$, system (2.19) cannot be globally controlled to the origin. This follows from the calculation:

$$\begin{aligned} \frac{d}{dt}(f_1(x_1) + x_2) &= f_1'(x_1)(f_1(x_1) + x_2) + f_1'(x_1)x_3^2 + x_3 \\ &\geq f_1'(x_1)(f_1(x_1) + x_2) - C \\ &\geq \epsilon(f_1(x_1) + x_2) - C \text{ when } f_1(x_1) + x_2 \geq 0 \end{aligned}$$

where we have used the fact that $f_1'(x_1)x_3^2 + x_3$ is bounded from below by a negative constant $-C$. If the initial condition of the system is such that

$\epsilon(f_1(x_{10}) + x_{20}) > C$, the quantity $f_1(x_1) + x_2$ diverges to $+\infty$ regardless of the control law. It means that such initial conditions (x_{10}, x_{20}) cannot be driven to the origin, even if f_1 is a simple linear function.

2.4 Limitations of slow control

The slow control design proposed in this chapter applies to a restricted class of nonlinear systems which simultaneously present destabilizing feedback connections and "rate limiting" feedforward connections. The restrictions on the feedforward connections (to be "rate limiting" rather than "gain limiting") and the boundedness of the f_i functions (and of their derivatives) have been justified by means of elementary (that is, scalar) controllability requirements.

A subtler limitation of slow control is the fact that the f_i's cannot depend on the states $x_1, \ldots, x_{i-1}$: the feedback connections are restricted to be "local" connections around each integrator. Thus our design can yield a slow control for the second order system

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& f_1(x_1) + x_2 \\ \dot{x}_2 &=& f_2(x_1, x_2) + u \end{array} \right.$$

but this does not mean that this slow control law can be "backstepped" to yield a slow control law for the augmented system

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& f_1(x_1) + x_2 \\ \dot{x}_2 &=& f_2(x_1, x_2) + x_3 \\ \dot{x}_3 &=& u \end{array} \right. \tag{2.21}$$

even if boundedness is assumed for f_2 and its derivatives. Indeed, the x_1 dependency of f_2 implies that

$$e_3 = x_3 + f_2(x_1, x_2) + \ \mathrm{sat}_{\epsilon_2}(K_2 e_2)$$

depends on x_1. Our control scheme ensures that, once e_3 is small, it stays small, and $\dot{x}_3$ and $\dot{e}_3$ are small. However, in this case, if e_3 is small with e_2 large, e_3 cannot be kept small because a large $\dot{x}_1$ term appears in the expression of $\dot{e}_3$: when f_2 depends on x_1, we cannot guarantee that $\dot{e}_3$ is small and near invariance of the manifold e_3 is lost.

This is in contrast with recent bounded backstepping results (Freeman & Praly 1998, Tsinias 1997) which show that a bounded control law with bounded rate can be backstepped to yield a new bounded control law with bounded rate (when all the nonlinearities are bounded with bounded derivatives). The difference with our result is that the bound on the control magnitude (or rate) is not a free design parameter in Freeman & Praly (1998) and

Tsinias (1997). In the present chapter, this flexibility is needed to account for the feedforward nonlinearities. For this reason, it seems plausible that the boundedness assumptions of the present chapter do not suffice to achieve global stabilization of a system as simple as

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & f_1(x_1) + x_2 + g(u), \qquad g(0) = 0 \\ \dot{x}_2 & = & f_2(x_1) + x_3 \\ \dot{x}_3 & = & u \end{array}\right. \tag{2.22}$$

which violates the structure covered by the present chapter only because f_2 depends on x_1 rather than x_2.

2.5 Robust stabilization of feedforward systems

Apart from allowing for the stabilization of feedforward systems with exponentially unstable Jacobian linearization, the control law (2.14) also allows for the robust global stabilization of feedforward systems in a form similar to (2.13) (see Marconi & Isidori (2000)). The class of systems

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & \mu_1 x_2 + g_1(\dot{x}_2, \dot{x}_3, \cdots, \dot{x}_n, \mu) \\ \dot{x}_2 & = & \mu_2 x_3 + g_2(\dot{x}_3, \cdots, \dot{x}_n, \mu) \\ & \vdots & \\ \dot{x}_{n-1} & = & \mu_{n-1} x_n + g_{n-1}(\dot{x}_n, \mu) \\ \dot{x}_n & = & \mu_n u \end{array}\right. \tag{2.23}$$

where μ_i is an uncertain, positive, time-varying parameter ($0 < \mu_i^L \leq \mu_i \leq \mu_i^U$, with μ_i^L and μ_i^U positive constants), and the feedforward nonlinearities are locally Lipschitz. The designed control law then has to be essentially robust to uncertainties on the gain at the input of each integrator. The control law

$$u(x) = -\ \mathrm{sat}_{\epsilon_n}(K_n e_n)$$

where

$$\begin{array}{lcl} e_1 & = & x_1 \\ e_i & = & x_i +\ \mathrm{sat}_{\epsilon_{i-1}}(K_{i-1} e_{i-1}) \quad \text{for } i \in \{2, \cdots, n\} \end{array}$$

with a suitable choice of ϵ_i and K_i, globally stabilizes the uncertain feedforward system (2.23). This choice of ϵ_i and K_i is similar to the one given in Lemma 1, and a μ-scaling of the ϵ_i parameters then ensures global stability.

The proof relies on the small-gain theorem of Teel (1996): considering the closed-loop system

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & \mu_1 x_2 + v_1 \\ \dot{x}_2 & = & \mu_2 x_3 + v_2 \\ & \vdots & \\ \dot{x}_{n-1} & = & \mu_{n-1} x_n + v_{n-1} \\ \dot{x}_n & = & \mu_n u(x) \end{array}\right.$$

it is shown that the gain between the inputs v_j and the outputs $g_j(\dot{x}_{j+1}, \cdots, \dot{x}_n)$ can be arbitrarily decreased by taking ϵ small enough. Therefore, once the outputs $g_j(\dot{x}_{j+1}, \cdots, \dot{x}_n)$ are fed back into v_j, global asymptotic stability is ensured by taking ϵ small enough.

2.6 Conclusion

We have considered the stabilization of structured systems: systems in feedback and feedforward form, and shown that the combination of both types of interconnections could lead to obstacles to stabilizability because the gain-limitation introduced at each integrator by the feedforward interconnections can be incompatible with the gain requirement of the feedback interconnections. Our result, which was published in Grognard et al. (1999), introduces a design that could be applied to a class of systems which simultaneously present "destabilizing" feedback connections and "rate limiting" feedforward connections. Under the restriction that the feedback connections are "local" connections around each integrator, a slow control design has been proposed which enforces a slow convergence towards a nested sequence of manifolds, the last of which is a stable manifold of the closed-loop system. Near invariance of the successive manifolds is achieved by allowing for enough gain in their neighborhood, yet keeping the control *slow* in the entire state space. The solution slowly evolve in the state-space; this control law only pursues stability without performance considerations. We have made a first step towards a solution of the problem of robustness of control laws for feedforward systems to the presence of feedback terms.

Chapter 3

Improving the performance of low-gain designs for bounded control of linear systems

The design of feedback control laws for linear systems subject to magnitude constraints on the control variable,

$$\dot{x} = Ax + bu \qquad u \in I\!R, \quad |u| \leq 1 \tag{3.1}$$

has long been recognized as a significant nonlinear control problem, and has given rise to the control methods exposed in Chapter 1. Notably, low-gain control laws have been designed to stabilize the origin with large regions of attractions. In Teel (1995) and Lin et al. (1996), such control laws have been developed from the algebraic Riccati equation, so that they have the form $u = -b^T P(\epsilon)x$; in this chapter, we will work on control laws based on that form. In such designs, the tuning of the low-gain parameter ϵ involves two conflicting objectives: on the one hand, large regions of attraction require small values of $\epsilon > 0$ so that the input bound is never attained along the solutions. Obviously, this leads to cautious designs, resulting in slow convergence. In contrast, local performance dictates a larger value of ϵ, resulting in a reduced guaranteed region of attraction. Two approaches have been taken in the literature to obtain more aggressive designs: gain-scheduling and low-and-high gain techniques.

Based on the rationale that a small value ϵ_0 is needed far from the origin for stability and that a larger value ϵ_f ($= 1$ without loss of generality) is needed close to the origin for performance, a scheduling of the parameter was first introduced in Megretski (1996): for each value of $\epsilon \in (0, 1]$, a level set of a Lyapunov function is used as estimate of the guaranteed region of attraction. The scheduling consists then in a choice of $\epsilon(x)$ such that, at every point, x lies on the boundary of the guaranteed region of attraction for the system steered by the controller $u = -b^T P(\epsilon)x$ with ϵ fixed. However, in most cases, such a design never makes full use of the available actuation.

The high-and-low gain method of Saberi et al. (1996), which explicitly uses the infinite gain margin of Riccati based linear control laws $u = -b^T P(\epsilon)x$, consists in multiplying this control law by a large constant $k > 1$ in order to achieve a better use of the available actuation without modifying the guaranteed region of attraction. This control law, $u = -\operatorname{sat}(kb^T P(\epsilon)x)$, results in a fast convergence of $b^T P(\epsilon)x$ to 0 (with $u = \pm 1$), followed by near invariance of the manifold $b^T P(\epsilon)x = 0$. The rate of convergence then strongly depends on the behavior of the system on this manifold. We will show, on an example, that this rate can be slow.

The observation of the shortcomings of those methods and that time-optimal control can be much faster motivates the introduction of the new gain-scheduling of Grognard et al. (2000a). Choosing an initial controller that ensures a sufficiently large region of attraction and a final controller that ensures good local performance, our objective is to design a scheduling ensuring the fastest possible transition between these two extreme controllers along the closed-loop solutions, while ensuring the satisfaction of the constraint. This is achieved by a constrained pointwise maximization of $\dot{\epsilon}$; Megretski (1996) enforces such a maximization with the form of the controller fixed to $u = -\operatorname{sat}(kb^T P(\epsilon)x)$. We will present a different form of transition that ensures such a maximization with an invariant other than the form of the control law.

The choice of this invariant results from the observation that the high-and-low gain philosophy enforces $b^T P(\epsilon)x \approx 0$ after some time, while ϵ is chosen small enough such that $|b^T P(\epsilon)x| \leq 1$ in the guaranteed region of attraction. The mismatch between what eventually occurs ($b^T P(\epsilon)x \approx 0$) and what is checked ($|b^T P(\epsilon)x| \leq 1$) indicates that the choice of ϵ is conservative, and that improvement can be brought by changing criterion. Our criterion checks that x lies in a region of the state space where invariance of the manifold $b^T P(\epsilon)x = 0$ can be forced with $|u| \leq 1$. Through that criterion, that is more relevant to the actual behavior of the system, a larger ϵ can be used in a larger region of the state-space, and $\dot{\epsilon}$ can be chosen to be larger so that $\epsilon = 1$ is reached earlier. The invariant of our scheduling is then $b^T P(\epsilon)x = 0$. An analytical example and simulations suggest that these heuristics lead to accelerated convergence of the closed-loop solutions.

This chapter is organized as follows. In Section 3.1, we expose conditions for the global stabilizability of linear systems with bounded input. Low-gain designs are then exposed in Section 3.2. The general algorithm is presented in Section 3.3, while the design of the scheduling controller is detailed in Section 3.4. Closed-loop convergence and performance considerations are given in Section 3.5. The algorithm is then illustrated on the double and triple integrators in Section 3.6 and 3.7, with a comparison with the earlier scheme proposed in Megretski (1996) and the time-optimal solution. In Section 3.8 and 3.9, we extend this result to linear systems with amplitude and rate limited input and to linear systems with affine constraints. Finally, we give some conclusions.

3.1 Global and semiglobal stabilizability

A simple example shows that all linear systems are not globally stabilizable when u is bounded. Let us consider the first order system

$$\dot{x} = x + u \tag{3.2}$$

with the constraint $|u| \leq 1$. If $|x(0)| > 1$, the solution obviously diverges towards $\pm\infty$, for any choice of bounded control $u(t)$. Only the initial conditions inside the open interval $(-1, 1)$ can be driven to the origin with bounded controls. The system is therefore not globally controllable to the origin, which means that it is certainly not globally stabilizable.

This notion leads to the following definition

Definition 6 *The domain of null-controllability $\mathcal{NC}$ is the set of initial conditions $x(0)$ for which there exists a control law $u(t)$ (defined for $t \in [0,T]$) such that $|u(t)| \leq 1$ for all t and $x(T) = 0$.*

In the example, the domain of null controllability is the interval $(-1, 1)$. In this chapter, we will focus on systems whose domain of controllability is the whole state-space; those systems are said to be asymptotically null-controllable. A characterization of asymptotically null-controllable systems is given in Schmittendorf & Barmish (1980).

Theorem 6 (Schmittendorf & Barmish (1980)) *System (3.1) is asymptotically null-controllable iff*

1. *the pair (A, b) is stabilizable*
2. *all eigenvalues of A have non positive real part.*

The origin of such systems is then globally stabilizable with saturated smooth feedback:

Theorem 7 (Sontag & Sussmann (1990)) *For system (3.1), the following two conditions are equivalent:*

1. *there is a smooth feedback* $k : \mathbb{R}^n \to \mathbb{R}$, $k(0) = 0$, *so that zero is a globally asymptotically stable equilibrium for the closed-loop system*

$$\dot{x} = Ax + b\ sat(k(x))$$

2. *system (3.1) is asymptotically null controllable.*

Those two theorems ensure the existence of a saturated smooth, globally stabilizing, feedback when the conditions of Theorem 6 are satisfied. In general, this feedback control law cannot simply be a saturated linear control law (Fuller 1969), and, in particular, this negative result is already valid for as simple a system as the third order chain of integrators. Therefore, globally stabilizing control laws need to be nonlinear (e.g. Teel (1992a)). In contrast, Lin & Saberi (1993) have shown that *semiglobal stabilization* by linear state feedback can be achieved through pole-placement under the conditions of Theorem 6.

Assumption 1 *In the remainder of this chapter, we will only consider asymptotically null controllable linear systems with bounded input.*

3.2 Explicit low-gain designs and tuning of the parameter

This section is devoted to the study of saturated linear low-gain control laws, which are in the form

$$u = -\ \mathrm{sat}\left(b^T P(\epsilon)x\right) \tag{3.3}$$

based on the quadratic Control Lyapunov Functions

$$V(x,\epsilon) = x^T P(\epsilon)x$$

The control law is given as an explicit function of x, so that such a low-gain design is said to be explicit.

The ϵ-family of control laws (3.3) can be generated in various ways; one method (see Teel (1995) and Lin et al. (1996)) uses the Riccati equation

$$P(\epsilon)A + A^T P(\epsilon) - P(\epsilon)bb^T P(\epsilon) = -Q(\epsilon), \quad \epsilon \in (0,\infty) \tag{3.4}$$

with $Q(\epsilon) > 0$ (positive definite), continuous, such that $\lim_{\epsilon\to 0} Q(\epsilon) = 0$ and $\frac{dQ(\epsilon)}{d\epsilon} > 0$. The Riccati matrix $P(\epsilon)$ generated from this equation then satisfies

Lemma 2 [Lin (1998)] If (A, b) is asymptotically null controllable then

1. For all $\epsilon \in (0, 1]$, equation (3.4) has a unique solution $P(\epsilon) > 0$, which is such that $A - bb^T P(\epsilon)$ is Hurwitz;

2. $\lim_{\epsilon \to 0} P(\epsilon) = 0$;

3. $\frac{dP(\epsilon)}{d\epsilon} > 0$ and is solution of the Lyapunov equation

$$\frac{dP(\epsilon)}{d\epsilon}(A - bb^T P(\epsilon)) + (A - bb^T P(\epsilon))^T \frac{dP(\epsilon)}{d\epsilon} = -\frac{dQ(\epsilon)}{d\epsilon}$$

The choice of $Q(\epsilon)$ that will ensure the best behavior of the closed-loop system is an inherent part of the design problem; it is not generically solved in the literature, and it is not tackled here. Therefore, unless the structure of the system suggests a particular choice that would ensure an easy calculation of $P(\epsilon)$ (like in the example below), there is no reason to make another choice than $Q(\epsilon) = \epsilon I$.

Remark 1 *Megretski (1996) suggests a simplified construction of $P(\epsilon)$ based on the Riccati inequality:*

$$P(\epsilon)A + A^T P(\epsilon) + P(\epsilon)W^{-2}P(\epsilon) - P(\epsilon)bb^T P(\epsilon) < 0 \tag{3.5}$$

with $W > 0$ a symmetric matrix. The advantage of (3.5) is that, if the region of attraction is only required to contain a given compact set Ω, one does not need to explicitly solve the Riccati inequality for every ϵ: one can calculate a solution P_0 of (3.5) such that $u = -\operatorname{sat}\left(b^T P_0 x\right)$ ensures Ω to be contained in the region of attraction, as well as P_1 $(> P_0)$ such that $u = -b^T P_1 x$ ensures satisfying local performance. Then the matrix:

$$P(\epsilon) = \left[(1 - \epsilon) P_0^{-1} + \epsilon P_1^{-1}\right]^{-1} \qquad \epsilon \in [0, 1] \tag{3.6}$$

is solution of (3.5), with $A - bb^T P(\epsilon)$ Hurwitz for all $\epsilon > 0$, and $\frac{dP(\epsilon)}{d\epsilon} > 0$. This construction leads to a control law that guarantees semiglobal stabilization only inside Ω. It is useful for practical implementation, because it does not require the online solution of the algebraic Riccati equation (3.4). The resulting function $P(\epsilon)$ is an interpolation between P_0, which is chosen for stabilization, and P_1, which is chosen for performance. Despite the implementation advantage of such a choice, we use (3.4) in the remainder of this chapter to build $P(\epsilon)$.

For a fixed $\epsilon > 0$, the maximal level set inside which $|b^T P(\epsilon)x| \leq 1$ provides a guaranteed region of attraction for $x = 0$. The choice of applying the control law only inside this level set $V(x,\epsilon) \leq \bar{V}(\epsilon)$ is the main source of conservatism of the Lyapunov methods because the actual region of attraction is usually much larger. This level is the solution of the optimization problem

$$\begin{array}{rcll} \bar{V}(\epsilon) & = & \min_{x \in I\!R^n} & x^T P(\epsilon) x \\ & & \text{s.t.} & b^T P(\epsilon) x = 1 \end{array} \tag{3.7}$$

(with the absolute value superfluous because of symmetry), which yields:

$$x_\epsilon = \frac{b}{b^T P(\epsilon) b}$$

and $|b^T P(\epsilon)x| \leq 1$ within the set

$$\mathcal{T}(\epsilon) = \{x \in I\!R^n | x^T P(\epsilon) x \leq \frac{1}{b^T P(\epsilon) b}\}$$

Finding $\epsilon(x_0)$ then amounts to look for the largest ϵ such that x_0 lies within the set $\mathcal{T}(\epsilon)$, that is:

$$\epsilon(x_0) = \max\{\eta \in (0,1] : (x_0^T P(\eta) x_0)(b^T P(\eta) b) \leq 1\} \tag{3.8}$$

With the choices (3.4), we have $\frac{dP(\epsilon)}{d\epsilon} > 0$, and large regions of attraction Ω require small values of the parameter $\epsilon > 0$. As a consequence, the resulting design is "cautious", that is, it uses little actuation near the origin and makes the convergence slow.

High-and-low gain A first improvement of the above low-gain design is the "high-low" gain modification based on the observation that the control laws (3.3) have infinite gain margin. As a consequence, the region of attraction achieved with the control law $u = -\,\text{sat}(k b^T P(\epsilon) x)$ still includes the set $\mathcal{T}(\epsilon)$ with any gain $k \geq 1$. The limiting case for $k \to \infty$ results in the sliding mode control

$$u = -\,\text{sign}(b^T P(\epsilon) x) \tag{3.9}$$

In this situation, full actuation is used throughout. However, the motion along the sliding surface $b^T P(\epsilon) x = 0$ induces chattering and can be very slow when ϵ is small. This will be illustrated on the following example.

Example 6 *The high-and-low gain philosophy is now illustrated on the double integrator:*

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \qquad |u| \leq 1 \end{cases}$$

The simplicity of this second order system allows for analytical calculations: if we consider that the behavior obtained with the pole placement in $\frac{-\sqrt{3}\pm i}{2}$ *(which is achieved with* $Q = I$*) is satisfying, we can solve the Riccati equation (3.4) with:*

$$Q(\epsilon) = \begin{pmatrix} \epsilon^4 & 0 \\ 0 & \epsilon^2 \end{pmatrix} \tag{3.10}$$

(where we do not use $Q(\epsilon) = \epsilon I$ *in order to obtain simpler expressions). This results in the Lyapunov matrix:*

$$P(\epsilon) = \begin{pmatrix} \sqrt{3}\epsilon^3 & \epsilon^2 \\ \epsilon^2 & \sqrt{3}\epsilon \end{pmatrix}$$

and the family of low-gain controls is then:

$$u = -b^T P(\epsilon)x = -\epsilon^2 x_1 - \sqrt{3}\epsilon x_2 \qquad \epsilon > 0$$

which is a typical low-gain control for second order systems in Brunovski form.

When the high-and-low gain control law (3.9) is applied, $b^T P(\epsilon)x$ *converges to zero, and the solution slides along*

$$x_2 = -\frac{\epsilon}{\sqrt{3}} x_1$$

while u *chatters between* $+1$ *and* -1*. The equivalent control (Utkin (1992)) to the chattering inducing the sliding is then*

$$u = -\frac{\epsilon x_2}{\sqrt{3}}$$

which places the poles of the closed-loop system in 0 *and* $-\frac{\epsilon}{\sqrt{3}}$*. The pole corresponding to the convergence to the origin is* $-\frac{\epsilon}{\sqrt{3}}$*. If* ϵ *is small because* $x(0)$ *is large, the system then evolves slowly once* $b^T P(\epsilon)x \approx 0$ *(even when* $k = +\infty$*). In this chapter, we will introduce an alternative to the choice (3.7) of* $\bar{V}(\epsilon)$ *so that the region of attraction corresponding to* ϵ *is enlarged. This in turn indicates that larger values of* ϵ *can be used for identical guaranteed regions of attraction. We will illustrate this for* $\epsilon = 0.5$ *of Figure 3.1. For* ϵ *fixed, the guaranteed region of attraction of the high-and-low gain control* *law is* $\mathcal{T}(\epsilon) \equiv x^T P(\epsilon)x \leq \frac{1}{\sqrt{3}\epsilon}$ *(dashed ellipse) inside which solutions are forced to* $b^T P(\epsilon)x = 0$ *(solid line) where convergence takes place to the origin. The level of* $\mathcal{T}(\epsilon)$ *is determined by the fact that* $|b^T P(\epsilon)x| = 1$ *at the stars on its boundary. If, instead, invariance of the manifold is considered to determine the level set, we will show that the guaranteed region of attraction is* $x^T P(\epsilon)x \leq \frac{6\sqrt{3}}{\epsilon}$ *(dotted ellipse) and* $b^T P(\epsilon)x = 0$*. The level of this region is fixed by the control that guarantees invariance of the manifold, which is equal to* 1 *at the* $+$*. Instead a criterion on the whole two dimensional state-space, we check a criterion in one dimension, the dimension of the manifold* $b^T P(\epsilon)x = 0$*. The problem of reaching this manifold is treated in the next section.*

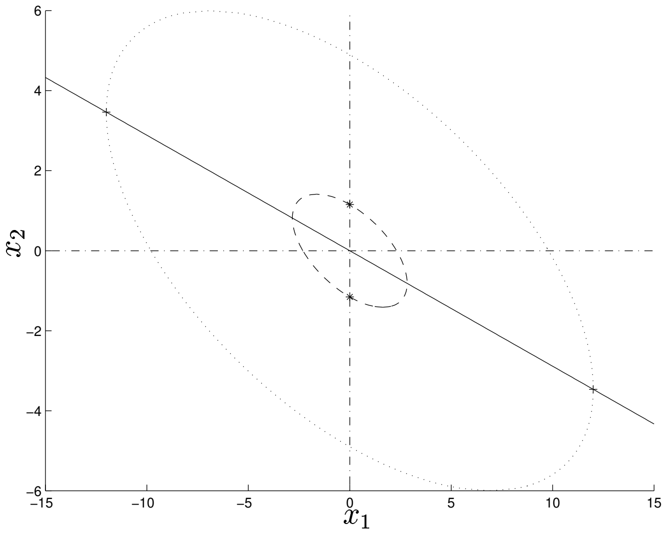

Figure 3.1: The manifold $b^T P(0.5)x = 0$ is drawn (solid line). Both the high-and-low-gain method and the invariance checking method force the solution to follow this manifold. This can be achieved in the dashed level set for the high-and-low gain method because $|b^T P(0.5)x| = 1$ at the stars. On the other hand the control that ensures invariance of $b^T P(0.5)x = 0$ reaches 1 at the $+$, and define the admissible level set for the invariance checking method (dotted ellipse). The rate of convergence corresponding to $\epsilon = 0.5$ can be achieved in a much large region

Scheduling The idea that ϵ should be small far from the origin (for stability) and larger near the origin (for performance) suggests that the performance of low-gain designs will improve with an on-line adaptation of ϵ.

Megretski (1996) proposes the choice

$$\epsilon(x) = \max\{\eta \in (0,1] : (x^T P(\eta)x)(b^T P(\eta)b) \leq 1\} \tag{3.11}$$

which corresponds to an online adaptation of the rule (3.8). It is shown in Megretski (1996) that $V(x, \epsilon(x))$ then decreases along the solutions in the region of attraction, which is the entire state space.

If the initial condition is far from the origin, the parameter $\epsilon(x)$ will be initially small. However, it will increase as the solution approaches the origin. This gain scheduling can be stopped once ϵ has reached a value judged acceptable for local performance ($\epsilon = 1$).

The design flexibility of multiplying the control law by any gain $k \geq 1$ can be combined with online adaptation of ϵ, a gain k proportional to $\frac{1}{\epsilon}$ being suggested by Lin (see e.g. Lin (1998)).

In the next section, we will develop a scheduling inspired by the observations made on the high-and-low gain design and forcing the fastest possible transition from the stabilizing controller ($u = -\operatorname{sat}(kb^T P(\epsilon_0)x)$) to the performing controller ($u = -b^T P(1)x$).

3.3 Implicit low-gain design

The online adaptation of ϵ presented in this section is different from (3.11). It is based on the following observation, made in Example 6: multiplying the low-gain control law by a large gain, in order to better use the available actuation, enforces the near-invariance of the subspace $\ker b^T P(\epsilon)$, at least in a neighborhood of the origin. In this region, making this subspace truly invariant will be less conservative and more relevant than ensuring $\dot{V} < 0$ in the entire set $\mathcal{T}(\epsilon)$. As a consequence, ϵ will be allowed to increase faster along the solutions. This suggests the following procedure in three phases:

- Controller 1: Reach the subspace $\ker b^T P(\epsilon)$ for some $\epsilon > 0$ in a region such that the invariance condition $b^T P(\epsilon)x = 0$ can be ensured for all future times with $|u| \leq 1$ (this can simply be achieved using low-and-high gain);

- Controller 2: Increase ϵ as fast as possible while maintaining $b^T P(\epsilon)x = 0$ and at the same time ensuring that, if the adaptation of ϵ is stopped, the control invariance of $b^T P(\epsilon)x = 0$ can be achieved with $|u| \leq 1$;

- Controller 3: Once x reaches $\mathcal{T}(1)$, apply $u = -\operatorname{sat}(kb^T P(1)x)$, which ensures asymptotic stability of the origin in $\mathcal{T}(1)$ and the desired local performance.

The scheduling, which is the main contribution of this chapter, consists in Controller 2. It requires to be preceded by Controller 1 because it is not always possible to find ϵ such that $b^T P(\epsilon)x = 0$ is satisfied and can be made control invariant with $|u| \leq 1$. Controller 1 steers $x(t)$ in such a region (that we will denote Ω).

Controller 2 is termed implicit because the control law does not explicitly appear in the defining equation $b^T P(\epsilon)x = 0$. It is important to observe that, when Controller 2 is applied, the function $V(x, \epsilon)$ is no longer ensured to decrease along the solutions. This feature justifies the need for the limit value $\epsilon = 1$, which ensures that ϵ cannot drift away to $+\infty$ without convergence of x to the origin. It also appears to be crucial for enabling a faster adaptation of ϵ.

A key point in this procedure is to characterize a reasonable set of initial conditions that can be steered to the origin under the two constraints $b^T P(\epsilon)x = 0$ and $|u(t)| \leq 1$. For a fixed $\epsilon > 0$, invariance of $b^T P(\epsilon)x = 0$ is sufficient to have convergence to the origin because $\ker b^T P(\epsilon)$ is a stable manifold (Anderson & Moore (1971)). Indeed,

$$\dot{V} = x^T P(\epsilon)Ax + x^T A^T P(\epsilon)x + 2x^T P(\epsilon)bu = -x^T Q(\epsilon)x < 0$$

because $2x^T P(\epsilon)bu = -x^T Pbb^T Px = 0$.

The following proposition identifies the set inside which $b^T P(\epsilon)x = 0$ can be kept invariant, for ϵ fixed, based on the quadratic Lyapunov estimate of the region of interest:

Proposition 1 *Let $\epsilon > 0$, the set*

$$\begin{aligned} \Omega_\epsilon &= \ker b^T P(\epsilon) \cap \{x \mid q(x,\epsilon) \leq g(\epsilon)\} \\ &= \ker b^T P(\epsilon) \cap \{x \mid V(x,\epsilon) \leq \bar{V}(\epsilon)\} \end{aligned} \tag{3.12}$$

with

$$q(x,\epsilon) = ((b^T PAP^{-1}A^T Pb)(b^T Pb) - (b^T PAb)^2)(x^T Px)$$

$$g(\epsilon) = (b^T Pb)^3$$

$$\bar{V}(\epsilon) = \frac{(b^T P(\epsilon)b)^3}{(b^T P(\epsilon)AP(\epsilon)^{-1}A^T P(\epsilon)b)(b^T P(\epsilon)b) - (b^T P(\epsilon)Ab)^2}$$

can be made controlled-invariant with $|u| \leq 1$

Proof: Invariance of the subspace $\ker b^T P(\epsilon)$ is achieved under the condition:

$$\frac{d}{dt}\left(b^T P(\epsilon)x\right) = b^T P(\epsilon)Ax + b^T P(\epsilon)bu = 0$$

This imposes the control

$$u = -\frac{b^T P(\epsilon)Ax}{b^T P(\epsilon)b}$$

When $\ker b^T P(\epsilon)$ is A-invariant, the whole subspace $\ker b^T P(\epsilon)$ can be made controlled invariant with $u = 0$ (≤ 1). Also, A-invariance implies $b^T P(\epsilon) = \alpha b^T P(\epsilon)A$ for some $\alpha \neq 0$. Then $q(x,\epsilon) = 0$ and $\Omega_\epsilon = \ker b^T P(\epsilon)$.

Otherwise, we look for the maximum level $\bar{V}(\epsilon)$ of the Lyapunov function, such that, when $V(x,\epsilon) \leq \bar{V}(\epsilon)$ and $b^T P(\epsilon)x = 0$, we have $|\frac{-b^T P(\epsilon)Ax}{b^T P(\epsilon)b}| \leq 1$. This amounts to find the minimal level set where the bound $|\frac{-b^T P(\epsilon)Ax}{b^T P(\epsilon)b}| = 1$ is attained when $b^T P(\epsilon)x = 0$, that is:

$$\begin{array}{rll} \bar{V}(\epsilon) = \min_{x\in I\!R^n} & x^T P(\epsilon)x & \\ \text{s.t.} & b^T P(\epsilon)Ax = b^T P(\epsilon)b & \\ & b^T P(\epsilon)x = 0 & \end{array} \tag{3.13}$$

Note that the absolute value in the first equality constraint is superfluous because of symmetry. The Linear Quadratic problem (3.13) has a unique solution, given by

$$x(\epsilon) = \frac{(b^T P(\epsilon)b)^2 P(\epsilon)^{-1}A^T P(\epsilon)b - (b^T P(\epsilon)b)(b^T P(\epsilon)Ab)b}{(b^T P(\epsilon)Ab)^2 - b^T P(\epsilon)b(b^T P(\epsilon)AP(\epsilon)^{-1}A^T P(\epsilon)b)}$$

where $(b^T PAb)^2 - b^T Pb(b^T PAP^{-1}A^T Pb) < 0$ when $\ker b^T P(\epsilon)$ is not A-invariant. This results in

$$\begin{array}{rcl} \bar{V}(\epsilon) & = & x(\epsilon)^T P(\epsilon)x(\epsilon) \\ & = & \frac{(b^T P(\epsilon)AP(\epsilon)^{-1}A^T P(\epsilon)b)(b^T P(\epsilon)b)^4 - (b^T P(\epsilon)Ab)^2(b^T P(\epsilon)b)^3}{[(b^T P(\epsilon)Ab)^2 - (b^T P(\epsilon)AP(\epsilon)^{-1}A^T P(\epsilon)b)(b^T P(\epsilon)b)]^2} \\ & = & \frac{(b^T P(\epsilon)b)^3}{(b^T P(\epsilon)AP(\epsilon)^{-1}A^T P(\epsilon)b)(b^T P(\epsilon)b) - (b^T P(\epsilon)Ab)^2} \end{array}$$

which concludes the proof. □

Remark 2 *The property that*

$$(b^T P(\epsilon)AP(\epsilon)^{-1}A^T P(\epsilon)b)(b^T P(\epsilon)b) - (b^T P(\epsilon)Ab)^2 > 0$$

if and only if $\ker b^T P(\epsilon)$ *is not* A*-invariant can be deduced from the Cauchy-Schwartz inequality. Indeed, defining the scalar product* $< .,. >_{P^{-1}}$ *on* $I\!R^n$ *as*

$$< y_1, y_2 >_{P^{-1}} = y_1^T P^{-1} y_2$$

the Cauchy-Schwartz inequality gives

$$< y_1, y_2 >_{P^{-1}}^2 \leq \|y_1\|_{P^{-1}}^2 \|y_2\|_{P^{-1}}^2$$

with the equality being satisfied if and only if y_1 *and* y_2 *are collinear. If we particularize this inequality with* $y_1 = b^T P(\epsilon) A$ *and* $y_2 = b^T P(\epsilon)$*, this yields:*

$$(b^T P(\epsilon) A b)^2 \leq (b^T P(\epsilon) A P(\epsilon)^{-1} A^T P(\epsilon) b)(b^T P(\epsilon) b)$$

and the equality is only valid if $y_1 = b^T P(\epsilon) A$ *and* $y_2 = b^T P(\epsilon)$ *are collinear, that is if* $b^T P(\epsilon) x = 0 \ \Leftrightarrow \ b^T P(\epsilon) A x = 0$*, which is equivalent to* $\ker b^T P$ *is* A*-invariant.*

Presumably, the additional constraint of

$$\begin{array}{rl} \bar{V}(\epsilon) = \min_{x \in I\!R^n} & x^T P(\epsilon) x \\ \text{s.t.} & b^T P(\epsilon) A x = b^T P(\epsilon) b \\ & b^T P(\epsilon) x = 0 \end{array}$$

compared to

$$\begin{array}{rl} \bar{V}(\epsilon) = \min_{x \in I\!R^n} & x^T P(\epsilon) x \\ \text{s.t.} & b^T P(\epsilon) x = 1 \end{array}$$

ensures a larger value of the level $\bar{V}(\epsilon)$ for the invariance criterion compared to the bound on the control law, so that similar performance can be obtained inside a larger level set.

In the following, we denote $\Omega = \cup_{\epsilon \in (0,1]} \Omega_\epsilon$. This is the set inside which we will check invariance of $b^T P(\epsilon) x = 0$ while maximizing $\dot{\epsilon}$. We will now define this set more precisely:

$$\Omega = \{x \in I\!R^n | \exists \epsilon \in (0,1] : b^T P(\epsilon) x = 0 \text{ and } V(x,\epsilon) \leq \bar{V}(\epsilon)\}$$

The first phase of the algorithm consists in bringing x inside the region Ω. We then define a second controller inside Ω. Because Ω does not need to be an open set, there can be a problem of definition of the solutions on the boundary of Ω. We will now characterize the elements of the state-space that can be on that boundary. From the definition of Ω, it directly appears that, when $\epsilon = 1$:

$$\{x \in I\!R^n | b^T P(1) x = 0 \text{ and } V(x,1) \leq \bar{V}(1)\}$$

or $V(x,\epsilon) = \bar{V}(\epsilon)$:

$$\{x \in I\!R^n | \exists \epsilon \in (0,1] : b^T P(\epsilon)x = 0 \text{ and } V(x,\epsilon) = \bar{V}(\epsilon)\}$$

the corresponding x can be on the boundary of Ω. However, those are not the only possible candidates: when $b^T P(\epsilon)x = 0$ and $V(x,\epsilon) \leq \bar{V}(\epsilon)$, to a small variation of x must correspond a small variation of ϵ such that $b^T P(\epsilon)x = 0$ stays satisfied: if it is not the case, x can be on the boundary of Ω. These small variations result in:

$$b^T \frac{dP}{d\epsilon} x d\epsilon + b^T P(\epsilon) dx = 0$$

If $b^T \frac{dP}{d\epsilon} x \neq 0$, any small variation of x can be compensated. Otherwise x can be a candidate to be on the boundary. In summary, if x is on the boundary of Ω, we have:

$$\epsilon = 1 \quad \text{or} \quad V(x,\epsilon) = \bar{V}(\epsilon) \quad \text{or} \quad b^T \frac{dP}{d\epsilon} x = 0$$

The proposed algorithm can be more precisely expressed as:

- Controller 1 (stabilizing controller): Steer x_0 to the interior of Ω with the static state feedback:

$$u = -\operatorname{sat}(k_0 b^T P(\epsilon_0)x) \qquad k_0 >> 1 \qquad 0 < \epsilon_0 < 1 \tag{3.14}$$

  This controller has a guaranteed basin of attraction $\mathcal{T}(\epsilon_0)$ that can be made arbitrarily large by selecting ϵ_0 small enough.

- Controller 2 (scheduling controller): for $x \in \Omega$ and $\epsilon \in (0,1]$, design a Lipschitz continuous dynamic feedback control law:

$$\left\{ \begin{array}{rcl} u & = & \alpha(x,\epsilon) \\ \dot{\epsilon} & = & \gamma(x,\epsilon) \end{array} \right. \tag{3.15}$$

  which maximizes $\dot{\epsilon}$ while making Ω invariant, under the constraints

$$b^T P(\epsilon)x = 0, \qquad |u| \leq 1, \qquad \epsilon \leq 1$$

  The construction of controller (3.15) is detailed in Section 3.4. Upon initialization with x in the interior of Ω and ϵ such that $b^T P(\epsilon)x = 0$, we prove convergence of $x(t)$ to the origin.

- Controller 3 (local controller): For $x \in \mathcal{T}(1) = \{x | x^T P(1)x \leq \frac{1}{b^T P(1)B}\}$, apply the stabilizing feedback $u = -\operatorname{sat}(b^T P(1)x)$. This controller has $\mathcal{T}(1)$ as guaranteed basin of attraction.

Along a closed loop solution, the control law undergoes at most two discontinuities, determined by the switching times between Controller 1 and Controller 2, then between Controller 2 and Controller 3.

Initialization of Controller 1 Controller 1 can be initialized for any $x \in \mathbb{R}^n$. It suffices to take ϵ_0 small enough. Controller 1 is applied until x enters Ω where the scheduling can be implemented.

Initialization of Controller 2 To make sure that Controller 2 is initialized in the interior of Ω, it is advisable to check the constraint

$$\begin{cases} V(x,\epsilon) \leq (1-\delta)\bar{V}(\epsilon) \\ b^T P(\epsilon)x = 0 \end{cases} \tag{3.16}$$

for $\epsilon \in (0,1)$ and for some small parameter $\delta > 0$. With such a choice, x can only be on the boundary of Ω if $b^T \frac{dP(\epsilon)}{d\epsilon} x = 0$.

Online checking that $x \in \Omega$ is not an obvious task because it requires to solve the nonlinear equation $b^T P(\eta)x = 0$. One way to initialize Controller 2 properly is to fix some $\tilde{\epsilon} > 0$ a priori and to wait until $x(T) \in \Omega_{\tilde{\epsilon}}$ to initialize Controller 2 with $\epsilon(T) = \tilde{\epsilon}$. In fact, T is guaranteed to be finite if $\tilde{\epsilon} = \epsilon_0$ and $u = -\operatorname{sign}(b^T P(\epsilon_0)x)$, which is the limit of Controller 1 for $k_0 \to \infty$.

This strategy is feasible, but does not take advantage of the fact that $x(t)$ will usually enter the set Ω before it reaches Ω_{ϵ_0}, which means that this strategy is too conservative. In order to check if some x belongs to Ω, we first need to find some η such that

$$b^T P(\eta)x = 0 \tag{3.17}$$

This nonlinear equation is not easy to solve online; therefore, a first practical recommendation is to check the condition $x(t) \in \Omega_{\epsilon_i}$ for a few ϵ_i in the interval $(0,1]$.

Another possible solution to avoid the conservatism consists in still applying the closed-loop control

$$u = -\operatorname{sat}(k_0 b^T P(\epsilon(x_0))x)$$

and, given the initial condition x_0, solve off-line the nonlinear equation (3.17). If a positive solution $\eta(x_0)$ exists, keep track of the solution of (3.17) while Controller 1 is applied, by using the adaptation law:

$$\frac{db^T P(\eta)x}{dt} = 0$$

which gives:

$$b^T P(\eta)Ax - \left(b^T P(\eta)b\right) \operatorname{sat}(k_0 b^T P(\epsilon(x_0))x) + b^T \frac{dP(\eta)}{d\eta} x\dot{\eta} = 0 \tag{3.18}$$

Extracting $\dot{\eta}$ from (3.18), dictates the evolution of η. It is then sufficient to check online whether the additional constraint $q(x,\eta) \leq (1-\delta)g(\eta)$ becomes satisfied for η, at a given time,in which case Controller 2 of the algorithm can be initialized with $\epsilon = \eta$ instead of waiting for $b^T P(\epsilon(x_0))x = 0$ to be satisfied. The solution of (3.17) may not exist for $x = x_0$ or may cease to exist along the closed-loop solution $x(t)$, in which case $\eta(t)$ escapes to infinity in finite time. In such situations, one will adopt the conservative solution of checking $x \in \Omega_\epsilon$ for $\epsilon = \epsilon(x_0)$ and a finite number of $\epsilon = \epsilon_i$.

Initialization of Controller 3 Controller 3 is initialized once $x(t)$ enters $\mathcal{T}(1)$, which is a neighborhood of the origin. In the next section, we will show that Controller 2 forces x to converge to the origin, which ensures initialization of Controller 3.

3.4 The scheduling controller

Following the observations made in the previous sections, the scheduling controller (3.15) that should be implemented is the solution of the following optimization problem:

$$
\begin{aligned}
&\max \dot{\epsilon} \qquad \text{s.t.} \\
&\quad \frac{d}{dt}(b^T P(\epsilon)x) = 0 \\
&\quad |u| \leq 1
\end{aligned}
$$

However, with this choice, $\dot{\epsilon}$ may be unbounded, which may result in unwanted discontinuities of the control law. Therefore, an upper bound is imposed on $\dot{\epsilon} \leq \dot{\epsilon}_{max}$.

Also, we wish to avoid the boundaries of Ω if $x(t)$ belongs to the interior of Ω, so that no problem of definition of solutions arises. In order to prevent ϵ from reaching 1, we impose

$$
\dot{\epsilon} \leq \ \mathrm{sat}_{[0,\dot{\epsilon}_{max}]}\left(\frac{\dot{\epsilon}_{max}}{\delta}(1-\epsilon)\right)
$$

Through that choice, when $\epsilon > 1-\delta$, ϵ cannot increase faster than exponentially towards 1. Finally, when $V(x,\epsilon) = \bar{V}(\epsilon)$, we impose $\dot{\epsilon} = 0$, so that $\dot{V}(x,\epsilon) < 0$ and the vector fields points towards the interior of Ω. This is imposed by taking:

$$
\dot{\epsilon} \leq \ \mathrm{sat}_{[0,\dot{\epsilon}_{max}]}\left(\frac{\dot{\epsilon}_{max}}{\delta}(1-\epsilon)\right) \ \mathrm{sat}_{[0,1]}\left(\frac{\bar{V}-V}{\delta\bar{V}}\right)
$$

We can however not prevent x from reaching the boundary where $b^T \frac{dP}{d\epsilon} x = 0$. The actual scheduling then is:

$$\max \dot{\epsilon} \qquad \text{s.t.} \tag{3.19}$$

$$\frac{d}{dt}(b^T P(\epsilon) x) = 0 \tag{3.20}$$

$$\dot{\epsilon} \leq L(\epsilon, \dot{\epsilon}_{max}) M(V, \bar{V}) \tag{3.21}$$

$$|u| \leq 1 \tag{3.22}$$

where

$$L(\epsilon, \dot{\epsilon}_{max}) = \ \text{sat}_{[0, \dot{\epsilon}_{max}]} \left(\frac{\dot{\epsilon}_{max}}{\delta} (1 - \epsilon) \right)$$

$$M(V, \bar{V}) = \ \text{sat}_{[0,1]} \left(\frac{\bar{V} - V}{\delta \bar{V}} \right)$$

The parameters $\dot{\epsilon}_{max} >> 1$ and $0 < \delta << 1$ ensure the Lipschitz continuity of the feedback controller inside Ω.

For any choice of $\dot{\epsilon}$, the control law always comes from (3.20), which results in

$$u = - \frac{b^T P(\epsilon) A x + b^T \frac{dP(\epsilon)}{d\epsilon} x \dot{\epsilon}}{b^T P(\epsilon) b}$$

and one of the constraints has to be active:

- If constraint (3.21) is active, we have

$$\dot{\epsilon} = L(\epsilon, \dot{\epsilon}_{max}) M(V, \bar{V})$$

- If constraint (3.22) is active, (3.20) can be rewritten as:

$$\dot{\epsilon} = - \frac{b^T P(\epsilon) A x + b^T P(\epsilon) b u}{b^T \frac{dP(\epsilon)}{d\epsilon} x}$$

  and the control that maximizes $\dot{\epsilon}$ is $u = -\ \text{sign}(b^T \frac{dP(\epsilon) x}{d\epsilon})$, which results in

$$\dot{\epsilon} = - \frac{b^T P(\epsilon) A x - b^T P(\epsilon) b \ \text{sign}(b^T \frac{dP(\epsilon)}{d\epsilon} x)}{b^T \frac{dP(\epsilon)}{d\epsilon} x}$$

As opposed to the scheduling (3.11) of Megretski (1996), this scheduling does not ensure the decrease of $V(x, \epsilon)$ along the whole solution. However, we show in the following theorem that $V(x, \epsilon)$ eventually decreases, and that $x(t)$ converges to the origin. The fact that we allow an increase of $V(x, \epsilon)$ along the solution is probably crucial in the performance improvement that is observed.

Theorem 8 *The solution of the optimization problem (3.19)-(3.22) is*

$$\dot{\epsilon} = \min\left(L(\epsilon, \dot{\epsilon}_{max}) M(V, \bar{V}), -\frac{b^T P A x - b^T P b \; sign(b^T \frac{dP}{d\epsilon} x)}{b^T \frac{dP}{d\epsilon} x} \right) \tag{3.23}$$

$$u = -\frac{b^T P(\epsilon) A x + b^T \frac{dP(\epsilon)}{d\epsilon} x \dot{\epsilon}}{b^T P(\epsilon) b} \tag{3.24}$$

When $\epsilon(0)$ is such that $b^T P(\epsilon(0)) x(0) = 0$, the controller (3.23)-(3.24) renders the origin $x = 0$ attractive with the interior of Ω as basin of attraction.

Proof: The control law (3.24) is determined by (3.20). In (3.23), $\dot{\epsilon}$ is selected as the minimum value that renders one of the constraints (3.21)-(3.22) active.

Invariance of Ω is direct from (3.20)-(3.21). Indeed, those ensure that $b^T P(\epsilon) x = 0$ and $V(x, \epsilon) \leq \bar{V}(\epsilon)$ stay satisfied.

A problem of definition of solutions could arise on the boundary of Ω because the vector field is not defined outside Ω. Because this vector field is continuous inside Ω, its definition can formally be extended to the whole state-space through Tietze's theorem, so that solutions are well defined in the classical sense on the boundary of Ω (invariance of Ω ensures that the exact knowledge of this extension is not necessary because $x(t)$ never quits Ω).

Next, we prove that a solution $x(t)$ is bounded. The derivative of the Lyapunov function is

$$\dot{V}(x, \epsilon) = -x^T Q(\epsilon) x + x^T \frac{dP(\epsilon)}{d\epsilon} x \dot{\epsilon}$$

Continuity of $\frac{dP(\epsilon)}{d\epsilon} > 0$ on $[\epsilon_0, \bar{\epsilon}]$ implies that one can find $\alpha > 0$ such that $x^T \frac{dP(\epsilon)}{d\epsilon} x \leq \alpha V(x, \epsilon)$, which implies

$$\dot{V}(x, \epsilon) \leq \alpha \dot{\epsilon} V(x, \epsilon)$$

and

$$V(x(t), \epsilon(t)) \leq V(x_0, \epsilon_0)e^{\int_0^t \alpha\dot{\epsilon}(\tau)d\tau}$$

One deduces the upperbound $V(x(t), \epsilon(t)) \leq V(x_0, \epsilon_0)e^{\alpha(\bar{\epsilon}-\epsilon_0)}$ for all $t \geq 0$, which guarantees boundedness of $x(t)$ along the solutions.

Next we prove that $\epsilon(t)$ is not decreasing along the closed-loop solutions. Indeed, $\dot{\epsilon} = 0$ is always admissible. The control law ensuring (3.20) is then $u = \frac{-b^T P(\epsilon)Ax}{b^T P(\epsilon)B}$, which satisfies (3.22) because $V(x, \epsilon) \leq \bar{V}(\epsilon)$. This guarantees that $\dot{\epsilon}$, solution of (3.19)-(3.22), is non negative.

Because $\epsilon(t)$ is an increasing function with an upper bound, there exists $\bar{\epsilon} \leq 1$ such that $\epsilon(t)$ converges to $\bar{\epsilon} \leq 1$ as $t \to \infty$.

Invariance of $\Omega_{\epsilon(x(t))}$ implies that $x(t)$ converges to the set $\Omega_{\bar{\epsilon}}$. Boundedness of $x(t)$ implies that the limit set of $x(t)$ is compact and invariant, and that it is contained in the largest invariant set of $\Omega_{\bar{\epsilon}}$, which is the origin. This completes the proof. □

Remark 3 *The proposed Controller 2 steers every initial condition to the origin. This steering property is not robust to errors on $\epsilon(0)$, since (3.24) enforces $b^T P(\epsilon(t)x(t) = b^T P(\epsilon(0))x(0)$ along the closed loop solution. A remedy to this lack of robustness is to replace the constraint (3.20) by*

$$\frac{d}{dt}(b^T P(\epsilon)x) = -\gamma(b^T P(\epsilon)x)$$

where $\gamma > 0$ is a damping parameter. It must also be noted that, in the proposed algorithm, the controller is only required to steer initial conditions in Ω to the set $\mathcal{T}(1)$. This property is robust to small errors on $\epsilon(0)$, even with the original constraint (3.20).

3.5 Closed-loop convergence and performance evaluation

We summarize the convergence properties of the proposed algorithm as follows:

- With any of the Controllers 1, 2, and 3, the equilibrium $x = 0$ is an attractor of the closed-loop system with basin of attraction $\mathcal{T}(\epsilon_0)$, Ω, and $\mathcal{T}(1)$, respectively.

- $x = 0$ is locally exponentially stable, and for any $x_0 \in I\!R^n$, there exists ϵ_0 such that $x_0 \in \mathcal{T}(\epsilon_0)$, so that $x(t)$ converges to the origin if the following switching algorithm is used: apply Controller 1 with ϵ_0 until $x(t)$ is in the interior of Ω; switch to Controller 2 with $\epsilon(0)$ such that (3.16) is satisfied and apply Controller 2 until $x \in \mathcal{T}(1)$; then switch to Controller 3 which guarantees local exponential stability.

Performance evaluation The proposed algorithm is designed to have the fastest possible scheduling from the "stabilizing controller" to the "local controller". The heuristics behind this performance criterion are that controllers $u = -\operatorname{sat}(kb^T P(\epsilon)x)$ are less cautious with large values of ϵ and perform "better" in that sense. Our algorithm is designed to speed up the transfer of ϵ from ϵ_0 to 1 (inside the set Ω). As it will be illustrated in Section 3.6, the set Ω extends well beyond $\mathcal{T}(1)$ and is not confined to a local neighborhood of the origin, which suggests an improvement of the behavior in a large domain of the state-space. .

Comparing (3.11) and (3.12), the algorithm proposed in the present paper will accelerate the adaptation of ϵ whenever

$$\frac{1}{(b^T P(\epsilon)b)} < \frac{(b^T P(\epsilon)b)^3}{(b^T P(\epsilon)AP(\epsilon)^{-1}A^T P(\epsilon)b)(b^T P(\epsilon)b) - (b^T P(\epsilon)Ab)^2} \tag{3.25}$$

because ϵ will then reach the neighborhood of 1 faster with our algorithm, which ensures a satisfying behavior earlier. This will be illustrated on two examples in Sections 3.6 and 3.7.

3.6 The double integrator

The algorithm is now illustrated on the double integrator:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \qquad |u| \leq 1 \end{cases}$$

Like in Example 6, the family of low-gain controls is:

$$u = -b^T P(\epsilon)x = -\epsilon^2 x_1 - \sqrt{3}\epsilon x_2 \qquad \epsilon > 0$$

which is a typical low-gain control for second order systems in Brunovski form. We consider that the behavior of the closed-loop system is satisfying when $\epsilon = 1$. The set Ω_ϵ is characterized by:

$$x_2 + \frac{\epsilon x_1}{\sqrt{3}} = 0 \text{ and } x^T P(\epsilon)x = \sqrt{3}\epsilon^3 x_1^2 + 2\epsilon^2 x_1 x_2 + \sqrt{3}\epsilon x_2^2 \leq \frac{6\sqrt{3}}{\epsilon} \tag{3.26}$$

or, equivalently

$$x_2 + \frac{\epsilon x_1}{\sqrt{3}} = 0 \text{ and } \frac{\epsilon |x_2|}{\sqrt{3}} \leq 1$$

The boundary of the region Ω is therefore characterized by

$$\begin{array}{lcll} V(x,\epsilon) = \bar{V}(\epsilon) & : & x_1 + x_2|x_2| = 0 & \text{and } |x_2| \geq \sqrt{3} \\ \epsilon = 1 & : & x_2 + \frac{x_1}{\sqrt{3}} = 0 & \text{and } |x_2| \leq \sqrt{3} \\ \epsilon = 0 & : & x_2 = 0 & \end{array}$$

This results in the region Ω, which is shaded on Figure 3.2.

For the first controller, implemented only for initial condition outside Ω, we follow (3.14) and use

$$u = -\ \mathrm{sat}(k_0(x_2 + \frac{\epsilon(x_0)x_1}{\sqrt{3}}))$$

with

$$\epsilon(x_0) = \max\{\eta \in (0,1] : \sqrt{3}\eta^3 x_{01}^2 + 2\eta^2 x_{01}x_{02} + \sqrt{3}\eta x_{02}^2 \leq \frac{1}{\sqrt{3}\epsilon}\}$$

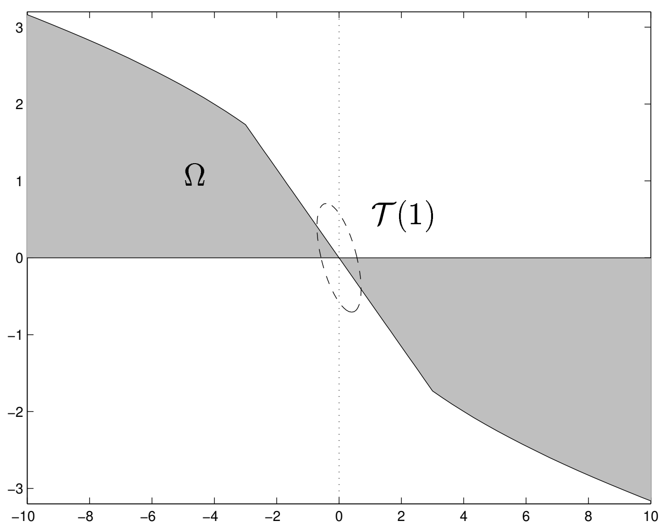

Figure 3.2: The regions Ω (shaded) and $\mathcal{T}(1)$ for the double integrator in the (x_1, x_2) state-space. The corner in the boundary at $(-3, \sqrt{3})$ in the boundary of the Ω region is due to the transition from the boundary defined by $V(x,\epsilon) = \bar{V}(\epsilon)$ to the boundary defined by $\epsilon = 1$

Depending on the initial condition, Controller 1 terminates when the solution reaches the axis $x_2 = 0$ or the manifold $x_1 + x_2|x_2| = 0$.

For Controller 2, in order to expose the geometry of the solution, we consider the limit case $\dot{\epsilon}_{max} = +\infty$ and $\delta = 0$. We use the maximum $\dot{\epsilon}$, while verifying

$$\frac{d}{dt}(x_2 + \frac{\epsilon x_1}{\sqrt{3}}) = 0 \equiv \frac{x_1 \dot{\epsilon}}{\sqrt{3}} + u + \frac{\epsilon x_2}{\sqrt{3}}$$

and $V(x,\epsilon) \leq \bar{V}(\epsilon)$ ($\frac{\epsilon |x_2|}{\sqrt{3}} \leq 1$).

The control algorithm (3.23)-(3.24) is then:

- When $V(x,\epsilon) < \bar{V}(\epsilon)$ and $\epsilon < 1$:

$$\left\{ \begin{array}{lllll} \dot{\epsilon} & = & \min\{+\infty, -\frac{-\sqrt{3}\,\mathrm{sign}(x_1)+\epsilon x_2}{x_1}\} & = & -\frac{-\sqrt{3}\,\mathrm{sign}(x_1)+\epsilon x_2}{x_1} \\ u & = & -\frac{\epsilon x_2 + x_1 \dot{\epsilon}}{\sqrt{3}} & = & -\,\mathrm{sign}(x_1) \end{array} \right. \tag{3.27}$$

- When $V(x,\epsilon) = \bar{V}(\epsilon)$ and $\epsilon < 1$ (that is $x_1 + x_2|x_2| = 0$):

$$\left\{ \begin{array}{lllll} \dot{\epsilon} & = & \min\{0, -\frac{-\sqrt{3}\,\mathrm{sign}(x_1)+\epsilon x_2}{x_1}\} & = & 0 \\ u & = & -\frac{\epsilon x_2 + x_1 \dot{\epsilon}}{\sqrt{3}} & = & -\,\mathrm{sign}(x_2) \end{array} \right. \tag{3.28}$$

- When $V(x,\epsilon) \leq \bar{V}(\epsilon)$ and $\epsilon = 1$ (that is $x_2 + \frac{x_1}{\sqrt{3}} = 0$):

$$\left\{ \begin{array}{lllll} \dot{\epsilon} & = & \min\{0, -\frac{-\sqrt{3}\,\mathrm{sign}(x_1)+x_2}{x_1}\} & = & 0 \\ u & = & -\frac{x_2 + x_1 \dot{\epsilon}}{\sqrt{3}} & = & -\frac{x_2}{\sqrt{3}} \end{array} \right. \tag{3.29}$$

When $\delta = 0$, the closed-loop vector field becomes discontinuous on $\partial\Omega$ and solutions chatter between (3.27) and (3.28) on $\partial\Omega$ (as long as $\epsilon(t) < 1$). This forces the solution to slide along the manifold $x_1 + x_2|x_2| = 0$ and results in the equivalent control (see Utkin (1992))

$$\left\{ \begin{array}{lll} \dot{\epsilon} & = & \frac{\epsilon^2}{2\sqrt{3}} \\ u & = & -\frac{1}{2}\,\mathrm{sign}(x_2) \end{array} \right.$$

The equivalent control is thus the control $u = \pm\frac{1}{2}$ that keeps the manifold $x_1 + x_2|x_2| = 0$ invariant (this equivalent control is what is used to draw Figure 3.3). The chattering phenomenon disappears when $\delta > 0$.

If no upper bound is chosen for ϵ, such that it can diverge towards $+\infty$, a finite escape time is observed for ϵ, which corresponds to finite time convergence of the solutions to the origin. The resulting control law is reminiscent of the time-optimal solution. The control law is bang-bang and the switching surface is the one of the time-optimal control for a constraint $|u| \leq \frac{1}{2}$.

If the adaptation of $\epsilon(t)$ is stopped once ϵ reaches 1, controller (3.29) is used, which forces invariance of $x_2 + \frac{x_1}{\sqrt{3}} = 0$, and convergence of $x(t)$ towards $\mathcal{T}(1)$.

Our implicit control algorithm and the explicit control algorithm proposed in Megretski (1996) can be compared based on the time taken by both schemes to reach the set $\mathcal{T}(1)$, that is:

$$\sqrt{3}x_1^2 + 2x_1x_2 + \sqrt{3}x_2^2 = \frac{1}{\sqrt{3}} \tag{3.30}$$

Indeed, once the solutions are inside this set, their behaviors are identical. The level set (3.30) is shown on Figure 3.2 and 3.3. The faster convergence of the present algorithm can be explained by a faster adaptation of ϵ (see Figure 3.3): for a solution to reach the level set (3.30) from the initial condition $(-10, 0)$, the stars indicate that it takes 6.79 for our control algorithm versus 10.25 for the explicit algorithm, to be compared with 5.64 for the time optimal solution. This results from (3.25), which is satisfied, so that, for a fixed x, the constraint (3.11) on ϵ, i.e.

$$x^T P(\epsilon) x = \sqrt{3}\epsilon^3 x_1^2 + 2\epsilon^2 x_1 x_2 + \sqrt{3}\epsilon x_2^2 \leq \frac{1}{\sqrt{3}\epsilon},$$

is tighter than (3.26) which leads to better performance.

3.7 Simulations

Figure 3.4 shows a simulation of the algorithm applied with $\dot{\epsilon}_{max} = +\infty$ and $\delta = 0$ to the triple integrator:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \qquad |u| \leq 1 \end{cases}$$

for the initial condition $(1, 1, 1)$. The family of low-gain control laws is generated by the Riccati equation (3.4), with the classical choice $Q(\epsilon) = \epsilon I$. An analytical solution cannot be found as easily as in the previous case, and the Riccati equation is therefore numerically solved online. Note that for higher dimensions, this calculation is expensive and justifies the use of (3.6), as suggested in Megretski (1996).

Figure 3.4 shows a comparison of the implicit algorithm presented in this chapter with two other control schemes: the time optimal solution (whose

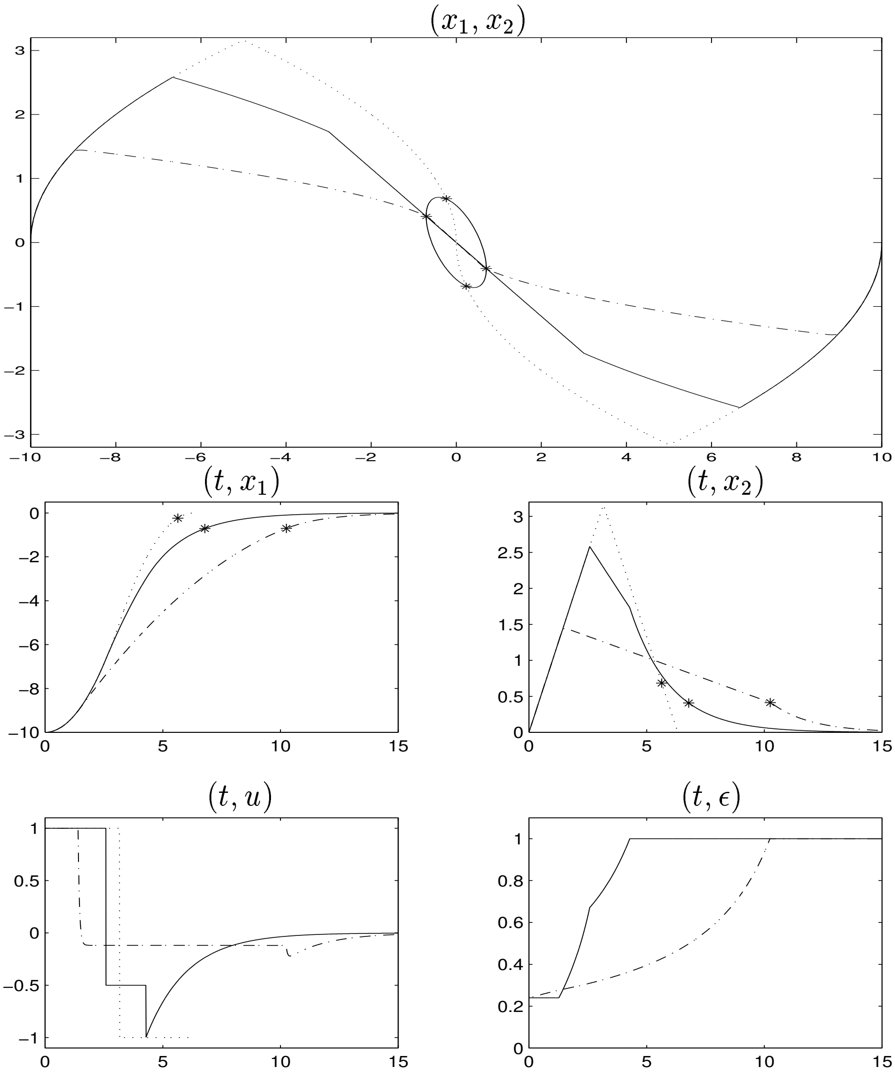

Figure 3.3: Phase plane and time evolutions for the double integrator with initial condition $(x_{10}, x_{20}) = (-10, 0)$: time optimal method (dotted), explicit controller of Megretski (1996) with $k = \frac{1}{\epsilon}$ (dash-dotted) and proposed implicit controller (solid). Stars indicate the time instants or the points in state-space at which the solutions reach $\mathcal{T}(1)$.

computation is still tractable for this example) and the explicit online adaptation from Megretski (1996) discussed in Section 3.2. The online adaptation is implemented with a large gain ($k = \frac{1}{\epsilon}$) because it yields superior performance in this example. The difference between the three control schemes can best be seen in the evolution of x_1. The convergence of x_1 with our method is intermediate between the time-optimal solution and the explicit solution of Megretski (1996). As in the previous example, the upper hand of the present algorithm is also illustrated by the time that it takes for solutions to reach $\mathcal{T}(1)$, the level set

$$x^T P(1) x = \frac{1}{b^T P(1) b}$$

The faster adaptation of ϵ, shown in Figure 3.4 allows for a faster convergence to this level set (7.56, to be compared with 9.83 for the algorithm of Megretski (1996) and 6.58 for the time-optimal solution, for the initial condition $(1, 1, 1)$). This can be explained by inequality (3.25), which is again satisfied for all $\epsilon > 0$. Finally, observe also that V does not decrease when Controller 2 is applied.

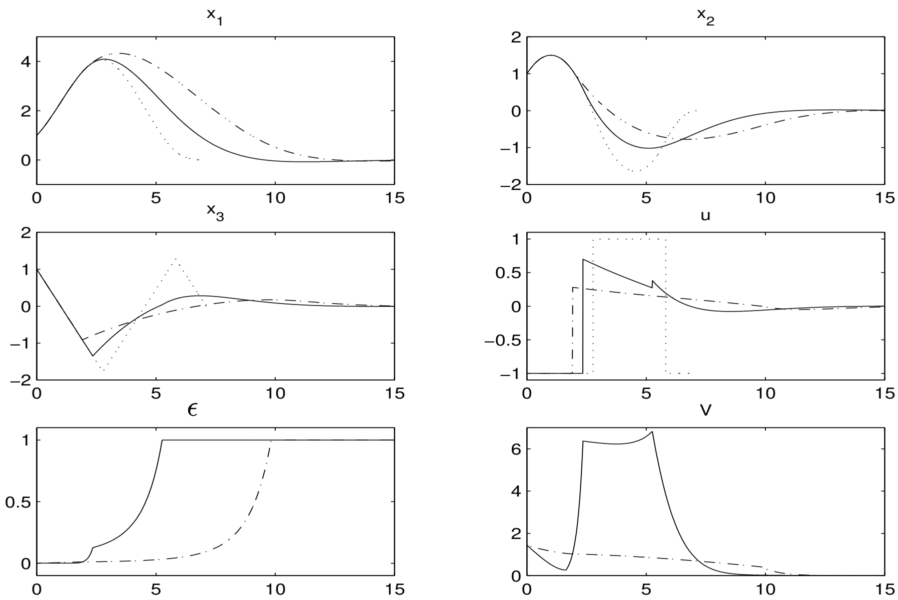

Figure 3.4: Control of the third order integrator: time optimal method (dotted), Lin-Megretski's method with $k = \frac{1}{\epsilon}$ (dash-dotted) and our method (solid) $((x_{10}, x_{20}, x_{30}) = (1, 1, 1))$

3.8 Input magnitude and rate constraints

Adding the rate constraint

$$|\dot{u}| \leq \dot{u}_{max}$$

to the input constraint considered in the previous section results in a relevant problem for which low-gain solutions have been presented in Lin (1997) and Stoorvogel & Saberi (1999). Both references produce a Riccati based family of Lyapunov functions. We choose to follow Lin (1997), who naturally takes into account the integral relation between $\dot{u}$ and u. The construction of the one-parameter family of Lyapunov functions uses the extended state-space model

$$\left\{ \begin{array}{rcl} \dot{x} &=& Ax + bu \\ \dot{u} &=& v \end{array} \right. \tag{3.31}$$

for which the constraints are now simply affine state and input constraints. The construction of the Lyapunov function relies on the construction of the Riccati matrix presented in Section 3.2: the Lyapunov function is the "backstepping" augmentation of the original Lyapunov function (Sepulchre et al. 1996)

$$V\left((x,u),\epsilon\right) = x^T P(\epsilon) x + (u + b^T P(\epsilon) x)^2$$

A family of controllers must be constructed such that the time derivative

$$\begin{array}{rcl} \dot{V} &=& x^T PAx + x^T A^T Px + 2x^T Pbu \\ && \quad +2\left(u + b^T Px\right)\left(v + b^T PAx + b^T Pbu\right) \\ &=& -x^T Qx - x^T Pbb^T Px \\ && \quad +2\left(u + b^T Px\right)\left(v + b^T PAx + b^T Pbu + b^T Px\right) \end{array}$$

is rendered negative. This is accomplished with the explicit backstepping controller

$$\begin{array}{rcl} v &=& -K(\epsilon)\begin{pmatrix} x \\ u \end{pmatrix} \\ &=& -(b^T PAx + b^T PBu + b^T Px + k(u + b^T Px)) \end{array} \tag{3.32}$$

with $k > 0$ or through the implicit specification

$$K(\epsilon)\begin{pmatrix} x \\ u \end{pmatrix} = u + b^T P(\epsilon) x = 0 \tag{3.33}$$

which is the limit of (3.32) for $k \to \infty$. Like in the previous sections, the first controller is termed explicit because the control law v is explicitly given as a function of the states x and u. The second expression does not contain v, which will be calculated to force the invariance of this relation. In both case, the objective is to bring ϵ to 1.

In both cases, estimates of the guaranteed region of attraction must be computed based on the Lyapunov function $V((x,u),\epsilon)$. The minimal level set inside which both the amplitude and the rate constraint are satisfied must be computed. In the explicit case, this results in

$$\bar{V}^e(\epsilon) = \min\left(\bar{V}_1^e(\epsilon), \bar{V}_2^e(\epsilon)\right)$$

with $\bar{V}_1^e$ corresponding to the minimal level set inside which the magnitude constraint is satisfied, and $\bar{V}_2^e$ corresponding to the minimal level set inside which the rate constraint is satisfied:

$$\bar{V}_1^e(\epsilon) = \min_x V((x, u_{max}), \epsilon)$$

$$\bar{V}_2^e(\epsilon) = \min_{(x,u)} V((x,u),\epsilon)$$

$$\text{s.t. } K(\epsilon)\begin{pmatrix} x \\ u \end{pmatrix} = \dot{u}_{max}$$

In the implicit case, for $\epsilon > 0$ fixed, invariance of the manifold $u = -b^T P(\epsilon)x$ then imposes

$$\begin{aligned} v &= -b^T P(\epsilon)Ax - b^T P(\epsilon)bu \\ &= -b^T P(\epsilon)(A - bb^T P(\epsilon))x = G^T(\epsilon)x \end{aligned}$$

so that $\bar{V}_1^i$ and $\bar{V}_2^i$ are now replaced by

$$\bar{V}_1^i(\epsilon) = \min_{(x,u)} x^T P(\epsilon)x + (u + b^T P(\epsilon)x)^2$$

$$\text{s.t. } \begin{cases} u + b^T P(\epsilon)x = 0 \\ u = u_{max} \end{cases}$$

which reduces to

$$\bar{V}_1^i(\epsilon) = \min_x x^T P(\epsilon)x$$

$$\text{s.t. } b^T P(\epsilon)x = u_{max}$$

and, similarly

$$\bar{V}_2^i(\epsilon) = \min_x x^T P(\epsilon)x$$

$$\text{s.t. } G^T(\epsilon)x = \dot{u}_{max}$$

whose solutions are $\bar{V}_1^i(\epsilon) = \frac{u_{max}^2}{b^T P(\epsilon) b}$ and $\bar{V}_2^i(\epsilon) = \frac{\dot{u}_{max}^2}{G^T(\epsilon)P^{-1}(\epsilon)G(\epsilon)}$.

The explicit adaptation rule is then in the same form as (3.11). At each (x, u), the maximal available ϵ is chosen:

$$\epsilon(x(t), u(t)) = \max\{\eta \in (0, 1] : V((x, u), \eta) \leq \bar{V}(\eta)\}$$

On the other hand, the implicit gain-scheduling consists in maximizing $\dot{\epsilon}$ under constraints similar to (3.20)-(3.22).

The implicit gain-scheduling (3.33) requires no "reaching phase", like Controller 1 of the bounded input case, if one assumes that the initial control variable u can be freely initialized at the value $-b^T P(\epsilon)x$.

The control v and the adaptation rule $\dot{\epsilon}$ are then obtained from a pointwise optimization similar to (3.19)-(3.22), which results in

$$\max \dot{\epsilon} \qquad \text{s.t.} \tag{3.34}$$

$$\frac{d}{dt}(u + b^T P(\epsilon)x) = 0$$

$$\dot{\epsilon} \leq L(\epsilon, \dot{\epsilon}_{max}) M(V, \bar{V})$$

$$|u| \leq u_{max} \quad |v| \leq \dot{u}_{max}$$

where

$$L(\epsilon, \dot{\epsilon}_{max}) = \ \text{sat}_{[0,\dot{\epsilon}_{max}]}\left(\frac{\dot{\epsilon}_{max}}{\delta}(1 - \epsilon)\right)$$

$$M(V, \bar{V}) = \ \text{sat}_{[0,1]}\left(\frac{\bar{V} - V}{\delta \bar{V}}\right)$$

Remark 4 *The application of our implicit control algorithm to the extended system (3.31) results in a control scheme which is exactly equivalent to the application of an explicit control scheme to the original system by taking both the amplitude and rate constraints into account. Indeed, the explicit control law applied on the original system is*

$$u = -b^T P(\epsilon)x$$

which, for a given ϵ, can be applied inside a level set of the original Lyapunov function such that neither the magnitude nor the rate constraint is violated. This is achieved by taking the minimum of both level sets:

$$\bar{V}_1(\epsilon) = \min_x x^T P(\epsilon)x$$

$$s.t.\ \ b^T P(\epsilon)x = u_{max}$$

and, noting that

$$\frac{d}{dt}(b^T P(\epsilon)x) = b^T P(\epsilon)Ax - b^T P(\epsilon)bb^T P(\epsilon)x = G^T(\epsilon)x$$

we have:

$$\bar{V}_2(\epsilon) = \min_x x^T P(\epsilon)x$$

$$s.t.\ \ G^T(\epsilon)x = \dot{u}_{max}$$

Therefore, the control law $u = -b^T P(\epsilon)x$ *is applied inside the level set* $V(x,\epsilon) \leq \min\left(\bar{V}_1(\epsilon), \bar{V}_2(\epsilon)\right)$*, which is exactly what we obtained with our control scheme. The implicit algorithm can then be seen as a dynamical controller:*

$$\begin{cases} u = -b^T P(\epsilon)x \\ \dot{\epsilon} \textit{ solution of (3.34)} \end{cases}$$

The application of these adaptation schemes are now illustrated on a simple example.

3.8.1 Example

On Figure 3.6, we compare the efficiency of different algorithms for the control of the double integrator with control rate and amplitude constraints:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \qquad |u| \leq 1,\ |\dot{u}| \leq 1 \end{cases}$$

with the extension $\dot{u} = v$. Solving the Riccati equation (3.4) with $Q(\epsilon)$ as in (3.10) again results in the Lyapunov matrix:

$$P(\epsilon) = \begin{pmatrix} \sqrt{3}\epsilon^3 & \epsilon^2 \\ \epsilon^2 & \sqrt{3}\epsilon \end{pmatrix}$$

and we arbitrarily consider that the target behavior of the closed-loop system is for $\epsilon = 1$.

In this case, the explicit controller (3.32) (with $J = 1$ and $k = 1$) is

$$v = -\epsilon^2(x_2 + 2x_1) - \sqrt{3}\epsilon(u + 2x_2) - u$$

and the implicit controller specification ($J = 0$ and $k = +\infty$) yields

$$u + \epsilon^2 x_1 + \sqrt{3}\epsilon x_2 = 0$$

$\bar{V}(\epsilon)$ is calculated and drawn on Figure 3.5 for both the explicit and the implicit case. We see that, for all ϵ between 0 and 1, $\bar{V}$ is larger for the implicit scheduling. This explains why we expect faster convergence with the implicit gain-scheduling. Our implicit controller never saturates so that it is clear that there is much room from improvement. We could force u to saturate by changing the invariance criterion to $u + \text{ sat}(kb^T P(\epsilon)x) = 0$ with k large. Though this will not influence $\bar{V}_1^i(\epsilon)$, the level $\bar{V}_2^i(\epsilon)$ will be reduced, so that ϵ will be smaller; what is gained with k is lost with ϵ.

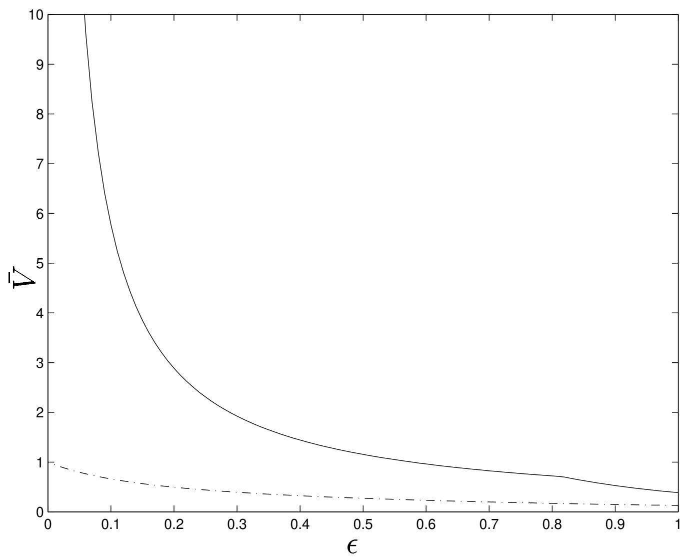

Figure 3.5: Evolution of $\bar{V}(\epsilon)$ for the explicit (dash-dotted line) and the implicit schedulings (solid line)

This expectation is confirmed by the simulations; three controllers are compared in Figure 3.6:

(i) a fixed low-gain explicit controller (dotted line curves);

(ii) an explicit gain scheduling (dash-dotted line curves);

(iii) an implicit gain-scheduling with $\delta = 0$ and $\dot{\epsilon}_{max} = +\infty$ (solid line curves).

The parameter $\delta = 0$ results in a discontinuous control law for the implicit scheme. For the initial condition $x_0 = (-10, 0)$, we choose $u_0 = -b^T P(\epsilon_0)x_0 = 0.57$ (with ϵ_0, the initial value of ϵ for the implicit algorithm) which ensures that the implicit scheduling is initialized on the manifold $u + b^T P(\epsilon)x = 0$. As expected, we see that the initial ϵ is larger for the implicit gain-scheduling, and that it increases faster. During the whole simulation, $\bar{V} = \bar{V}_2$ in the explicit scheduling, which means that the rate constraint is the limiting value. In the implicit scheduling, $\bar{V} = \bar{V}_1$ until $\epsilon = 0.81$, and $\bar{V} = \bar{V}_2$ afterwards, that is u_{max} is first the limiting value, and then $\dot{u}_{max}$; this transition is especially visible in the change of slope at $\epsilon = 0.81$ on Figure 3.5 and in the first discontinuity in the v graph in Figure 3.6. The second discontinuity is due to the interruption of the adaptation which eliminates the $b^T \frac{\partial P}{\partial \epsilon} x \dot{\epsilon}$ term in the expression of v. Figure 3.6 shows that the implicit scheduling allows for a higher peak for x_2, which accelerates the convergence of x_1 to the origin.

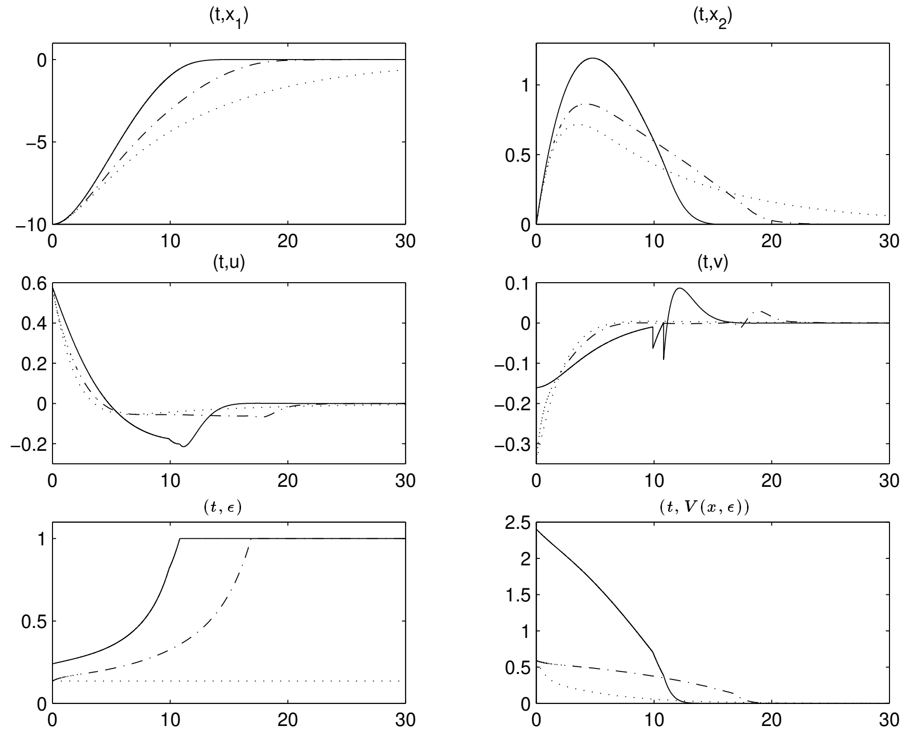

Figure 3.6: Solution of the controlled double integrator with amplitude and rate constraint on the control variable ($u_{max} = 1$ and $v_{max} = 1$). The application of a low-gain control law without gain-scheduling (dotted line) is compared to a control law with explicit (dash-dotted line) and implicit gain-scheduling (solid line)

3.9 Scheduling of parameterized control laws in the presence of affine constraints

In this section, we will formally generalize the control schemes of the previous sections to the case of linear systems

$$\dot{x} = Ax + bu \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R} \tag{3.35}$$

subject to p affine constraints

$$Fx + Gu \leq H \tag{3.36}$$

with $F \in \mathbb{R}^{p\times n}$, $G \in \mathbb{R}^p$, $H \in \mathbb{R}^p$.

The input constraints treated in the previous sections are particular cases of such systems for which the Lyapunov approach can be efficient. The approach can be generalized to (3.35)-(3.36). In the case of actual state constraints, a quadratic Lyapunov functions approach can result in very conservative results. However, such a generalization can lead to the extension of our method to meaningful problem in the class (3.35)-(3.36).

The *realistic* assumption that we adopt as a starting point is that the constraints (3.36) are not active locally, so that any smooth stabilizing feedback satisfies the constraints in a neighborhood of the origin $x = 0$:

Assumption 2 *The origin $x = 0$ is in the interior of the set defined by $Fx \leq H$, and $u = 0$ is in the interior of the set defined by $Gu \leq H$.*

In order to apply the scheduling presented in the previous sections, we need to have a one-parameter family of linear controllers of the explicit or the implicit type available. We will also assume that this family contains an "initial" controller that ensures a large region of attraction for the closed-loop system and a "final" controller that ensures satisfying local performance. The control definitions

$$u = -K(\lambda)x \qquad \text{or} \qquad K(\lambda)x = 0 \tag{3.37}$$

ensure, for each $\lambda \in (0, 1]$, the decrease of a quadratic Lyapunov function

$$V(x, \lambda) = x^T P(\lambda)x \tag{3.38}$$

along the solutions of (3.35) in the absence of the constraints (3.36) (with $K(\lambda)$ and $P(\lambda) > 0$ continuously differentiable). Our design will ensure a scheduling between the initial and the final controller.

We replace ϵ of the previous sections by λ because the family of controllers does not need to be of the low-gain type, that is $K(\lambda)$ does not need to satisfy

$lim_{\lambda \to 0} K(\lambda) = 0$ (which is psychologically implied when using the parameter ϵ). A procedure to generate families of Lyapunov functions has been exposed in Section 3.2 for the input magnitude constraints, and in Section 3.8 for the input magnitude and rate constraints.

As a convention, the value $\lambda = 1$ will correspond to the target controller that yields good local performance. However, we will be interested in dealing with initial conditions x_0 that initially force a smaller value of λ in order to satisfy the constraints (3.36). Our gain-scheduling design will show how to adapt the parameter λ along the closed-loop solutions such as to guarantee the fastest possible transition to the target controller while satisfying the constraints and ensuring convergence to the origin.

3.9.1 Controller gain

We have illustrated, on the bounded input case, that the forms (3.37)

$$u = -K(\lambda)x \qquad K(\lambda)x = 0$$

are a valid starting point for schedulings achieving improved performance, whether the explicit approach $u = -K(\lambda)x$ is chosen, like in Megretski (1996), or the implicit approach $K(\lambda)x = 0$, like in Grognard et al. (2000a). We will now explain in the general case (3.35)-(3.36) the reason why this distinction is rather important for the proposed gain-scheduling.

By assumption, any fixed controller (3.37) will satisfy the constraints (3.36) in a neighborhood of the origin. A Lyapunov estimate of this region is given by the minimum level set $\bar{V}(\lambda)$ where one of the constraints becomes active.

The online requirement $V(x, \lambda) \leq \bar{V}(\lambda)$ will guarantee closed-loop convergence to the origin, but it is the main source of conservatism in the adaptation of λ.

In the explicit case, $\bar{V}(\lambda)$ is computed as

$$\begin{aligned} &\bar{V}(\lambda) = \min_{x \in I\!R^n} x^T P(\lambda) x \\ &\quad \text{s.t. } \exists i : (F_i - G_i K(\lambda))x = N_i \end{aligned}$$

In the implicit case, we assume the relative degree one condition $K(\lambda)b \neq 0$, which was always satisfied in the input saturation case. Therefore, when λ is fixed, the invariance condition $K(\lambda)x = 0$ is ensured by the control law

$$u = -\frac{K(\lambda)Ax}{K(\lambda)b}$$

$\bar{V}(\lambda)$ is then computed as

$$\bar{V}(\lambda) = \min_{x\in I\!R^n} x^T P(\lambda)x \\ \text{s.t.} \begin{cases} \exists i : (F_i - G_i \frac{K(\lambda)A}{K(\lambda)b})x = H_i \\ K(\lambda)x = 0 \end{cases} \tag{3.39}$$

In both the explicit and implicit cases, $\bar{V}(\lambda)$ results from the minimization of the quadratic function under affine constraints. Presumably, the additional equality constraint in (3.39) will result in a larger value $\bar{V}(\lambda)$, thereby reducing the conservatism of the Lyapunov estimate.

The specification of the controller through the implicit relation $K(\lambda)x = 0$ may of course pose a problem for the initialization of the control scheme. If no λ_0 exists such that $K(\lambda_0)x_0 = 0$, an initial phase of the control algorithm is necessary to bring the solution in an admissible region of the state-space (see Controller 1 in Section 3.3). This part of the algorithm is somewhat decoupled from the gain-scheduling problem and will not be discussed any further in the general case. The local controller (Controller 3) will not be treated in detail either. In the explicit case, it obviously is $u = -K(1)x$. In the implicit case, it has to be a stabilizing controller $u = -Kx$, which ensures satisfying local performance.

3.9.2 Gain-scheduling

In this section, we will consider scheduling controllers (like Controller 2 of Section 3.3) in the more general framework of stabilization of the origin of linear system (3.35) under the affine constraints (3.36).

Consider the feasibility region Ω determined by

$$\Omega = \left\{ x \in I\!R^n | \exists\lambda, u : x^T P(\lambda)x \leq \bar{V}(\lambda) \text{ and (3.37) is satisfied } \right\}$$

By definition of Ω, for any $x_0 \in \Omega$ there exists λ_0 such that the fixed parameter controller (3.37) yields a decrease of $V(x, \lambda_0)$ and closed-loop convergence without constraint violation. Our gain-scheduling algorithm will determine an adaptation rule $\dot{\lambda} \geq 0$ and the accompanying control law such as to maximize $\dot{\lambda}$ along the closed-loop solutions and satisfy the constraints (3.36), while ensuring the closed-loop invariance of Ω and the satisfaction of

$$x^T P(\lambda)x \leq \bar{V}(\lambda) \text{ and } \begin{array}{rcl} u & = & -K(\lambda)x \\ & \text{or} & \\ 0 & = & K(\lambda)x \end{array} \tag{3.40}$$

along the solution. Invariance of Ω and the satisfaction of (3.40) along the solution implies that the adaptation of λ can be stopped at any time, the convergence of $x(t)$ to the origin being then guaranteed by the preceding argument.

Explicit controller In the explicit case $u = -K(\lambda)x$, Ω reduces to

$$\Omega = \left\{x \in \mathbb{R}^n | \exists\lambda : x^T P(\lambda)x \leq \bar{V}(\lambda)\right\}$$

because the additional constraint $u = -K(\lambda)x$ can always be satisfied. Invariance of the feasible region Ω guarantees that the constraints are satisfied along the closed-loop solutions because the definition of $\bar{V}(\lambda)$ implies that $Fx - GK(\lambda)x \leq H$ when $(x, \lambda) \in \Omega$.

Invariance of Ω and satisfaction of the constraints is then guaranteed by the feedback rule, inspired by (3.11):

$$\lambda(x(t)) = \max\{\eta \in (0,1] : V(x,\eta) \leq \bar{V}(\eta)\} \tag{3.41}$$

or through the adaptation rule

$$\begin{array}{l} \max \dot{\lambda} \quad \text{s.t.} \\ \quad \frac{d}{dt}(V(x,\lambda) - \bar{V}(\lambda))_+ \leq 0 \text{ if } \ V(x,\lambda) = \bar{V}(\lambda) \end{array} \tag{3.42}$$

Rewriting the differential constraint of (3.42) as

$$\frac{\partial V}{\partial x}\dot{x} + \left(\frac{\partial V}{\partial \lambda} - \left(\frac{\partial \bar{V}}{\partial \lambda}\right)_+\right)\dot{\lambda} \leq 0$$

we see that, for an initial condition (x_0, λ_0) satisfying $V(x_0, \lambda_0) = \bar{V}(\lambda_0)$, the adaptation rule is uniquely determined as

$$\dot{\lambda} = -\left(\frac{\partial V}{\partial \lambda} - \left(\frac{\partial \bar{V}}{\partial \lambda}\right)_+\right)^{-1}\frac{\partial V}{\partial x}\dot{x}$$

under the monotonicity assumption

$$\frac{\partial V(x,\lambda)}{\partial \lambda} - \left(\frac{\partial \bar{V}(\lambda)}{\partial \lambda}\right)_+ > 0 \tag{3.43}$$

Assumption (3.43) was satisfied for the explicit controller in the input magnitude constraint case, and it guarantees a continuous evolution of $\lambda(t)$, in which case the feedback rule (3.41) is just the integral form of the adaptation rule (3.42).

In absence of the monotonicity assumption, the feedback rule (3.41) still guarantees a monotonic evolution of $\lambda(t)$. This is because, if $V(x_0, \lambda_0) \leq \bar{V}(\lambda_0)$, this inequality stays satisfied along the solution, which means that $\lambda(t)$ can only increase: $\dot{\lambda} = 0$ is a feasible solution of (3.42) at any point of Ω. In case $\frac{\partial V(x,\lambda)}{\partial \lambda} - \left(\frac{\partial \bar{V}(\lambda)}{\partial \lambda}\right)_+$ goes through zero, $\lambda(t)$ can present discontinuities, and thus a discontinuous control law $u = -K(\lambda)x$ is applied. This discontinuity can be eliminated in the adaptation rule (3.42) by adding a bound on $\dot{\lambda}$ and replacing the rule by

$$\begin{array}{ll} \max \dot{\lambda} & \text{s.t.} \\ & \frac{d}{dt}(V(x,\lambda) - \bar{V}(\lambda))_{+} \leq 0 \text{ if } \; V(x,\lambda) = \bar{V}(\lambda) \\ & \dot{\lambda} \leq \dot{\lambda}_{max} \end{array} \tag{3.44}$$

which guarantees that $u(t)$ is continuous. The adaptation rule (3.44) will be used for comparison with the implicit gain-scheduling developed below.

Implicit controller In this case, the feasibility region can be defined as in Section 3.3: we first define

$$\Omega_{\lambda} = \ker K(\lambda) \cap \{x | x^T P(\lambda) x \leq \bar{V}(\lambda)\}$$

and then $\Omega = \cup_{\lambda \in (0,1]} \Omega_{\lambda}$.

The adaptation rule (3.41) and (3.44) are no longer valid in the case of an implicit gain-scheduling. The control law enforcing the invariance condition $K(\lambda)x = 0$ is given by

$$u = \frac{1}{K(\lambda)b}\left(-K(\lambda)Ax - \frac{\partial K}{\partial \lambda} x \dot{\lambda}\right) \tag{3.45}$$

Because of the additional term $\frac{\partial K}{\partial \lambda} x \dot{\lambda}$ in (3.45), the constraints are no longer guaranteed to be satisfied when $x \in \Omega$ and $K(\lambda)x = 0$ (which guarantees that $u = -\frac{K(\lambda)Ax}{K(\lambda)b}$ allows for the constraints to be satisfied).

To ensure closed-loop invariance of Ω and satisfaction of the constraints, the adaptation rule $\dot{\lambda}$ must now be determined as the solution of the pointwise maximization

$$\begin{array}{ll} \max \dot{\lambda} & \text{s.t.} \\ & \frac{d}{dt}(K(\lambda)x) = 0 \\ & \dot{\lambda} \leq L(\lambda, \dot{\lambda}_{max}) M(V(x,\lambda), \bar{V}(\lambda)) \\ & Fx + Gu \leq H \end{array} \tag{3.46}$$

where

$$L(\lambda, \dot{\lambda}_{max}) = \text{sat}_{[0, \dot{\lambda}_{max}]}\left(\frac{\dot{\lambda}_{max}}{\delta}(1 - \lambda)\right)$$

$$M(V, \bar{V}) = \text{sat}_{[0,1]}\left(\frac{\bar{V} - V}{\delta \bar{V}}\right)$$

which is directly derived from (3.19)-(3.22). It must again be emphasized that the solution $\dot{\lambda} = 0$ is feasible at any point of Ω, which ensures that the solution of (3.46) is non negative. The bound $\dot{\lambda}_{max} (> 0)$ is arbitrary, but prevents jumps in the evolution of $\lambda(t)$ and guarantees that the control (3.45) is Lipschitz continuous. Like in the bounded input case, the implicit gain-scheduling is designed to allow for the parameter λ to converge towards 1 as fast as possible, while maintaining invariance of Ω.

The next theorem guarantees closed-loop convergence of $x(t)$ to the origin for both the explicit and implicit gain schedulings, which is a direct adaptation of Theorem 8.

Theorem 9 *Consider a family of Lyapunov functions*

$$V(x,\lambda) = x^T P(\lambda) x \qquad \lambda \in (0,1] \qquad P(\lambda) > 0$$

whose time derivative, with λ fixed, along the solutions of the linear system $\dot{x} = Ax + bu$ is rendered negative definite by the explicit control law $u = -K(\lambda)x$ or through the invariance condition $K(\lambda)x = 0$. Then, the adaptation rule (3.44) with

$$u = -K(\lambda)x \tag{3.47}$$

guarantees convergence of $x(t)$ to the origin for any solution with initial condition $x_0 \in \Omega$ and λ_0 satisfying $V(x_0,\lambda_0) \leq \bar{V}(\lambda_0)$. Likewise, the adaptation rule (3.46) with

$$u = \frac{1}{K(\lambda)b}\left(-K(\lambda)Ax - \frac{\partial K}{\partial \lambda} x \dot{\lambda}\right) \tag{3.48}$$

guarantees convergence to the origin of $x(t)$ for any initial condition $x_0 \in \Omega$ and λ_0 satisfying $V(x_0,\lambda_0) \leq \bar{V}(\lambda_0)$ and $K(\lambda_0)x_0 = 0$.

Proof: In the explicit case, (3.44) ensures that the feedback control law $u = -K(\lambda)x$ is Lipschitz continuous in time and that $V \leq \bar{V}$ stays satisfied, so that Ω is invariant.

In the implicit case, the choice (3.46) guarantees that $K(\lambda)x = 0$ and $V(x,\lambda) \leq \bar{V}(\lambda)$ stay satisfied, so that Ω is invariant.

Let us suppose that $\lambda(t)$ does not reach the target value $\lambda_f = 1$ in finite time (for both the implicit or explicit algorithm). $\lambda(t)$ is then an increasing function with an upper bound; therefore, there exists $\bar{\lambda} \leq 1$ such that $\lambda(t)$ converges to $\bar{\lambda} \leq 1$ as $t \to \infty$. The derivative of the Lyapunov function $V(x,\lambda)$ is:

$$\dot{V}(x,\lambda) = x^T P(\lambda) A x + x^T A^T P(\lambda) x + 2x^T P(\lambda) b u + x^T \frac{dP(\lambda)}{d\lambda} x \dot{\lambda}$$

with u satisfying (3.47) or (3.48). In both cases, one has

$$\dot{V}(x,\lambda) = -x^T R(\lambda)x + x^T S(\lambda)x\dot{\lambda} \tag{3.49}$$

with $R(\lambda) > 0$. Continuity of $S(\lambda)$ on $[\lambda_0, \bar{\lambda}]$ implies that one can find $\alpha > 0$ such that $x^T S(\lambda)x \leq \alpha V(x,\lambda)$, which results in

$$\dot{V}(x,\lambda) \leq \alpha\dot{\lambda}V(x,\lambda)$$

and

$$V(x(t),\lambda(t)) \leq V(x_0,\lambda_0)e^{\int_0^t \alpha\dot{\lambda}(\tau)d\tau}$$

One deduces the upperbound $V(x(t),\lambda(t)) \leq V(x_0,\lambda_0)e^{\alpha(\bar{\lambda}-\lambda_0)}$ for all $t \geq 0$, which guarantees boundedness of $x(t)$ along the solutions. Let us now suppose that $x(t)$ does not converge to $x = 0$ as $t \to \infty$. Integrating (3.49) yields

$$V(x(t),\lambda(t)) = -\int_0^t x(\tau)^T R(\lambda(\tau))x(\tau)d\tau + \int_0^t x(\tau)^T S(\lambda(\tau))x(\tau)\dot{\lambda}(\tau)d\tau$$

Taking the limit for $t \to \infty$, the first term diverges to $-\infty$ and the second term is finite (because $x^T S(\lambda)x \leq \alpha V(x_0,\lambda_0)e^{\alpha(\bar{\lambda}-\lambda_0)}$ and $\int_0^{+\infty}\dot{\lambda}(\tau)d\tau = \bar{\lambda} - \lambda_0$). This is in contradiction with the positiveness of V. We conclude that $x(t)$ converges to the origin, even when $\lambda(t)$ does not reach 1 in finite time.

In the implicit case, $\lambda(t)$ is designed such as not to reach 1 in finite time, so that $x(t)$ always converges to the origin.

In the explicit case, either $\lambda(t)$ does not reach 1 in finite time, and we have shown that $x(t) \to 0$ or $\lambda(t)$ reaches 1, and $u = -K(1)x$ ensures asymptotic stability of the origin. □

3.10 Conclusion

In this chapter, we have introduced a new control algorithm aimed at improving the control performance of linear systems with bounded input. Starting from existing explicit low-gain designs $u = -b^T P(\epsilon)x$ with infinite gain margin, it takes advantage of the fact that, with high gain, the condition $b^T P(\epsilon)x \approx 0$ holds after a finite time. We explicitly use this observation in the adaptation rule $\dot{\epsilon}$, which leads to less conservative designs. Our design can be seen as a sliding mode design for which the sliding surface is calculated online. This result is presented in Grognard et al. (2000a). We then have extended this implicit scheduling to the computation of an efficient control law for the stabilization of linear systems with input amplitude and rate limitation. Finally, we have generalized the scheduling methods of Megretski (1996) and Grognard et al. (2000a) to linear systems with affine constraints (see Grognard et al. (2000b)) .

For a given family of Lyapunov functions, two schedulings have been compared: an explicit gain-scheduling based on the control law $u = -K(\lambda)x$ and an implicit gain scheduling based on the invariance condition $K(\lambda)x = 0$. They allow for the satisfaction of both stability and performance specifications for the control of linear systems subject to affine constraints. The Lyapunov-based scheduling provides online interpolation between an initial controller chosen from stability specifications and a target controller, chosen for local performance.

Chapter 4

Dynamical systems that compute time-optimal switchings

In this chapter, like in Chapter 3, we will consider the problem of steering a solution from an initial condition z_0 to the origin for single-input linear systems

$$\dot{z} = Az + bv \tag{4.1}$$

subject to the input constraint

$$|v| \leq 1$$

We have shown, in the previous chapter, that feedback control laws $v(z)$ could be designed to efficiently stabilize the origin of system (4.1). Moreover, the utilization of *closed-loop* control ensures robustness of this convergence. However, it has also appeared on examples, that the resulting solutions converge to the origin much slower than the solutions obtained with *open-loop* time-optimal control.

Also, we have shown in Chapter 1 that performance of open-loop optimal control and robustness of closed-loop control can be achieved by nesting the optimal control problem into an MPC scheme. This requires the possibility of efficiently computing the open-loop optimal control law online for any given initial condition z_0. We will focus on that problem in the special case of time-optimal control.

The switchings of the time-optimal controller occur on so called "switching curves" in the state space. The computation of those curves is equivalent to computing a feedback control law $v^*(z)$. For second and third order systems, this synthesis can easily be achieved through the analytic calculation of those switching curves. For larger dimensions, this calculation quickly becomes untractable, and the calculation of the time-optimal controller has to rely on numerical algorithms. This justifies the use of an MPC scheme instead of directly tackling the design of feedback time-optimal controllers.

Time-optimal control is especially well-suited for MPC implementation. Indeed, if $v^*(t)$ ($t \in [s, T^*]$) is the solution of the time-optimal control problem with $z(s)$ as initial condition, then $v^*(t)$ ($t \in [s+\tau, T^*]$) is the solution of the time-optimal control problem with $z(s+\tau)$ as initial condition in the absence of disturbances and inaccuracies. On the other hand, if disturbances are present, $v^*(t)$ ($t \in [s+\tau, T^*]$) will be a good initial guess for any algorithm that attempts to compute the time-optimal solution with $z(s+\tau)$ as initial condition.

The challenge then consists in designing efficient iterative schemes to compute the time-optimal control law $v^*(t)$ for any given z_0. Several gradient-based iterative methods have been proposed since the starting point of the time-optimal control problem itself (Bellman et al. 1956, Desoer 1959). These gradient methods are exposed in Section 4.2; they typically iterate on the adjoint initial or final state together with the time of response (see for instance Neustadt (1960), Ho (1962), Fadden & Gilbert (1964), Gilbert (1964) and, for a summary of those methods, Plant (1968)). It is known that these methods are, in general, sensitive to the starting condition (initial guess) and have poor convergence properties.

In contrast, the algorithms discussed in the present chapter are based on particular heuristics, to be explained in Section 4.3. They do not require the computation of the gradient and good convergence properties have been exposed in Yastreboff (1969a), Yastreboff (1969b), and De Doná (2000). They are based on the following observations concerning time-optimal control.

When the pair (A, b) is controllable, it is well-known from the maximum principle that the optimal control law $v(t)$ that steers an initial state $z(0) = z_0$ to a final state $z(T) = z_f$ in minimum time is bang-bang, i.e. it switches between the extreme values $v = +1$ and $v = -1$.

We see the computation of the time-optimal control as the computation of the optimal sequence of switchings times $0 = t_0 < t_1, \cdots < t_N = T$ (where $N-1$ is the number of switchings) or, equivalently, the optimal sequence of time intervals $\bar{x}_1 = t_1 - t_0, \bar{x}_2 = t_2 - t_1, \ldots, \bar{x}_N = t_N - t_{N-1}$. In this chapter, we construct continuous time-systems $\dot{x} = f(x)$ which 'produce' the optimal sequence $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_N)^T$, in the sense that they possess a unique equilibrium at $x = \bar{x}$ and that this equilibrium is asymptotically stable. These systems are positive, i.e. they are only defined in the open positive orthant $\mathcal{O}_N^+ = \{x | x_i > 0, i = 1, \cdots, N\}$ and the dynamics leave this open set invariant. The main result shows that, when the eigenvalues of A are real, and under proper time-scale decomposition, the convergence of solutions to the desired equilibrium $x = \bar{x}$ can be enforced in a rather transparent way, through a sequence of nested invariant manifolds. This manifold structure connects the convergence properties of the algorithm to the geometry of the time-optimal control problem and supports the excellent convergence properties that are observed in simulations, even when the time scales are no longer enforced.

Discrete-time versions of the algorithms investigated in this chapter include two algorithms previously proposed in Yastreboff (1969a)-Yastreboff (1969b) and, more recently, in De Doná (2000). No convergence analysis is proposed in those references but simulations suggest strong (and, in fact, global) convergence properties. The continuous-time algorithm proposed here is more amenable to a time-scale decomposition analysis but the geometry displayed in the present chapter is roughly retained in discrete-time versions as well.

The main contribution of this chapter consists in the introduction of a continuous version of the algorithms of Yastreboff (1969a)-Yastreboff (1969b)-De Doná (2000), the introduction of the particular choice of f_i (linear) and the semiglobal singular perturbation convergence analysis.

This chapter is organized as follows: in Section 4.1, we expose the time-optimal problem, as well as basic results of optimal control theory that are necessary for the understanding of this chapter. Section 4.2 is devoted to exposing some of the existing synthesis methods for our time-optimal control problem. The heuristics, upon which our synthesis algorithm is based, are exposed in Section 4.3. In Section 4.4, the main features of the algorithm and some of its convergence issues are illustrated with an example. In Sections 4.5, the time-scale separation and the convergence of the fast and slow subsystem of a particular continuous-time version of the algorithm are analyzed, before putting those analyses together to obtain a semiglobal convergence result. Implementation issues are discussed in Section 4.6. The application of the algorithm to systems with complex eigenvalues is then exposed in Section 4.7. Conclusions are given in Section 4.8.

4.1 Time-optimal control

The time-optimal problem that is treated in this chapter is the following: given $A \in I\!R^{n\times n}, b \in I\!R^n, z_0 \in I\!R^n$ with (A, b) controllable, solve

$$\boxed{\begin{array}{ll} T^* = \min T & \\ \quad \text{s.t.} & \dot{z} = Az + bv \\ & z(0) = z_0 \\ & z(T) = 0 \\ & |v(t)| \leq 1 \end{array}} \qquad (\mathcal{TO})$$

4.1.1 Open-loop control

The major breakthroughs in optimal control theory date back to the 1950's and 60's with the definition of Dynamic Programming (Bellman 1957) and the Maximum Principle (Pontryagin et al. 1962), which still are the basis of most optimal control designs today. The maximum principle provides necessary conditions that must be satisfied when a control law is optimal. Using the definition of the Hamiltonian function

$$H(\eta, z, v) = \eta^0 + \eta^T Az + \eta^T bv$$

and the adjoint system

$$\left\{ \begin{array}{lcl} \dot{\eta}^0 & = & 0 \\ \dot{\eta} & = & -A^T\eta \end{array} \right. \tag{4.2}$$

which is independent of x, the maximum principle for the time-optimal control problem $\mathcal{T}0$ then is:

Theorem 10 (The Maximum Principle) *If $v^*(t)$ is a time-optimal solution of $\mathcal{TO}$ with response $z^*(t)$ (such that $z^*(0) = z_0$ and $z^*(T^*) = 0$), then it is maximal, that is, there exists a non trivial adjoint response $(\eta^{0*}, \eta^*(t))$ of system (4.2) such that*

$$H(\eta^*(t), z^*(t), v^*(t)) = \max_{u\in\Omega} \eta^{0*} + \eta^{*T} Az + \eta^*(t)^T Bu = 0 \tag{4.3}$$

almost everywhere and $\eta^{0*} \leq 0$.

Any optimal control law satisfies the maximum principle; any control law that satisfies the maximum principle is said to be maximal (not necessarily optimal).

Observation of equation (4.3) then indicates that the optimal control satisfies

$$\begin{array}{rcll} v^*(t) & = & \operatorname{sign}(\eta^*(t)^T b) = \operatorname{sign}(\eta^*(0)^T e^{-At} b) & \text{if } \eta^*(t)^T b \neq 0 \\ & = & ? & \text{if } \eta^*(t)^T b = 0 \end{array} \qquad (4.4)$$

which is independent of the evolution of z because (4.2) is independent of z. The form of (4.4) indicates that, if $\eta^*(0)^T e^{-At} b \neq 0$ almost always, $v^*(t)$ is piecewise constant, and it only takes the extreme values of the control interval (almost always), that is, the controller is *bang-bang* (it is the case when the pair (A, b) is controllable). Moreover, all the information about the controller $v^*(t)$ lies in the choice of $\eta^*(0)$ (and T^*). In particular, the number of switchings depends on the number of roots of $\eta^*(0)^T e^{-A^T t} b$ lying between $t = 0$ and $t = T^*$.

Two fundamental uniqueness theorems are given in Athans & Falb (1966):

Theorem 11 (Athans & Falb (1966)) *If the pair (A, b) is controllable then the time-optimal control solution of $\mathcal{TO}$ is unique (if it exists).*

We know from the maximum principle that this controller is maximal. The next theorem indicates that it is the only maximal controller

Theorem 12 (Athans & Falb (1966)) *If the pair (A, b) is controllable and if a time-optimal controller $v^*(t)$ solution of $\mathcal{TO}$ exists, there is a unique maximal controller (the time-optimal controller).*

These theorems show that maximality and optimality are equivalent in the case of problem $\mathcal{TO}$. These theorems, coupled with the following theorem, will prove crucial in our synthesis scheme.

Theorem 13 (Lee & Markus (1967)) *Suppose that the pair (A, b) is controllable and that there exists an optimal controller $v^*(t)$, solution of $\mathcal{TO}$.*

1. *If all the eigenvalues of A are real, this controller is bang-bang and switches at most $n - 1$ times (from $+1$ to -1 or from -1 to $+1$)*

2. *If all the eigenvalues of A have a nonzero imaginary part, the number of switches is unbounded in the sense that, for any specified number of switches N, there exists z_0 such that the corresponding time-optimal controller presents at least N switches.*

Idea of the proof of 1: This proof relies on the form (4.4) of the control law:

$$v^*(t) = \operatorname{sign}(\eta^*(0)^T e^{-At} b)$$

and it uses the following result:

Lemma 3 [Lee & Markus (1967)] Let $P_j(t)$ be a real polynomial of degree $\leq n_j - 1$ and let $\lambda_1 < \cdots < \lambda_r$ be distinct real numbers, then

$$\eta(t) = P_1(t)e^{\lambda_1 t} + \cdots + P_r(t)e^{\lambda_r t} \tag{4.5}$$

has at most $n_1 + \cdots + n_r - 1$ roots. □

The function $\eta^*(0)^T e^{-A^T t} b$ is in the form of (4.5), with $n_1 + \cdots + n_r = n$, which means that it has at most $n-1$ roots, and the control law $v^*(t)$ presents at most $n - 1$ switchings. □

This theorem is at the basis of the synthesis algorithms of Yastreboff (1969a), Yastreboff (1969b), De Doná (2000), and Grognard et al. (2001b). It results in the following property, which is the main result that allows for the development of those algorithms.

Proposition 2 *If (A, b) is controllable, with all the eigenvalues of A real, any bang-bang control law that steers $z(t)$ from z_0 to the origin of system*

$$\dot{z} = Az + bv$$

with $n - 1$ switchings or less is solution of $\mathcal{TO}$.

Proof: Let $v(t)$ $(t \in, [0, T])$ be a bang-bang control law that steers $x(t)$ from x_0 to the origin with less than $n - 1$ switchings. Let $t = t_j$ $(j = 1, \cdots, N \leq n - 1)$ be the switching times. Then, one can find a non trivial initial condition for the adjoint system (4.2) such that $\eta(0)^T e^{-At_j} b = 0$ and $\eta(0)^T e^{-At} b \neq 0$ for all other $t \in [0, T]$. The control $v(t)$ then satisfies the maximum principle, which is sufficient for optimality (see Theorem 12) □

In the real eigenvalues case, we look for the optimal controller by simply looking for a controller that switches at most $n - 1$ times. The second part of Theorem 13 indicates why the approach is not as simple in the complex eigenvalues case (see Section 4.7).

4.1.2 Closed-loop control

The most appealing formulation of the solution of the time-optimal control is the expression of the "switching surfaces", that is the surfaces in the state-space where the control switches from $+1$ to -1 or from -1 to $+1$. Indeed, those surfaces divide the state-space in regions where the control $v = -1$ or $v = +1$ is applied, which means that obtaining those switching surfaces results in obtaining the solution of the optimal control problem in feedback form $v^*(z)$. In Athans & Falb (1966), the switching surfaces of numerous time-optimal control problems of small dimension are given (second order linear integrator, oscillator, damped oscillator and system with two time-scales, first order nonlinear systems, third order integrator, $\cdots$). This is achieved by backward integration, a technique which quickly become untractable when the dimension of the system is large, and which we illustrate on two examples.

Example 1: the double integrator

An example of the time-optimal problem is a mass which moves horizontally along a frictionless track (e.g. a train), with an initial position y_0 and an initial speed $\dot{y}_0$. Suppose that we want to stop this mass at a specified position $y = 0$ (e.g. the next station) in minimum time. The force that can be applied to the train is obviously limited ($|F| \leq F_{max}$), and the simplified dynamics can be expressed from Newton's law:

$$m\ddot{y} = F \quad |F| \leq F_{max}$$

Through the definition:

$$(z_1, z_2) = \left(\frac{my}{F_{max}}, \frac{m\dot{y}}{F_{max}}\right)$$

and $v = \frac{F}{F_{max}}$, the equations of motion are:

$$\left\{\begin{array}{lll} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & v \qquad |v| \leq 1 \end{array}\right.$$

and the control problem is now to find the control law that minimizes the time of the transfer from $z_0 = (\frac{my_0}{F_{max}}, \frac{m\dot{y}_0}{F_{max}})$ to $z_f = (0, 0)$.

In this example, a control law $v(t)$ can easily be derived for any initial condition in the form $z_0 = (z_{10}, z_{20})$, but the simplicity of the system allows to go further: the calculation of the switching curves through the method of backward integration of both the original system and the adjoint system:

$$\left\{\begin{array}{lll} \frac{dz_1}{d\tau} & = & -z_2 \\ \frac{dz_2}{d\tau} & = & -v \end{array}\right.$$

$$\begin{cases} \frac{d\eta_1}{d\tau} & = & 0 \\ \frac{d\eta_2}{d\tau} & = & \eta_1 \end{cases}$$

The solution of the adjoint system is $\eta_1(\tau) = \eta_{1f}$ and $\eta_2(\tau) = \eta_{1f}\tau + \eta_{2f}$, which means that the control law

$$v(\tau) = \operatorname{sign}(\eta(\tau)^T b) = \operatorname{sign}(\eta_{1f}\tau + \eta_{2f})$$

can switch at most one time (which confirms Theorem 13). Because η_{1f} and η_{2f} are free parameters, this switch can take place at any time, and from $+1$ to -1 or from -1 to $+1$. Independently of the actual control switching sequence, the final portion of a minimum-time trajectory for this example must coincide with either the $v = +1$ or the $v = -1$ trajectory that reaches the origin. These two terminal trajectories can be determined by integrating the differential equations backward from the origin with $\bar{v} = \pm 1$ constant, which yields the solution

$$\begin{array}{rcl} z_1 & = & \frac{\bar{v}\tau^2}{2} \\ z_2 & = & -\bar{v}\tau \end{array}$$

Eliminating τ from those equations, we see that the terminal portion of the forward time solutions that reaches the origin under a constant control is given by

$$z_2^2 - 2\bar{v}z_1 = 0$$

which, because $\bar{v} = \pm 1$, can be rewritten as

$$z_2|z_2| + 2z_1 = 0$$

that is, the two branches of parabolas illustrated on Figure 4.1 ($A+$ corresponds to $\bar{v} = +1$ and $A-$ to $\bar{v} = -1$). This is the switching surface and results in the control law

$$v^*(z) = \operatorname{sign}(z_1 + \frac{z_2|z_2|}{2})$$

Under the curve $A^+ \cup A^-$, the control law $v = +1$ is applied, and above the curve, the control $v = -1$ is applied, which results in the phase plane illustrated on Figure 4.1.

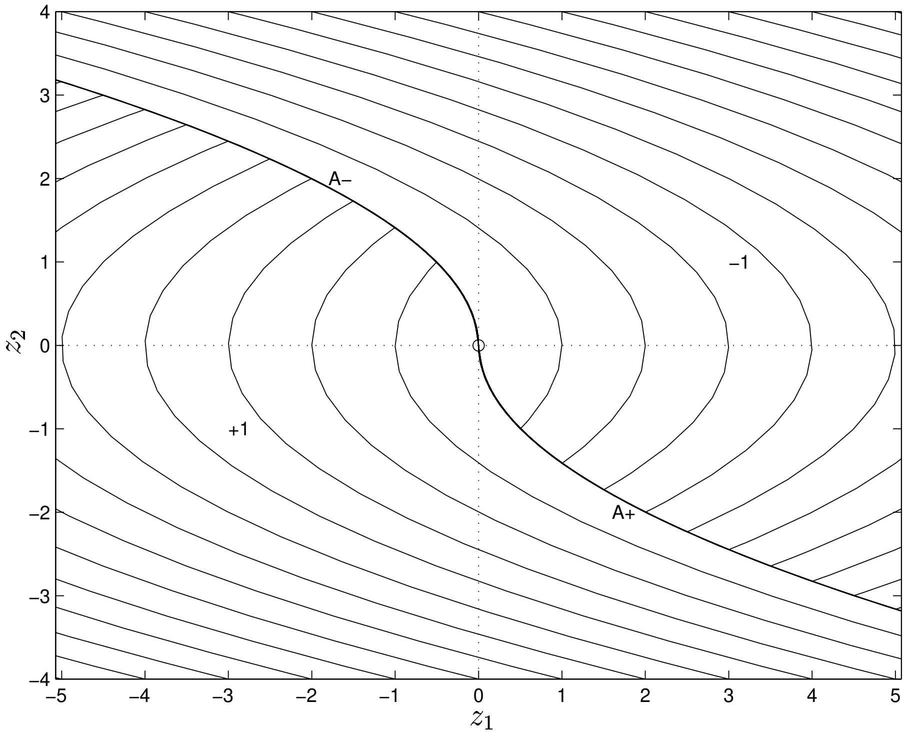

Figure 4.1: Phase plane of the time-optimal control of the double integrator

Example 2: the harmonic oscillator

Backward integration is very easy to implement for small order systems with real eigenvalues. The backward integration of the adjoint system yields more complex solutions when considering systems with complex eigenvalues. In the case of the controlled harmonic oscillator.

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & -z_1 + v \qquad |v| \leq 1 \end{array} \right.$$

The backward system is then given by

$$\left\{ \begin{array}{rcl} \frac{dz_1}{d\tau} & = & -z_2 \\ \frac{dz_2}{d\tau} & = & z_1 - v \end{array} \right.$$

The solution of the backward adjoint system

$$\left\{ \begin{array}{rcl} \frac{d\eta_1}{d\tau} & = & -\eta_2 \\ \frac{d\eta_2}{d\tau} & = & \eta_1 \end{array} \right.$$

is:

$$\eta_1(\tau) = \rho \sin(\tau + \varphi) \qquad \eta_2(\tau) = \rho \cos(\tau + \varphi)$$

where ρ and φ are fixed constants depending on the final condition of the adjoint system. We have seen that the optimal control law had to be in the form:

$$v(\tau) = \operatorname{sign}(\eta^t b) = \operatorname{sign}(\rho \cos(\tau + \varphi))$$

which means that the control switches every π time instants. As stated in the case of the integrator, the final portion of a minimum-time trajectory for this example must coincide with either the $\bar{v} = +1$ or the $\bar{v} = -1$ trajectory that reaches the origin (for a duration of at most π). These two terminal trajectories can be determined by integrating the differential equations backward from the origin with $v = \pm 1$ constant:

$$\begin{aligned} z_1(\tau) &= \bar{v}(1 - \cos(\tau)) \\ z_2(\tau) &= -\bar{v} \sin(\tau) \end{aligned}$$

and we see that the terminal portion of the forward time solutions that reaches the origin under a constant control is given by

$$z_2^2 + (z_1 - \bar{v})^2 = \bar{v}^2 = 1$$

with, when $\bar{v} = -1$, $z_2 \geq 0$ (because $t \leq \pi$ and $z_2 = -\bar{v}\sin(t)$), and when $\bar{v} = +1$, $z_2 \leq 0$. The switching curve is then defined by two half circles. This curve is the switching curve that defines the last switch. Because we know that the previous switch took place π units of time earlier, we can now build the surface defining the previous switch. This leads to the switching curve that is exposed in Figure 4.2.

When the dimension of the system increases, the method of backward integration quickly becomes untractable, which justifies the search for efficient iterative methods.

We will now expose some of the existing synthesis methods for open-loop time-optimal control laws.

4.2 Existing synthesis methods

Several synthesis methods for optimal controllers for problem $\mathcal{TO}$ have been derived since the first formulation of the (time-)optimal control problem. Those iterative methods, produce the time-optimal control $v^*(t)$ for any given initial condition z_0. Those synthesis methods were an important topic of

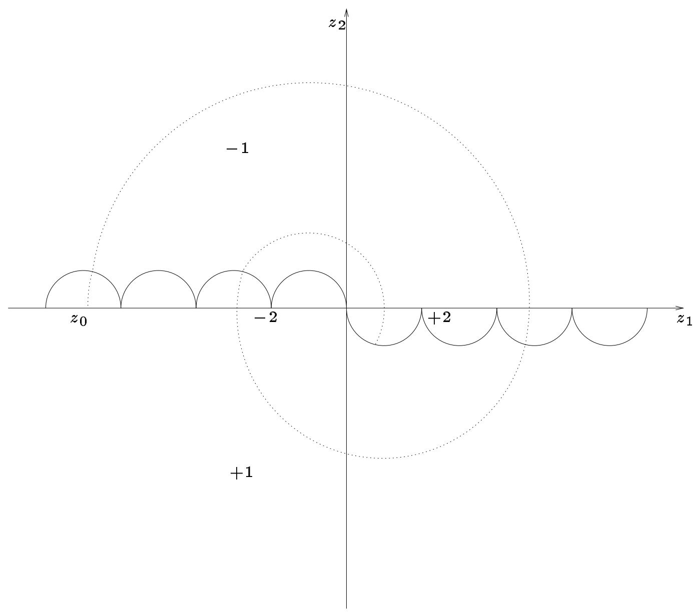

Figure 4.2: Switching curve for the harmonic oscillator (solid line) and time-optimal solution for the initial condition z_0 (dotted line), which presents 4 switches

research in the 1960's and a clear summary can be found in Plant (1968). Essentially, most methods are based on the special form (4.4) of the control law:

$$v^*(t) = \ \text{sign}(\eta(0)e^{-At}b)$$

This control law is entirely defined by $\eta(0)$ and the duration of its application T. The resulting end-point of the solution of system (4.1) with z_0 as initial condition and $v(t)$ as input is:

$$z_f = e^{AT}z_0 + \int_0^T e^{-A(\tau-T)}b \ \text{sign}(\eta(0)e^{-At}b)d\tau$$

Because the desired-end point is the origin, the objective is to find $\eta(0)$ and T such that

$$0 = e^{AT}z_0 + \int_0^T e^{-A(\tau-T)}b \ \text{sign}(\eta(0)e^{-At}b)d\tau = F(\eta(0), T) \tag{4.6}$$

Theorems 11 and 12 ensure that a choice of $\eta(0)$ and T, which forces (4.6) to be satisfied, define the time-optimal controller. The computation of the time-optimal controller then resumes to the computation of the roots of the nonlinear equation (4.6).

The two following methods take different approaches to the search of roots of (4.6).

4.2.1 Gradient and Newton's method

Equation (4.6) is a set of n scalar equations with $n+1$ unknowns (the n elements of $\eta(0)$ and T). It does not have an isolated root because, if a given $(\eta(0), T)$ is solution of (4.6), then $(\alpha\eta(0), T)$ is solution of (4.6) for any $\alpha > 0$. This can be eliminated by adding a constraint on $\eta(0)$:

$$\|\eta(0)\| = 1 \tag{4.7}$$

Equations (4.6) and (4.7) then constitutes a set of $n+1$ equations with $n+1$ unknowns to which a classical gradient or Newton's methods can directly be applied.

When the optimal control law contains exactly $n-1$ switchings, the equations (4.6)-(4.7) have a single root, and this root corresponds to the time-optimal solution. The application of a gradient or Newton's method then ensures local convergence of $(\eta(0), T)$ to the time-optimal equilibrium. Newton's method can lead to very fast convergence, but global or semiglobal convergence cannot be guaranteed.

4.2.2 Iteration on the adjoint initial state

This method was developed in Neustadt (1960), Neustadt (1961), and Plant (1968) and relies on the properties of the set $\mathcal{C}(T)$ of points that can be steered to the origin in time T with the control satisfying $|v(t)| \leq 1$:

Proposition 3 (Plant (1968)) *1. $\mathcal{C}(T)$ is convex and compact*

2. $\mathcal{C}(T_1) \subset \mathcal{C}(T_2)$ if $T_1 < T_2$

3. $\mathcal{C}(T)$ grows continuously with T

The boundary of the set $\mathcal{C}(T)$ is the set of points whose time-optimal control brings them to the origin in time T. This boundary is a minimum cost surface.

Neustadt's method starts from a given $\eta(0)$. It also uses the fact that, for each T, there exists $\hat{z}(T)$ belonging to the boundary of $\mathcal{C}(T)$ such that the optimal controller that steers $z(t)$ to the origin from $\hat{z}(T)$ uses $\eta(0)$ as initial condition. The estimate T of T^* is then the value such that z_0 belongs to the tangent subspace of $\mathcal{C}(T)$ at $\hat{z}(T)$. If $\eta(0) = \eta^*(0)$, this result in the choice $T = T^*$; otherwise, this results in an estimate of the optimal time $T < T^*$ because the convexity of $\mathcal{C}(T)$ implies that the tangent space (and thus z_0) is outside $\mathcal{C}(T)$. $\eta(0)$ is then updated in order to increase T, which forces convergence of T to T^*.

First note that $\eta^*(0)$ is an inward orthogonal to the tangent hyperplane to $\mathcal{C}(T^*)$ at z_0. Convexity then implies that $\eta^*(0)^T z_0 < 0$. The algorithm then reads as:

1. Choose $\eta(0)$ such that

$$\eta(0)^T z_0 < 0$$

2. Calculate the function

$$\hat{z}(t, \eta(0)) = -\int_0^t e^{-As} b \ \mathrm{sign}(\eta(0)^T e^{-As} b) ds$$

which is an element of the boundary of $\mathcal{C}(t)$. For any given t, $\hat{z}$ is a state whose time-optimal controller is $v(s) = \mathrm{sign}(\eta(0)^T e^{-As} b)$ (for $s \in [0, t]$). Let us now consider the function

$$g(t, \eta(0)) = \eta(0)^T \hat{z}(t, \eta(0))$$

This function is strictly decreasing with t. Indeed, we have:

$$\begin{array}{rcl} g(t, \eta(0)) & = & -\eta(0)^T \int_0^t e^{-As} b \ \mathrm{sign}(\eta(0)^T e^{-As} b) ds \\ & = & -\int_0^t \left| \eta(0)^T e^{-As} b \right| \end{array}$$

which results in $\frac{d}{dt} g(t, \eta(0)) < 0$ almost always. Therefore, there is a unique $T(\eta(0))$ such that

$$\eta(0)^T \hat{z}(T(\eta(0)), \eta(0)) = \eta(0)^T z_0 \tag{4.8}$$

Note that this means that there is a unique T such that z_0 belongs to the hyperplane Π tangent to $\mathcal{C}(T)$ at $\hat{z}(T(\eta(0)), \eta(0))$ (this plane is defined by the normal vector $\eta(0)$). This $T(\eta(0))$ is smaller than T^* because the convexity of $\mathcal{C}(T)$ implies that Π is outside $\mathcal{C}(T)$ (see Figure 4.3 for geometric interpretation). This time $T(\eta(0))$ is then taken as the estimate of the minimum time.

3. Improve $\eta(0)$:

$$\eta(0)^{k+1} = \eta(0)^k + \alpha_k(\hat{z}(T(\eta(0)^k), \eta(0)^k) - z_0)$$

The update law aims at the increase of the function

$$F(\eta(0)) = \eta(0)^T(\hat{z}(T(\eta(0)^k), \eta(0)) - z_0)$$

with $T(\eta(0)^k)$ fixed. Indeed, the gradient of $F(\eta)$ is $\hat{z}(T(\eta(0)^k), \eta(0)) - z_0$. The update law then enforces a steepest ascent strategy on $F(\eta(0))$. If α_k is small enough, we then have $(\eta(0)^{k+1})^T(\hat{z}(T(\eta(0)^k), \eta(0)^{k+1}) - z_0) > 0$, and equation (4.8) is satisfied with $T(\eta(0)^{k+1}) > T(\eta(0)^k)$ because $g(T, \eta(0)^{k+1})$ is a decreasing function.

We have seen that the update law forces the increase of $T(\eta)$. It was shown in Eaton (1962) that, for α_k small enough, the sequence $\eta(0)^k$ converges to $\eta^*(0)$. Because α_k is required to be small, the convergence is slow. Moreover $T(\eta(0))$ is very flat, which indicates that, when T is close to T^*, η_0 can still be very far from the optimal one.

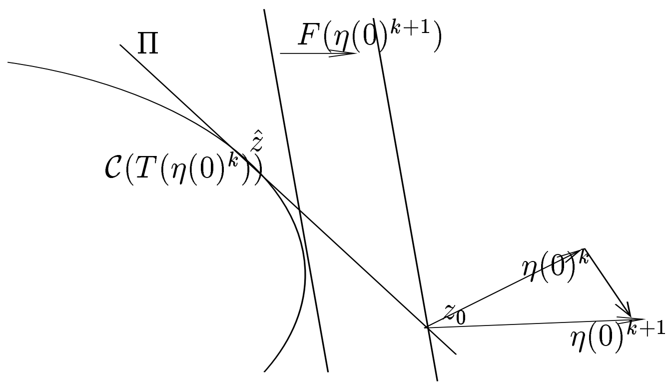

Figure 4.3: Description of one step of Neustadt's algorithm

4.3 Iterative computation of switching times for bang-bang controllers

The algorithms described in the previous section as well as other existing algorithms (Ho 1962) do not achieve fast convergence to the optimal solution, or do not ensure global or semiglobal convergence. Therefore we have tackled the problem of analyzing the convergence properties of a new family of algorithms.

Our algorithm does not fit in any of the previously described methods. It does not calculate the switching curve and it does not directly rely on the maximum principle as the methods of Section 4.2; it relies on heuristics, which prove successful when the problem $\mathcal{TO}$ is defined with the pair (A, b) controllable and the eigenvalues of A real. We assume that z_0 is in the null-controllable set, i.e. the time-optimal problem has a solution. Proposition 2 then indicates that any bang-bang policy $v(t)$ that steers z_0 to the origin with at most $n-1$ switchings is solution of $\mathcal{TO}$.

As a consequence of this property, the search for the optimal control can be restricted to the steering controls that are defined by a sequence of n time intervals $x_i \triangleq t_i - t_{i-1}$ and the corresponding sequence of constant control values u_i. This class of piecewise constant controls (Figure 4.4) is characterized by a pair of vectors (x, u), where x denotes the vector of time intervals and u denotes the vector of control values. The time-optimal solution is then defined by $(\bar{x}, \bar{u})$, with $|\bar{u}_i| = 1$.

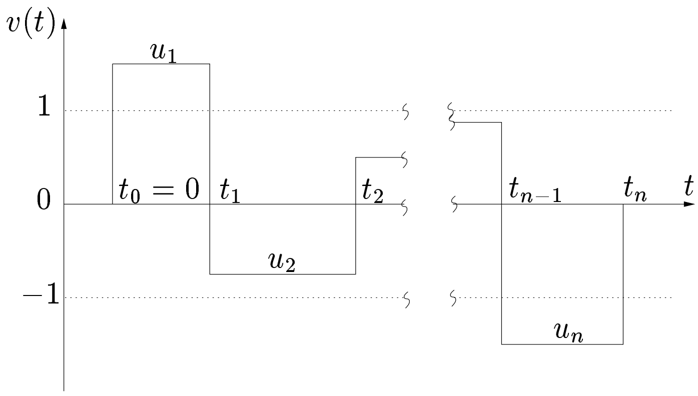

Figure 4.4: Control sequence

From the solution of the linear system for $t \geq t_0 = 0$

$$z(t) = e^{At}\left(z(0) + \int_0^t e^{-A\tau} b v(\tau) d\tau\right),$$

it is seen that a control defined by the pair (x, u) will steer z_0 to $z = 0$ if it satisfies the 'steering equation'

$$\Phi(x)\, u = -z_0 \tag{4.9}$$

where the i-th column of the matrix Φ is

$$\Phi_{(:,i)}(x) \triangleq \int_{t_{i-1}}^{t_i} e^{-A\tau} b d\tau = \int_{\sum_{k=1}^{i-1} x_k}^{\sum_{k=1}^{i} x_k} e^{-A\tau} b d\tau$$

The equation $\Phi(x)\,\bar{u} = -z_0$ is the nonlinear equation to be solved to determine the optimal control. In contrast, (4.9) is linear in u and is easily solved for a given x:

Lemma 4 When the pair (A, b) is controllable, and when all the eigenvalues of A are real, the matrix $\Phi(x)$ is of full rank when x is inside the positive orthant.

Proof: Let $x > 0$. Let us suppose that $\Phi(x)$ is not of full rank. Then, there exists $w \in I\!R_0^n$ such that

$$w^T \Phi(x) = (0\ 0\ \cdots\ 0\ 0)$$

Separating the columns, we obtain n equations in the form:

$$w^T \int_{t_{i-1}}^{t_i} e^{-A\tau} b d\tau = 0$$

and Rolle's Theorem implies that there exists $s_i \in (t_{i-1}, t_i)$ such that:

$$w^T e^{-As_i} b = 0$$

$w^T e^{-At} b$ is then an exponential polynomial with n distinct roots, which is not possible because the eigenvalues of A are real. Therefore, $\Phi(x)$ is of full rank. $\square$

Lemma 4 implies that a unique solution $u(x)$ of (4.9) exists for any x in the open positive orthant. A natural class of iterative methods thus consists in updating the time intervals vector x such as to enforce convergence of the corresponding control vector $u(x)$ to a bang-bang sequence of magnitude $|u_i| = 1$.

A first possible approach is gradient based: scalar costs can be built such as to be minimal only at the optimal solution:

$$V_1(x) = \sum_{i=1}^{n} (u_i^2(x) - 1)^2$$

or

$$V_2(x) = \sum_{i=1}^{n} (u_i(x) + u_{i+1}(x))^2$$

where we fix $u_{n+1}(x) = +1$ or $u_{n+1}(x) = -1$ (only one of the two will yield a result, the one such that $\bar{u}_n = -u_{n+1}(x)$) . Those functions are suited to apply a gradient algorithm, which leads x to the optimal solution. However, the existence of local minima for those scalar functions cannot be excluded, and a gradient method requires the computation of a gradient at each step. Instead, we decided to consider the following heuristics.

The heuristics considered in Yastreboff (1969a)–Yastreboff (1969b) and more recently in De Doná (2000) are the "decentralized" adaptation of the vector x: if $|u_i(x)|$ is larger than one, increase the length of the corresponding time interval x_i; if $|u_i(x)|$ is smaller than one, decrease the length of the corresponding time interval x_i.

In continuous-time, these heuristics yield the decentralized adaptation

$$\dot{x}_i = f_i(|u_i(x)| - 1)x_i, \; i = 1, \ldots, n \tag{4.10}$$

where f_i should be a (smooth) scalar function with its image in the first and third quadrant and should only vanish at zero. x_i multiplies f_i in order to guarantee the positive invariance of the open positive orthant. If z_0 is not on a switching surface of the time-optimal control, the unique equilibrium of (4.10) in $\mathcal{O}_n^+$ is $\bar{x}$, the vector of optimal time intervals.

In Yastreboff (1969a)-Yastreboff (1969b)-De Doná (2000), the same heuristics were considered in discrete-time, yielding the decentralized adaptation

$$x_i(k+1) = x_i(k) + f_i(|u_i(x(k))| - 1)x_i(k), \; i = 1, \ldots, n \tag{4.11}$$

which is the Euler discretization of (4.10) with sampling interval $\Delta t = 1$.

In Yastreboff (1969b), f_i is chosen as

$$\begin{array}{rcll} f_i(|u_i| - 1) & = & q \ln |u_i| & \text{if} \;\; |u_i(k)| \geq 1 \\ & & -q \; \ln(2 - |u_i|) & \text{if} \;\; |u_i(k)| < 1 \end{array}$$

where $q > 0$ is a constant parameter. In De Doná (2000), f_i is chosen as $f_i(|u_i| - 1) = |u_i|^p - 1$, with $p > 0$ a constant parameter. Both algorithms were derived independently but it is apparent that they are closely related. Simulation studies in De Doná (2000) show that both algorithms perform similarly, provided that the respective tuning parameters p and q, are properly chosen. Several examples are included in Yastreboff (1969a)–Yastreboff (1969b), in which the algorithm is compared with other existing methods (Smith 1961, Fadden & Gilbert 1964, Fadden 1965). Extensive simulations suggest that both the continuous-time algorithm (4.10) and its discrete-time version (4.11) converge (globally) to the time-optimal solution and that the convergence properties are not very sensitive to the details of the function f_i.

The present chapter provides a global analysis of the continuous-time system (4.10) with the functions f_i selected as saturated linear functions, yielding the algorithm:

$$\epsilon_i \dot{x}_i = \ \mathrm{sat}_M(|u_i(x)| - 1)x_i, \quad i \in \{1, \cdots, n\},\ x_i(0) > 0 \tag{4.12}$$

With $0 < \epsilon_n << \epsilon_{n-1} << \cdots << \epsilon_1$, a time-scale separation can be enforced between the different x_i dynamics, and the different control values $|u_i|$ successively converge to 1 (starting with $|u_n|$). These different time-scales are illustrated on Figure 4.5.

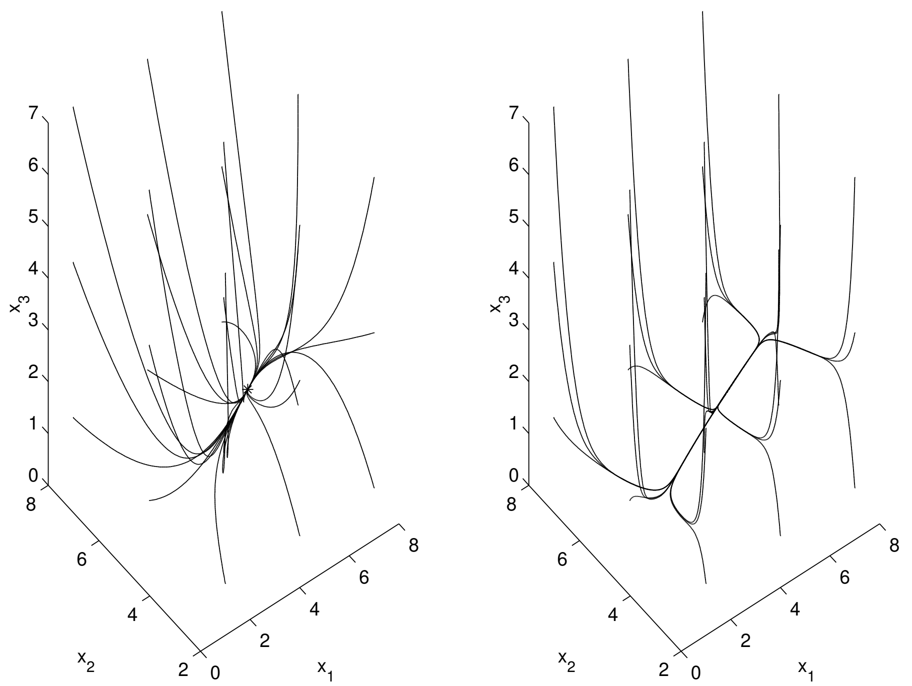

Figure 4.5: Simulations of (4.12) for the triple integrator $\frac{d^3y}{dt^3} = u$. The initial condition in the state-space is $y(0) = \dot{y}(0) = \ddot{y}(0) = 2$. The time-scale separation is ensured with, respectively, $\epsilon_1 = 2\epsilon_2 = 4\epsilon_3 = 1$ (left) and $\epsilon_1 = 10\epsilon_2 = 100\epsilon_3 = 1$ (right). The strong time-scales separation appears on the right, while it is clear that convergence takes place on the left, even with a weak separation.

4.4 An example

For the double integrator

$$\left\{ \begin{array}{lcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & v \end{array} \right. \qquad |v| \leq 1, \quad z(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

the dynamical system (4.12), with $\epsilon_1 = 1$, takes the form

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & \mathrm{sat}(|u_1| - 1)x_1 \\ \epsilon\dot{x}_2 & = & \mathrm{sat}(|u_2| - 1)x_2 \end{array} \right. \tag{4.13}$$

with $u = (u_1, u_2)^T$ solution of the steering equation (4.9):

$$\begin{pmatrix} -\frac{x_1^2}{2} & -\frac{(x_1^2+x_2^2)-x_1^2}{2} \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = - \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$$

The phase portrait of (4.13) is illustrated on Figure 4.6 (top) for ϵ very small. Fast convergence to the slow manifold $|u_2| \approx 1$ is followed by slow convergence to the equilibrium (indicated by a circle).

The manifold $|u_2| = 1$ has two branches: the branch $u_2 = +1$ goes through the equilibrium but exists only for $x_1 > 1$. The branch $u_2 = -1$ only exists for $x_1 \in (0, 1)$. The two branches connect at $(x_1, x_2) = (1, 0)$, on the boundary of the positive orthant.

If the initial condition of (4.13) is chosen such that $x_1 \in (0, 1)$, the solution first converges to the manifold $u_2 = -1$. In the phase plane of the original system, this corresponds to a steering control of the form $(u_1, u_2) = (-\hat{u}, -1)$ where $\hat{u} > 1$, see Figure 4.6.a. The solution of (4.13) then "slides" along the manifold $u_2 = -1$ up to the point $(x_1, x_2) = (1, 0)$ before jumping to the manifold $u_2 = +1$ where it is attracted to the equilibrium. The evolution of the corresponding steering control is illustrated in Figure 4.6 (bottom). The point $(x_1, x_2) = (1, 0)$ corresponds to a transition from a steering control $(u_1, u_2) = (-\hat{u}, -1)$ to a steering control $(u_1, u_2) = (-\hat{u}, +1)$. The transition takes place at $(x_1, x_2) = (1, 0)$ which produces the steering control $(-2, 0)$ illustrated on Figure 4.6.b. The phase plane of the time-optimal solution is then illustrated on 4.6.c.

On this simple example, the difficulties that will be encountered in the singular perturbation analysis are already present. When x_1 belongs to $(0, 1)$ or when $x_1 > 1$, the analysis of convergence to the manifold $u_2 = -1$ or $u_2 = 1$, followed by the sliding along this manifold is clear. The difficulty arises at $x_1 = 1$, which we call a *singular point*; at that point, no manifold $|u_2| = 1$ is defined inside the positive orthant. It corresponds to Figure 4.6.b, that is a solution that is steered to the origin without switching (and with

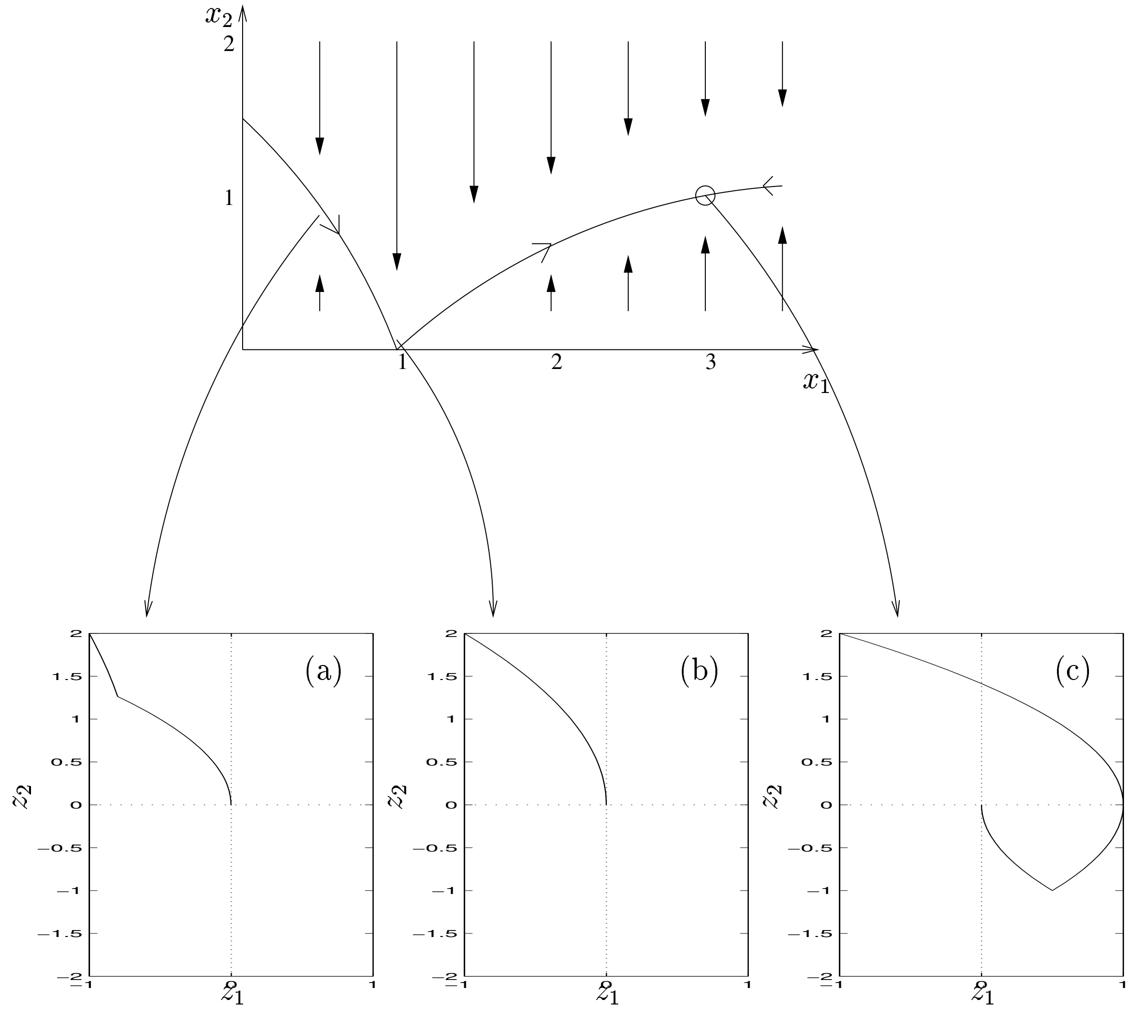

Figure 4.6: Top: Phase portrait of (4.13) for $\epsilon \to 0$. Bottom: Evolution of the solution $(z_1(t), z_2(t))$ along the application of the algorithm: (a) $\{x, u\} = \{(0.1228, 1.2649)^T, (-6, -1)^T\}$; (b) $\{x, u\} = \{(1, 0)^T, (-2, 0)^T\}$ (c) $\{\bar{x}, \bar{u}\} = \{(3, 1)^T, (-1, 1)^T\}$. Figure (b) represents the singularity, where the origin is reached without switching, and Figure (c) represents the time-optimal solution.

u_1 not satisfying the constraint). The singular perturbation analysis breaks down at that point because the sliding solutions do not have any manifold inside the domain of definition to follow anymore. In higher dimensions, the problem becomes even more intricate. There can even be a lack of continuity between the two branches of the manifold. This explains why a lot of effort will be devoted to the definition and treatment of singularities, in showing that solutions go through singularities and jump from one branch of manifold to the next.

The next sections show that the behavior displayed in the above example generalizes to higher dimensions.

4.5 Convergence analysis

The choice of $\epsilon_1 >> \cdots >> \epsilon_n$ induces a time-scales decomposition. Writing the system in the time-scale $\tau_i = \frac{t_i}{\epsilon_i}$ results in

$$\left\{\begin{array}{lcl} \frac{\epsilon_1}{\epsilon_i}\dot{x}_1 & = & \mathrm{sat}(|u_1(x)|-1)x_1 \\ & \vdots & \\ \frac{\epsilon_{i-1}}{\epsilon_i}\dot{x}_{i-1} & = & \mathrm{sat}(|u_{i-1}(x)|-1)x_{i-1} \\ \dot{x}_i & = & \mathrm{sat}(|u_i(x)|-1)x_i \\ \frac{\epsilon_{i+1}}{\epsilon_i}\dot{x}_{i+1} & = & \mathrm{sat}(|u_{i+1}(x)|-1)x_{i+1} \\ & \vdots & \\ \frac{\epsilon_n}{\epsilon_{n-1}}\dot{x}_n & = & \mathrm{sat}(|u_n(x)|-1)x_n \end{array}\right.$$

and $(x_1, \cdots, x_i)$ are the slow variables while $(x_{i+1}, \cdots, x_n)$ are the fast variables. The analysis will be done by induction from one time-scale to the other. Supposing that the equilibrium set of the fast variables $(x_{i+1}, \cdots, x_n)$ is SGAS when $(x_1, \cdots, x_i)$ is fixed, we will show that the equilibrium set of the variables $(x_i, x_{i+1}, \cdots, x_n)$ is SGAS when $(x_1, \cdots, x_{i-1})$ is fixed, this being the stability requirement for the fast variables in the time-scale $\tau_{i-1} = \frac{t_{i-1}}{\epsilon_{i-1}}$. The analysis will therefore concentrate on the subsystem

$$\left\{\begin{array}{lcl} \dot{x}_i & = & \mathrm{sat}(|u_i(x)|-1)x_i \\ \frac{\epsilon_{i+1}}{\epsilon_i}\dot{x}_{i+1} & = & \mathrm{sat}(|u_{i+1}(x)|-1)x_{i+1} \\ & \vdots & \\ \frac{\epsilon_n}{\epsilon_{n-1}}\dot{x}_n & = & \mathrm{sat}(|u_n(x)|-1)x_n \end{array}\right. \tag{4.14}$$

with $(x_1, \cdots, x_{i-1})$ fixed. We will describe the equilibrium set of the fast subsystem for x_i fixed and assume that it is SGAS. We will then show that, when the fast variables follow their equilibrium (forcing $|u_{i+1}|, \cdots, |u_n| = 1$), the equilibrium of the slow dynamics is GAS-LES. A singular perturbation analysis then concludes to the semiglobal stability of the whole system.

4.5.1 Fast dynamics equilibria

In the time-scale $\tau_i = \frac{t}{\epsilon_i}$, the fast dynamics of (4.14) are

$$\left\{ \begin{array}{rcl} \frac{\epsilon_{i+1}}{\epsilon_i}\dot{x}_{i+1} & = & \text{sat}(|u_{i+1}(x)| - 1)x_{i+1} \\ & \vdots & \\ \frac{\epsilon_n}{\epsilon_i}\dot{x}_n & = & \text{sat}(|u_n(x)| - 1)x_n \end{array} \right. \tag{4.15}$$

with $(x_1, \cdots, x_i)$ fixed. The slow dynamics of (4.14) then depend on the equilibria of (4.15). In this section, we will show that these equilibria are solutions representing the unique solution of the following time-optimal control problem:

$$\begin{array}{ll} \min T & \\ \quad \text{s.t.} & \dot{z} = Az + bv \qquad |v| \leq 1 \\ & z(0) \in \Pi_i = e^{At_i}z_0 + e^{At_i}\text{span}\{\Phi_{(:,1)}, \cdots, \Phi_{(:,i)}\} \\ & z(T) = 0 \end{array} \tag{$\mathcal{P}_i$}$$

with $t_j > t_{j-1}$ for $j \in \{1, \cdots, i\}$. Note that $\mathcal{P}_i$ is the time-optimal control problem of reaching the origin $z = 0$ from the affine subspace Π_i.

Proposition 4 *The solution of Problem $\mathcal{P}_i$ is unique and bang-bang with at most $n - i - 1$ switches. Moreover, any bang-bang strategy with at most $n - i - 1$ switches that satisfies $z(0) \in \Pi_i$ and $z(T) = 0$ is optimal.*

Proof: • Let $v^*(t)$ be an optimal solution of $\mathcal{P}_i$. Because it is maximal, we can find $\eta^*(t) = e^{-A^T t}\eta^*(0)$ solution of (4.2) such that

$$v^*(t) = \text{sign}(\eta^*(t)^T b)$$

almost everywhere, as well as the transversality condition

$$\eta^*(0) \perp T_{z^*(0)}\Pi_i$$

which comes directly from the maximum principle applied to the reverse time system. This transversality condition results in

$$\eta^*(0)^T e^{At_i} \int_{t_{j-1}}^{t_j} e^{-A\tau} b d\tau = 0$$

for $j \in \{1, \cdots, i\}$. Because $t_j > t_{j-1}$ for all j, this forces the existence of a τ_j inside each interval (t_{j-1}, t_j) such that

$$\eta^*(0)^T e^{-A(\tau_j - t_i)} b = 0$$

and $\eta^*(0)^T e^{-At} b$ has i negative roots, which means that $\eta^*(t)^T b = \eta^*(0)^T e^{-At} b$ can have at most $n-i-1$ positive roots and $v(t)$ can switch at most $n-i-1$ times between $t = 0$ and $t = T$ (see Lemma 3).

• Let $v(t)$ be bang-bang and drive $z(t)$ from Π to 0 with at most $n-i-1$ switches (at times s_k). Noting that Theorems 11 and 12 can be extended to Problem $\mathcal{P}_i$, we see that a maximal solution $v(t)$ of Problem $\mathcal{P}_i$ is unique and is the only optimal solution. If we show that $v(t)$ is maximal, it is then optimal. In order to show maximality, we have to find a non trivial η_0 such that

$$\eta_0^T e^{-As_k} b = 0$$

and the transversality conditions are satisfied:

$$\eta_0^T e^{At_i} \int_{t_{j-1}}^{t_j} e^{-A\tau} b d\tau = 0$$

Those two sets of inequalities result in a set of $n-1$ equations with n unknowns (the elements of η_0). Therefore, there exists a nontrivial solution η_0 of this set of equations such that $v(t) = \operatorname{sign}(\eta_0 e^{-At} b)$. Therefore, $v(t)$ is maximal and thus optimal. □

Because switching solutions are optimal, and the optimal solution is unique, there is a unique switching solution of problem $\mathcal{P}_i$. Regular solutions (for generic $(x_1, \cdots, x_i)$) involve $n-i-1$ switches and determine a unique sequence of switching intervals $\Omega(\mathcal{P}_i) = \{(x_{i+1}, \cdots, x_n)\}$ $(T = x_{i+1} + \cdots + x_n)$, with the corresponding control sequence $(\tilde{u}_{i+1}, \cdots, \tilde{u}_n)$ alternating between $+1$ and -1 (Grognard et al. 2001b). Singular solutions involve less than $n-i-1$ switches. In this case, the sequence of optimal switching intervals is not determined by a unique $(x_{i+1}, \cdots, x_n)$ of $I\!R^{n-i}$. For instance, if $n = 2$, a singular steering control $\left\{ \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \end{pmatrix} \right\}$ is equally represented by $\left\{ \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$ provided that $\bar{x}_1 + \bar{x}_2 = \bar{x}$. The set of vectors $(x_{i+1}, \cdots, x_n)^T$ that represent the same singular solution is denoted by $\Omega(\mathcal{P}_i)$; i.e.

$$\Omega(\mathcal{P}_i) = \left\{ \begin{array}{l} (x_{i+1}, \cdots, x_n)^T \in \mathcal{O}^+_{n-i} : \ \exists (u_{i+1}, \cdots, u_n)^T \text{ such that} \\ \{(x_{i+1}, \cdots, x_n)^T, (u_{i+1}, \cdots, u_n)^T\} \text{ is solution of } \mathcal{P}_i \end{array} \right\} \tag{4.16}$$

Definition 7 *When $\Omega(\mathcal{P}_i)$ is a singleton, we call the vector $(x_{i+1}, \cdots, x_n)^T$ regular. Otherwise, we call the vector $(x_{i+1}, \cdots, x_n)^T$ singular.*

We now characterize singularities more precisely.

Lemma 5 Let $(x_1, \cdots, x_i)$ be a singular point.
Then, either $\Phi(x_1, \cdots, x_i, 0, \cdots, 0)u = -z_0$ has a solution, in which case, we call $(x_1, \cdots, x_i)$ a singularity of order $n - i$,
or there exists an integer k, $1 \leq k < n - i$, and a unique $(x^0_{i+1}, \cdots, x^0_{n-k}) > 0$ such that $|u_j| = 1$ for $j \in \{i+1, \cdots, n-k\}$ and $u_j u_{j+1} = -1$ for $j \in \{i+1, \cdots, n-k-1\}$, and

$$\Phi(x_1, \cdots, x_i, x^0_{i+1}, \cdots, x^0_{n-k}, 0, \cdots, 0)u = -z_0$$

In the latter case, we call $(x_1, \cdots, x_i)$ a singularity of order k. □

The proof of Lemma 5 follows from the fact that the optimal solution of $\mathcal{P}_i$ is bang-bang and unique. All the elements of $\Omega(\mathcal{P}_i)$ lead to steering controls that are equivalent to the steering control defined in Lemma 5.

Proposition 5 *Let $\Omega(\mathcal{P}_i)$ be defined by (4.16) for a fixed $(x_1, \cdots, x_i)$.*

(i) *$\Omega(\mathcal{P}_i)$ characterizes the equilibrium set of (4.15) in $\mathcal{O}^+_{n-i}$.*

(ii) *Fix $(\bar{x}_1, \cdots, \bar{x}_{i-1}) > 0$. Then for each x_i, $u_i(x)$, solution of $\Phi(x)u = -z_0$, is identical for all $(x_{i+1}, \cdots, x_n) \in \Omega(\mathcal{P}_i)$, i.e.*

$$u_i(\bar{x}_1, \cdots, \bar{x}_{i-1}, x_i, x_{i+1}, \cdots, x_n) = \hat{u}_i(x_i)$$

(iii) *$\hat{u}_i$ is continuous on $(0, +\infty)$*

Proof: **(i)** $(x_{i+1}, \cdots, x_n)$ is an equilibrium of (4.15) in $\mathcal{O}^+_{n-i}$ if and only if $|u_k| = 1$ for all $k \in \{i+1, \cdots, n\}$. The resulting steering control switches at most $n - i - 1$ times. By uniqueness of the solution of $\mathcal{P}_i$, the steering control is optimal and $(x_{i+1}, \cdots, x_n) \in \Omega(\mathcal{P}_i)$
(ii) Because all points of $\Omega(\mathcal{P}_i)$ represent the same steering control, the impact point z^i in Π_i is identical for all $(x_{i+1}, \cdots, x_n) \in \Omega(\mathcal{P}_i)$. Because $(u_1, \cdots, u_i)$ is determined by the steering equation

$$e^{At_i} z_0 + \sum_{j=1}^{i} e^{At_i} \Phi_{(:,i)} u_j = z^i \tag{4.17}$$

it follows that $u_i(\bar{x}_1, \cdots, \bar{x}_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is identical for all elements $(x_{i+1}, \cdots, x_n) \in \Omega(\mathcal{P}_i)$. Define $\hat{u}_i(x_i) = u_i(\bar{x}_1, \cdots, \bar{x}_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ with $(x_{i+1}, \cdots, x_n) \in \Omega(\mathcal{P}_i)$.
(iii) The solution $(u_1, \cdots, u_i)$ of (4.17) is a continuous function of z^i. It is then a continuous function of x_i if z^i is a continuous function of x_i. It is shown in Lee & Markus (1967) (Sec. 2.5, Theorem 22) that the optimal time and the optimal control are continuous with respect to the initial condition. The initial set is here continuous with respect to x_i; therefore, the optimal time and the control law are continuous with respect to x_i. Continuity of the initial point z^i with respect to x_i follows directly. □

Point (ii) of Proposition 5 is fundamental in our singular perturbation analysis. Indeed, when x_i evolves with $|u_{i+1}| = \cdots = |u_n| = 1$ staying satisfied, and reaches a singular point $(x_1, \cdots, x_{i-1}, x_i^s)$, where $(x_{i+1}, \cdots, x_n)$ in $\Omega(\mathcal{P}_i)$ is not unique, the unicity of $\hat{u}_i(x_i^s)$ for all the elements of $\Omega(\mathcal{P}_i)$ forces the unicity of $\dot{x}_i = \text{sat}(|u_i| - 1)x_i$ and x_i crosses the singularity (see Figure 4.7).

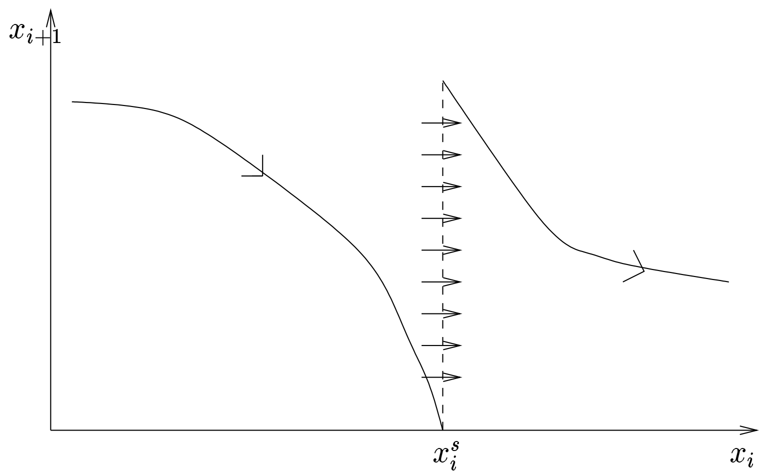

Figure 4.7: A solution following a manifold where $|u_{i+1}| = \cdots = |u_n| = 1$ until x_i^s where the manifold is not uniquely defined, but is characterized by the dashed line; the solution then jumps on to the second branch of the manifold.

4.5.2 Slow dynamics analysis

Fast convergence of the solutions of (4.15) to $\Omega(\mathcal{P}_i)$ will be proven in Section 4.5.3. In the time-scale $\tau_i = \frac{t}{\epsilon_i}$, the slow dynamics of (4.12) then reduces to the scalar equation:

$$\dot{x}_i = \operatorname{sat}(|\hat{u}_i(x_i)| - 1)x_i \tag{4.18}$$

with the continuous function $\hat{u}_i(x_i)$ determined as in Proposition 5. In Corollary 1 and 2 of this section, we will show that the equilibrium set of (4.18) is GAS.

Proposition 6 *Denote by Γ_i the equilibrium set of (4.18) in $(0, \infty)$.*

- *If $(x_1, \cdots, x_{i-1})$ is regular, then $\Gamma_i = \{\bar{x}_i\}$ where $\bar{x}_i$ is uniquely determined by the solution of $\mathcal{P}_{i-1}$.*
- *If $(x_1, \cdots, x_{i-1})$ is singular of order k $(1 \leq k < n - i + 1)$, then $\Gamma_i = (0, x_i^0]$, where x_i^0 is determined by Lemma 5.*
- *If $(x_1, \cdots, x_{i-1})$ is singular of order $n-i+1$, then $\hat{u}_i(x_i) \equiv 0$, and $\Gamma_i = \emptyset$ ($x_i = 0$ is the unique attractor of (4.18) which reduces to $\dot{x}_i = -x_i$).*

The remainder of the section is devoted to the stability analysis of Γ_i. If $(x_1, \cdots, x_{i-1})$ is regular, global attractivity of the equilibrium $\bar{x}_i$ is a consequence of the following result.

Proposition 7 *If $(x_1, \cdots, x_i)$ is regular, then*

$$\frac{d\hat{u}_i}{dx_i}(x_i)(\hat{u}_i(x_i) - \tilde{u}_{i+1}) < 0 \tag{4.19}$$

where the sign of $\tilde{u}_{i+1} = \pm 1$ is determined by the optimal solution of $\mathcal{P}_i$.

Proof: The control $(x_i, \hat{u}_i(x_i))$ is the unique steering control from the hyperplane Π_{i-1} to the switching surface

$$S_{i+1} = S_{i+1}^+ \cup S_{i+1}^- \text{ with}$$
$$S_{i+1}^{\pm} = \{z \in I\!R^n \mid z = \pm \textstyle\sum_{j=i+1}^n \int_{s_{j-1}}^{s_j} e^{-A\tau} b d\tau (-1)^j\}$$

$s_j = \sum_{k=i+1}^j x_k$. The initial condition in Π_{i-1} is

$$z^{i-1} = e^{At_{i-1}} z_0 + \sum_{j=1}^{i-1} e^{At_{i-1}} \Phi_{(:,j)} u_j$$

Because $(x_1, \cdots, x_i)$ is regular, the final condition in S_{i+1} is

$$z^{i+1} = -\sum_{j=i+1}^{n} \int_{s_{j-1}}^{s_j} e^{-A\tau} b d\tau \tilde{u}_j$$

The steering equation can then be rewritten as:

$$-\Phi_{(:,i)}(x)\hat{u}_i = e^{-At_{i-1}}z^{i-1} + e^{-At_i}z^{i+1}$$

that is,

$$-\Phi_{(:,i)}(x)\hat{u}_i \ \in \ Q_i = e^{-At_{i-1}}\Pi_{i-1} + e^{-At_i}S_{i+1}$$

A small variation of x_i implies a small variation of $\hat{u}_i$ such that

$$-d(\Phi_{(:,i)}(x)\hat{u}_i) \ \in \ e^{-At_{i-1}}T_{z^{i-1}}\Pi_{i-1} + e^{-At_i}T_{z^{i+1}}S_{i+1} - e^{-At_i}Az^{i+1}dx_i \quad (4.20)$$

where T_xM denotes the tangent space to M at x. An explicit calculation yields:

$$e^{-At_{i-1}}T_{z^{i-1}}\Pi_{i-1} = \ \text{span}\{\Phi_{(:,1)}(x), \cdots, \Phi_{(:,i-1)}(x)\}$$

$$e^{-At_i}T_{z^{i+1}}S_{i+1} = \ \text{span}\{e^{-At_{i+1}}b, \cdots, e^{-At_n}b\}$$

$$e^{-At_i}Az^{i+1}dx_i = -\sum_{j=i+1}^{n}(e^{-At_j}b - e^{-At_{j-1}}b)\tilde{u}_j dx_i$$

which implies that

$$e^{-At_i}Az^{i+1}dx_i \ \in \ e^{-At_i}T_{z^{i+1}}S_{i+1} + e^{-At_i}b\tilde{u}_{i+1}dx_i$$

Also

$$d(\Phi_{(:,i)}(x)\hat{u}_i) = e^{-At_i}b\hat{u}_i dx_i + \int_{t_{i-1}}^{t_i} e^{-A\tau}bd\tau d\hat{u}_i$$

so that (4.20) is equivalent to

$$\begin{aligned} &-e^{-At_i}b(\hat{u}_i - \tilde{u}_{i+1})dx_i - \textstyle\int_{t_{i-1}}^{t_i} e^{-A\tau}bd\tau d\hat{u}_i \\ &\quad \in E_i = \ \text{span}\{\Phi_{(:,1)}(x), \cdots, \Phi_{(:,i-1)}(x), e^{-At_{i+1}}b, \cdots, e^{-At_n}b\} \end{aligned}$$

Choosing a nonzero vector $w \in E_i^{\perp}$, we obtain:

$$w^T\int_{t_{i-1}}^{t_i} e^{-A\tau}bd\tau d\hat{u}_i + w^Te^{-At_i}b(\hat{u}_i - \tilde{u}_{i+1})dx_i = 0 \quad (4.21)$$

Define $f(t) = w^Te^{-A\tau}b$. Orthogonality of w to E_i implies $w^T\int_{t_{j-1}}^{t_j} f(\tau)d\tau = 0$ for $j \in \{1, \cdots, i-1\}$, which means that $f(t)$ has $i-1$ roots in the open interval $(0, t_{i-1})$, and also $f(t_j) = 0$ for $j \in \{i+1, \cdots, n\}$, which means that $f(t)$ has $n-i$ roots in the closed interval $[t_{i+1}, t_n]$. Because $f(t)$ has at most

$n-1$ roots (see Theorem 13), $f(t_i) \neq 0$ and $\frac{f(t_i)}{\int_{t_{i-1}}^{t_i} f(\tau)d\tau} > 0$. From (4.21), we conclude

$$\frac{d\hat{u}_i}{dx_i} = -\frac{f(t_i)}{\int_{t_{i-1}}^{t_i} f(\tau)d\tau}(\hat{u}_i - \tilde{u}_{i+1})$$

which proves (4.19). □

Corollary 1 *If $(x_1, \cdots, x_{i-1})$ is regular, the unique equilibrium $\bar{x}_i$ of (4.18) is locally exponentially stable and globally attractive in $(0, +\infty)$.*

Proof: Linearization of (4.18) at $x_i = \bar{x}_i$ yields

$$\delta\dot{x}_i = \frac{d\hat{u}_i}{dx_i}(\bar{x}_i)\ \mathrm{sign}(\hat{u}_i(\bar{x}_i))\bar{x}_i\delta x_i$$

which is stable if $\frac{d\hat{u}_i}{dx_i}\ \mathrm{sign}(\hat{u}_i(x_i)) < 0$. Noting that $\tilde{u}_i = -\ \mathrm{sign}(\hat{u}_i)$ at the equilibrium, (4.19) becomes

$$2\frac{d\hat{u}_i}{dx_i}\ \mathrm{sign}(\hat{u}_i(x_i)) < 0$$

and local exponential stability is proven. For a scalar system with a single equilibrium, global attractivity follows from local stability. □

In order to prove asymptotic stability of the equilibrium set $[0, x_i^0]$ when $(x_1, \cdots, x_{i-1})$ is singular, we need the following proposition

Proposition 8 *If $(x_1, \cdots, x_{i-1})$ is regular (resp. singular of order smaller than $n-i+1$), the values $x_i > 0$ (resp. $x_i > x_i^0$) such that $(x_1, \cdots, x_i)$ is singular are isolated.*

Proof: Let $(x_1, \cdots, x_i)$ be singular of order k. Then $(x_i, \hat{u}_i)$ is a steering control from Π_{i-1} to S_{i+1+k} with impact point z^{i+1+k} and the steering equation yields

$$-\Phi_{(:,i)}(x)\hat{u}_i \quad \in \quad Q_i = e^{-At_{i-1}}\Pi_{i-1} + e^{-At_i}S_{i+1+k}$$

If the singularity is not isolated, there exists a variation dx_i such that the transfer $-\Phi_{(:,i)}(x)\hat{u}_i$ stays in Q_i:

$$-d(\Phi_{(:,i)}(x)\hat{u}_i) \in e^{-At_{i-1}}T_{z^{i-1}}\Pi_{i-1} + e^{-At_i}T_{z^{i+1+k}}S_{i+1+k} - e^{-At_i}Az^{i+1+k}dx_i$$

with a calculation similar to the proof of Proposition (7), this is equivalent to:

$$\begin{aligned} &-e^{-At_i}b(\hat{u}_i \pm 1)dx_i - \int_{t_{i-1}}^{t_i} e^{-A\tau}bd\tau d\hat{u}_i \\ &\quad \in\ E_i \ \ \text{span}\{\Phi_{(:,1)}(x), \cdots, \Phi_{(:,i-1)}(x), e^{-At_{i+1}}b, \cdots, e^{-At_{n-k}}b, \} \end{aligned} \tag{4.22}$$

Because $dim(E_i) < n-1$, there exists a nonzero vector w orthogonal to E_i and to $\int_{t_{i-1}}^{t_i} e^{-A\tau}bd\tau$, but not orthogonal to $e^{-At_i}b$. Equation (4.22) then yields

$$w^T e^{-At_i}b(\hat{u}_i \pm 1) = 0$$

This is a contradiction if $|u_i| \neq 1$, which holds true whenever $(x_1, \cdots, x_{i-1})$ is regular or singular of order smaller than $n-i+1$ and $x_i > x_i^0$. □

Corollary 2 *If $(x_1, \cdots, x_{i-1})$ is singular of order $< n-i+1$, the equilibrium set $\Gamma_i = (0, x_i^0]$ is globally asymptotically stable in $(0, +\infty)$.*

Proof: By Proposition 6, $|\hat{u}_i| = 1$ only if $x_i \in [0, x_i^0]$. By Proposition 8, there exists $\epsilon^* > 0$ such that $(x_1, \cdots, x_{i-1}, x_i + \epsilon)$ is regular for all $0 < \epsilon < \epsilon^*$. For $x_i \in I_{\epsilon^*} = (x_i^0, x_i^0 + \epsilon^*)$, the optimal control value $\tilde{u}_{i+1}(x_i) = \pm 1$ is constant.

We prove that $\tilde{u}_{i+1}(x_i) = -\hat{u}_i(x_i^0)$ for $x_i \in I_{\epsilon^*}$. By contradiction, suppose that $\tilde{u}_{i+1}(x_i) = \hat{u}_i(x_i^0)$. If $\hat{u}_i(x_i) > \tilde{u}_{i+1}(x_i)$, (4.19) forces $\frac{d\hat{u}_i}{dx_i} < 0$ in the whole interval I_{ϵ^*}, which, by continuity of $\hat{u}_i(x_i)$, contradicts that $\hat{u}_i(x_i) > \tilde{u}_{i+1}(x_i)$. The same reasoning can be held for $\hat{u}_i(x_i) < \tilde{u}_{i+1}(x_i)$ inside the interval. The only possibility is then that $\hat{u}_i(x_i) = \hat{u}_i(x_i^0)$, which contradicts Proposition 6.

We conclude that $\tilde{u}_{i+1}(x_i) = -\hat{u}_i(x_i^0)$ and that $|\hat{u}_i(x_i)| < |\tilde{u}_{i+1}(x_i)|$ for $x_i \in I_{\epsilon^*}$. This implies $|\hat{u}_i(x_i)| < 1$ for all $x_i > x_i^0$. The set Γ_i is then globally asymptotically stable in $(0, \infty)$.

□

4.5.3 Global convergence analysis

This section contains the main result of the chapter and its proof. It puts together the fast and slow dynamics analyses of the previous two sections. It handles the problems illustrated by x_i^s in Figure 4.7.

Theorem 14 *Let $1 \leq i \leq n$ and fix $(x_1, \cdots, x_{i-1}) > 0$ if $i > 1$*
Then the equilibrium set $\Omega(\mathcal{P}_{i-1})$ of

$$\begin{cases} \dot{x}_i & = & sat(|u_i(x)| - 1)x_i \\ & \vdots & \\ \epsilon_n \dot{x}_n & = & sat(|u_n(x)| - 1)x_n \end{cases} \tag{4.23}$$

is asymptotically stable. It is exponentially stable if $\Omega(\mathcal{P}_{i-1})$ is a singleton.

Moreover, the region of attraction of $\Omega(\mathcal{P}_{i-1})$ in the positive orthant is enlarged at will by proper separation of the time-scales $\tau_n = \frac{t}{\epsilon_n}, \cdots, \tau_i = \frac{t}{\epsilon_i}$

Proof: The proof goes by induction:
Let $i = n$ If $(x_i, \cdots, x_{n-1})$ is regular, then $u_n(x) = \hat{u}_n(x_n)$ and the unique equilibrium $\bar{x}_n$ of $\dot{x}_n = \text{sat}(|\hat{u}_n(x_n)| - 1)x_n$ is exponentially stable and globally attractive, see Corollary 1.

If $(x_i, \cdots, x_{n-1})$ is singular, then Π_{n-1} contains the origin $z = 0$ and $u_n(x) = \hat{u}_n(x_n) = 0$. Then $x_n = 0$ is globally exponentially stable.
Induction step Consider (4.23) with $1 \leq i < n$. Denote by Γ_i the equilibrium set of

$$\epsilon_i \dot{x}_i = \text{sat}(|\hat{u}_i(x_i)| - 1)x_i$$

and by Σ_i the subset of $I\!R_0^+ \setminus \Gamma_i$ such that $x_i \in \Sigma_i$ implies that $(x_1, \cdots, x_i)$ is singular.

Assume first that $\Sigma_i = \emptyset$. Then the fast subsystem $(x_{i+1}, \cdots, x_n)$ of (4.23) has a unique equilibrium for all $x_i \in I\!R_0^+ \setminus \Gamma_i$, and this equilibrium is exponentially stable with an arbitrarily large region of attraction from the induction hypothesis. As a consequence, there exists an invariant slow manifold in the neighborhood of $I\!R_0^+ \setminus \Gamma_i \times \Omega(\mathcal{P}_i)$. On the other hand, the equilibrium set Γ_i of (4.23) is globally asymptotically stable (see Corollaries 1 and 2).

From standard singular perturbations theory (e.g. Theorem 3.18 in Sepulchre et al. (1996)), the equilibrium set $\Gamma_i \times \Omega(\mathcal{P}_i)$ of (4.23) is locally asymptotically stable and its region of attraction can be made arbitrarily large by making $\frac{\epsilon_{i+1}}{\epsilon_i}$ small enough.

Next suppose that $\Sigma_i \neq \emptyset$. Denote by Σ_i^γ a closed neighborhood of Σ_i such that $\Sigma_i^\gamma \cap \Gamma_i = \emptyset$ and such that Σ_i^γ is made of disjoint closed intervals I_l^γ of length γ $(l \geq 1)$. By Proposition 8, $\gamma > 0$ can be selected arbitrarily small. Consider a solution that satisfy $x_i(t) \in I_l^\gamma$ for all $t > 0$ and let $\tilde{x}_i$ be a singularity of maximal order in I_l^γ. Then $(x_{i+1}(t), \cdots, x_n(t))$ converges to a γ neighborhood $\Omega^\gamma(\mathcal{P}_i)(\tilde{x}_i)$ of $\Omega(\mathcal{P}_i)(\tilde{x}_i)$. Because $|\, |\hat{u}_i(\tilde{x}_i)| - 1 \,| \geq \delta$, for some $\delta > 0$ and $u_i(x_i, \cdots, x_n)$ is continuous, there exists γ small enough such that $|\,|u_i(\tilde{x})| - 1\,| \geq \frac{\delta}{2}$ for all $x_i \in I_l^\gamma$ and $(x_{i+1}(t), \cdots, x_n(t)) \in \Omega^\gamma(\mathcal{P}_i)(\tilde{x}_i)$. But then $\dot{x}_i$ is sign definite for t large enough and does not converge to zero, which contradicts the fact that $x_i(t) \in I_l^\gamma$ for all $t > 0$. The slow dynamics (4.18) being valid in $(I\!R^+ \setminus \Gamma_i) \cup \Sigma_i^\gamma$, we conclude that all solutions converge to $\Gamma_i \times \Omega(\mathcal{P}_i)$.

□

This theorem completes the global convergence analysis of the continuous flow (4.12). This analysis rested on the time-scale separation of the dynamics into n scalar dynamics, each one characterized as the transfer from an hyperplane to a switching surface. The next section studies the implementation of the algorithm.

4.6 Implementation of the algorithm

The convergence analysis of the previous sections was performed for the system

$$\left\{ \begin{array}{rcl} \epsilon_1 \dot{x}_1 & = & \mathrm{sat}_M(|u_1(x)| - 1)x_1 \\ & \vdots & \\ \epsilon_n \dot{x}_n & = & \mathrm{sat}_M(|u_n(x)| - 1)x_n \end{array} \right.$$

with $\epsilon_1 >> \cdots >> \epsilon_n > 0$. Theorem 14 ensures semiglobal convergence of x to the optimal solution for x_0 inside $\mathcal{O}_n^+$ with a sufficient separation of the ϵ_i parameters.

Choice of the discretization algorithm

This continuous version can obviously not be implemented as is on a computer. Therefore, it is necessary to discretize the algorithm. Two discretizations have been considered:

- a variable step Runge-Kutta algorithm (e.g. through `ode45` of Matlab);
- a fixed step Euler discretization.

Runge Kutta methods are designed to achieve simulations of continuous systems with a solution that is very close to the solution of the continuous system (this closeness is measured by an upper bound m on the error that is given as input to the algorithm). On the other hand, an Euler discretization is the most basic discretization: if we fix the discretization step at the value δ, the Euler approximation of the continuous algorithm is:

$$\left\{\begin{array}{rcl} x_1(k+1) & = & x_1(k) + \frac{\delta}{\epsilon_1}\,\mathrm{sat}_M(|u_1(x(k))| - 1)x_1(k) \\ & \vdots & \\ x_n(k+1) & = & x_n(k) + \frac{\delta}{\epsilon_n}\,\mathrm{sat}_M(|u_n(x(k))| - 1)x_n(k) \end{array}\right. \tag{4.24}$$

which, in the particular case where $\delta = 1$ and $\epsilon_i = 1$, amounts to the algorithm of De Doná (2000).

Remark 5 *Note that, for the Euler discretization, it is advised to take a discretization step $\delta < \epsilon_n$. Indeed, this step size ensures invariance of $\mathcal{O}_n^+$. Indeed, if we consider any time-interval x_j, the algorithm gives*

$$x_j(k+1) = x_j(k) + \frac{\delta}{\epsilon_j}\; sat_M(|u_j(x(k))| - 1)x_j(k)$$

The value of u_j that forces the largest decrease of $x_j(k)$ is $u_j = 0$, which means that, for any u_j, we have

$$x_j(k+1) \geq x_j(k) - \frac{\delta}{\epsilon_j}x_j(k) = (1 - \frac{\delta}{\epsilon_j})x_j(k) > 0$$

because $1 - \frac{\delta}{\epsilon_j} > 1 - \frac{\delta}{\epsilon_n} > 0$.

The choice $\delta < \epsilon_n$ does not guarantee stability, but at least, it guarantees that x stays in $\mathcal{O}_n^+$.

Let us now consider an example: the double integrator:

$$\left\{\begin{array}{rcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & u \end{array}\right.$$

with the initial condition $z_0 = (-1\ 2)^T$ as considered in Section 4.4. Semi-global convergence of the algorithm is ensured through a sufficient time-scale separation (Theorem 14). Therefore, we take $\epsilon_1 = 1$ and $\epsilon_2 = 0.1$, which results in the continuous algorithm

$$\begin{array}{rcl} \dot{x}_1 & = & \mathrm{sat}_M(|u_1(x)| - 1)x_1 \\ 0.1\dot{x}_2 & = & \mathrm{sat}_M(|u_2(x)| - 1)x_2 \end{array}$$

In order to compare the Runge-Kutta algorithm and the Euler discretization we pick $x = \{(0.1\ \ 0.1)^T, (0.1\ \ 3)^T, (5\ \ 0.1)^T, (5\ \ 3)^T)\}$ as initial conditions, sufficiently remote from the equilibrium $x = (3\ \ 1)^T$, and decide to stop the algorithm when u is sufficiently close to a bang-bang strategy ($\|u -$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \| < 10^{-3}$). We see on Figure 4.6 that the algorithm has to go through a singularity (in $x_1 = 2$) before reaching the equilibrium when $x_{10} = 0.1$, which means that the algorithm has to go through what we identified as a critical point of the convergence.

The question is now posed of the tolerance level m and the time-step δ. It is obvious that the smaller we will take m and δ, the closer the solution will be to the solution of the continuous time system; small m and δ also obviously result in slow convergence. We a priori pick the tolerance level $m = 10^{-3}$, and the limit time-step given in Remark 5: $\delta = 0.1$ (simulations have shown that for larger δ, the solution exits $\mathcal{O}_2^+$). The four initial conditions resulted in the following number of Floating Point Operations (`flops` in Matlab).

x_0	Runge-Kutta	Euler
$(0.1\ 0.1)^T$	158224	37432
$(0.1\ 3)^T$	145200	37480
$(5\ 0.1)^T$	107543	32142
$(5\ 3)^T$	108173	31738

The upper hand of the Euler algorithm is obvious in this table. The corresponding (x_1, x_2) phase plane is shown in Figure 4.8. We see in this figure that, as expected, the behavior of the Runge-Kutta algorithm (solid line) results in a solution similar to what was shown on Figure 4.6: the solution first reaches the manifold where $u_2 = -1$, then gets close to the singularity ($x = (2\ 0)^T$) while jumping to the manifold $u_2 = +1$, and finally converges to the equilibrium solution. This figure also shows that the Euler discretization oscillates around the manifold $u_2 = -1$ because of the combination of the high-gain with the fixed-step, but those oscillations are damped, and the manifold is eventually reached. A larger fixed-step δ generates unstable oscillations because of the factor $\frac{\delta}{0.1}$ in (4.24), and the algorithm does not converge to the equilibrium. On the other hand, a smaller δ ensures a closer following of the continuous-time solution, which forces a slower convergence.

Reduction of the time-scale separation

Though we have shown that the Euler discretization could result in a faster convergence to the equilibrium than the Runge Kutta method, we have noticed a dangerous oscillating behavior around the manifold. This oscillation comes from the combination of high gain with a large discretization step. This behavior can be avoided by diminishing the time-step, but this results in slower convergence. We can also diminish the time-scale separation, by

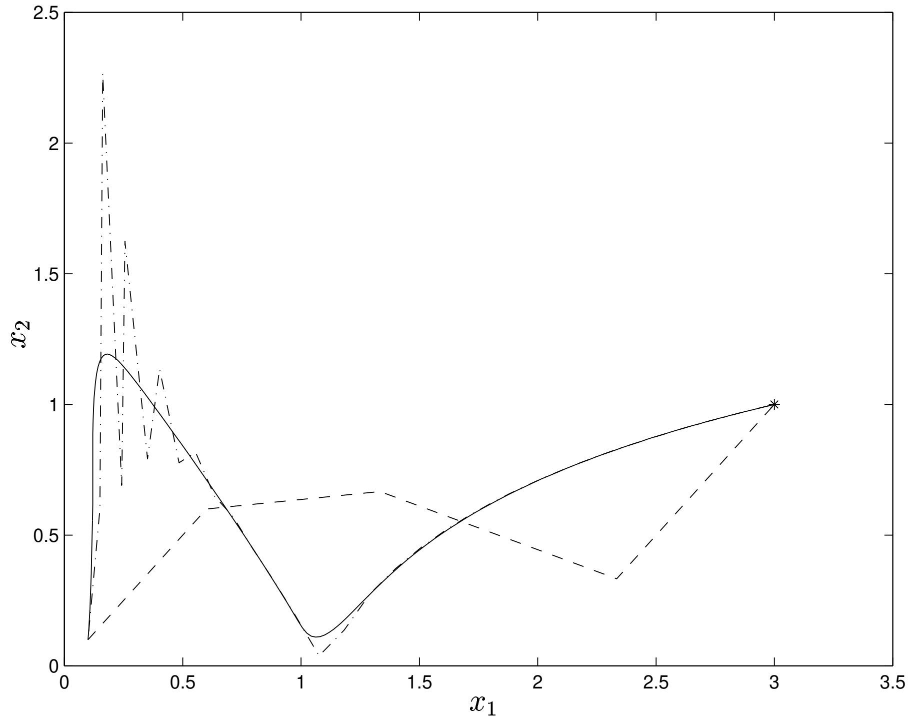

Figure 4.8: Phase plane evolution of the algorithm: (solid line) Runge-Kutta's method with a time-scale separation; (dash-dotted) Euler with time-scale separation; (dashed) Euler without time-scale separation.

taking $\epsilon_2 = 1$. If we now take the corresponding allowed time step (by Remark 5), that is $\delta = 1$, we obtain

$$\left\{ \begin{array}{rcl} x_1(k+1) & = & x_1(k) + \ \mathrm{sat}_M(|u_1(x(k))| - 1)x_1(k) \\ x_2(k+1) & = & x_2(k) + \ \mathrm{sat}_M(|u_2(x(k))| - 1)x_2(k) \end{array} \right.$$

which is exactly the algorithm of De Doná (2000). Considering the double integrator with the same z_0 and the same initial conditions x_0, the Euler discretization presented in the previous section and the one presented here compare as follows:

x_0	Euler 1	Euler 2
$(0.1\ 0.1)^T$	37432	2326
$(0.1\ 3)^T$	37480	2406
$(5\ 0.1)^T$	32142	2066
$(5\ 3)^T$	31738	1248

which shows that a suppression of the time-scale separation results in improved convergence. Figure 4.8 (dashed line) shows that convergence takes place towards the equilibrium without actual convergence to the manifold $u_2 = \pm 1$ However, Figure 4.9 illustrates the comparison of the phase planes between the continuous algorithm and the Euler algorithm without time-scale separation. Though the manifold $|u_2| = 1$ is not invariant in the latter case, we see that this convergence has a pattern similar to the one exhibited in the continuous case with time-scale separation, which indicates that, dropping the time-scale separation is not dangerous for convergence.

Those results show that the the convergence of the algorithm can be very fast, especially if an Euler discretization with a large time-step is chosen. However, because of those two choices the semiglobal result of Theorem 14 is not valid anymore.

4.7 Application of the algorithm to systems with complex eigenvalues

The proofs of convergence of the algorithm have been exposed for the case where the eigenvalues of A are real. We will now show that the algorithm can still be efficient when those eigenvalues are complex, as well as the obstacles to convergence that arise in such a case.

Example 7 *We will consider the controlled harmonic oscillator:*

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & -z_1 + v \qquad |v| \leq 1 \end{array} \right.$$

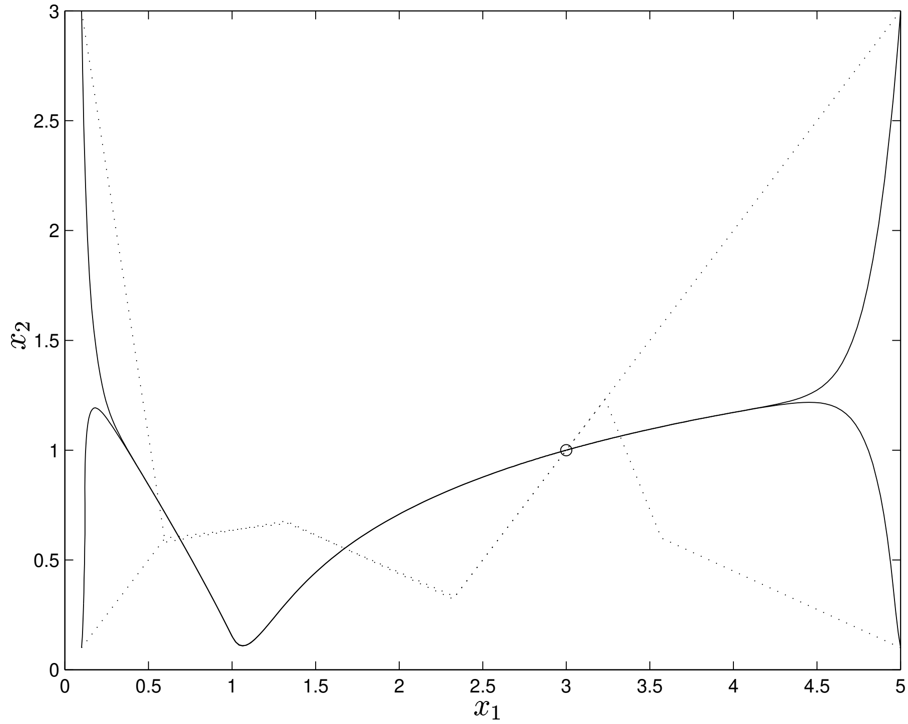

Figure 4.9: Phase plane evolution of the algorithm: (solid line) Runge-Kutta's method with a time-scale separation; (dotted line) Euler without time-scale separation.

whose switching curve has been analytically calculated in Section 4.1.2 and is illustrated on Figure 4.10: it is the plain line forming successive half-circles; above the curve, the control $v = -1$ *is applied, and the control* $v = +1$ *is applied under it. The phase-plane evolution of a solution starting far from the origin* (z_{0a}) *is shown (dotted line). The control switches each time the solution crosses a switching curve (in this case four times). After the last switch, the solution stays on the last branch of the switching curve to converge to the origin. The hatched zone* $\mathcal{A}$ *is the set of initial conditions which only require one switch to reach the origin.*

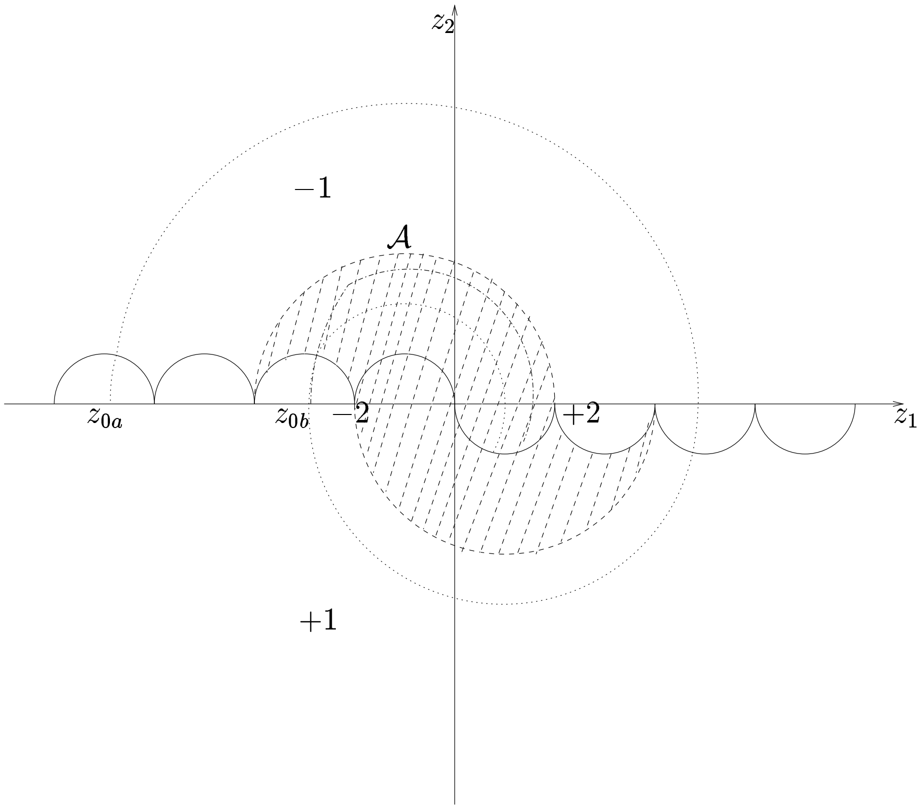

Figure 4.10: Phase plane of the time-optimal control of the harmonic oscillator: (solid line) switching curve; (dotted line) the time-optimal solution initiating at z_{0a} and going through z_{0b}; (dash-dotted line) a bang-bang solution with two switches starting at z_{0b}

If we take $z_0 = (1\ 1)^T$*, an initial condition such that the time-optimal solution only presents one switch, the phase plane of the evolution of the algorithm is as shown on Figure 4.11 (with* $\epsilon_1 = 1$ *and* $\epsilon_2 = 0.1$*): four* *equilibria are clearly attractive, that is the time-optimal equilibrium* $x = (0.9305\ 1.5709)^T$*, and* $x = (0.9305 + 2\pi\ 1.5709)^T$, $x = (0.9305\ 1.5709 + 2\pi)^T$*, and* $x = (0.9305 + 2\pi\ 1.5709 + 2\pi)^T$ *(represented by circles on the figure). Regions of attraction for those points are roughly apparent on the figure. Among others, the dotted lines are separatrix of those regions. The lower one is* $x_1 + x_2 = \pi$ *and is a line of equilibria (corresponding to* $u_1 = u_2 = +1$*). Those equilibria are mostly repulsive, except in the region where* $x_1 > 4$*, where they are attractive. The upper one is* $x_1 + x_2 = 3.5\pi$ *and is also a line of equilibria (corresponding to* $u_1 = u_2 = +1$*).*

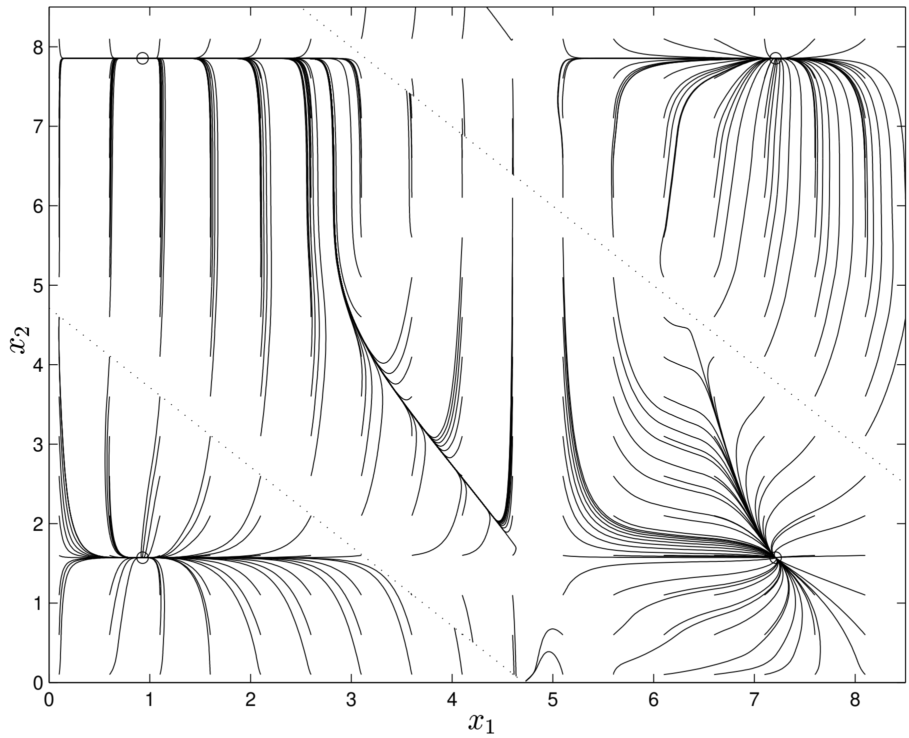

Figure 4.11: Phase plane of the evolution of the continuous algorithm with $z_0 = (1\ 1)^T$. Four equilibria, including the time optimal solution $x = (0.9305\ 1.5709)^T$, are represented. They are all attractive and their region of attraction is apparent on the figure

The efficiency of the algorithm results from the fact that the proofs given in this chapter do not so much rely on the fact that the eigenvalues of A are real as they rely on the number of switchings, which must be less than $n-1$.

Beyond that positive result, there are two obstacles to semiglobal convergence of the algorithm:

- Theorem 13 indicates that *there is no a priori upper bound on the number of switchings of the time-optimal solution.* Because the algorithm can only be applied when there are less than $n-1$ switches, we don't know a priori if the application of the algorithm for a given initial condition will converge to a time-optimal solution.

- Property 2 is not valid anymore: a bang-bang solution is not forced to be optimal. This can be illustrated on the controlled harmonic oscillator:

  1. Consider the optimal solution starting at z_{0b} for the harmonic oscillator: it switches twice, that is n times (each time it crosses the switching curve). However, any solution starting with $+1$ at least until z belongs to $\mathcal{A}$, and switching to -1 before z leaves $\mathcal{A}$, and then back to $+1$ once the switching curve is reached again (e.g. the dash-dotted line), is an admissible bang-bang solution that steers the solution from z_{0b} to the origin. Therefore, uniqueness of the bang-bang controller is not ensured. This controller is not even isolated in the x state-space, which means that no convergence result can be hoped for, for such an initial condition. This confirms that we should not try to apply the algorithm to time-optimal solutions with more than $n-1$ switchings.

  2. Let us now consider an initial condition inside $\mathcal{A}$. We have shown on the example that convergence to the time-optimal solution occurred. However, no global convergence is achieved. Indeed, even in this case, there is not a unique bang-bang solution: if the strategy $\{\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}\}$ is the time-optimal strategy, we see that the strategies $\{\begin{pmatrix} \bar{x}_1 + 2k_1\pi \\ \bar{x}_2 + 2k_2\pi \end{pmatrix}, \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}\}$ with k_1, k_2 positive integers, also are bang-bang strategies with the origin as endpoint, but they certainly are not time-optimal. This is illustrated on Figure 4.11, which represent the phase plane of the continuous time algorithm with $z_0 = (1\ 1)^T$. (with $u_1 + u_2 = 1$).

The algorithm is efficient in the region surrounding the origin, characterized by the number of switchings of the time-optimal solution which is inferior to $n-1$ (the region $\mathcal{A}$ in the case of the controlled harmonic oscillator). For points outside this region, another controller must be applied to bring the solution inside $\mathcal{A}$, for example, one of the scheduled low-gain controllers presented in the previous section. In the case of the harmonic oscillator, we can evidence the interest of combining a global stabilizing feedback with a time-optimal solution once z reaches $\mathcal{A}$. For example, one can use a damping control law based on the quadratic Lyapunov function

$$V(z_1, z_2) = \frac{z_1^2 + z_2^2}{2} \qquad \frac{dV}{dt} = z_2 v$$

that is

$$v = - \operatorname{sign}(z_2)$$

which generates a decrease of $V(z_1, z_2)$ until $z_2 = 0$ and $z_1 \in [-1, 1]$, where chattering equilibria are generated. This chattering is not harmful because, by the time z reaches such points, the algorithm is able to compute the time-optimal solution and the damping control is not active any more.

This is equivalent to a modification of the switching curve presented in Athans & Falb (1966) and illustrated on Figure 4.12, which, for any initial condition on the z_1 axis, results in an increase of the total time inferior to 5% with respect to the optimal time. Therefore, we see that using the exact time-optimal controller far from the origin is not always necessary in the case of systems with oscillating modes.

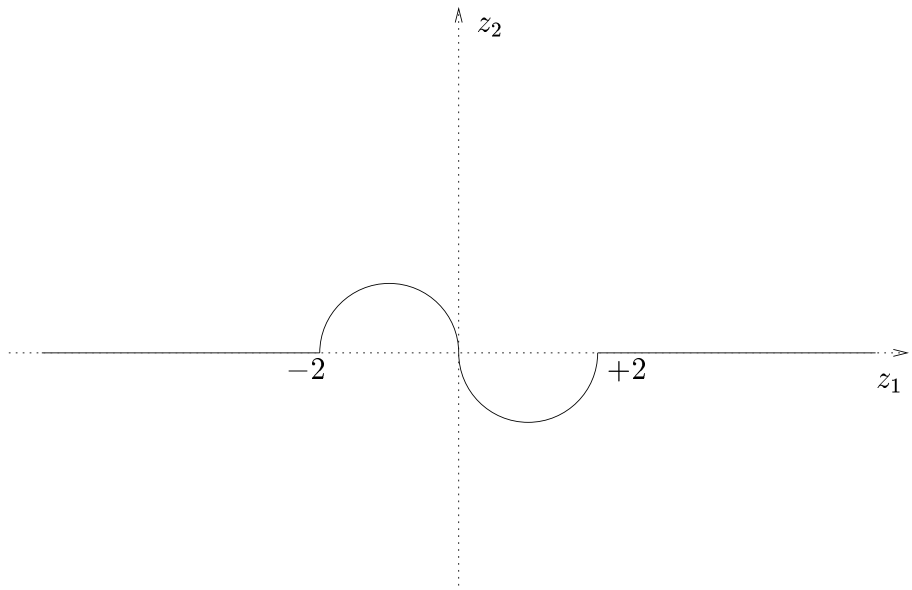

Figure 4.12: Modified switching curve

4.8 Conclusion

In this chapter, we presented the results of (Grognard & Sepulchre (2001a)) and (Grognard et al. (2001b)): an algorithm for the synthetis of time-optimal control laws for linear processes. We have first given the necessary tools for understanding time-optimal control. We have then shown that the calculation of the time-optimal controller in feedback form is usually not tractable and hinted that an efficient algorithm to solve the open-loop time-optimal control problem could be useful to be implemented in an MPC scheme, which closes the loop. We have then presented two families of methods to compute the open-loop control for a given initial condition. For each of these methods we have exposed its qualities and its drawbacks. This has lead us to the construction of a synthesis method which ensures semiglobal convergence to the optimal controller, and doing so rapidly (this is shown empirically here and in Yastreboff (1969a) and De Doná (2000)). The implementation of this method has also been exposed, as well as its utilization for systems with complex eigenvalues. In order to validate this method for its use in an MPC scheme, its local convergence properties should be analyzed.

Conclusion

In this thesis, we have addressed the control of systems presenting constraints. Those constraints result from feedforward interconnections, which are present in many mechanical systems in feedforward form, or from input constraints, commonly encountered in applications. Both are gain-limiting, and we have described low-gain methods that could be used to stabilize the origin of those systems. As illustrated in Chapter 1, existing stabilizing methods for those systems do not sufficiently take into account robustness to the presence of feedback terms or performance requirements.

In Chapter 2, existing stabilizing designs for structured nonlinear systems in feedback form (backstepping) and in feedforward form (forwarding) have been exposed. We have shown that backstepping was not robust to the presence of gain-limiting feedforward interconnections, and that forwarding was not robust to the presence of gain-demanding feedback interconnections. In Chapter 2, we have presented conditions on the feedback and feedforward interconnections, so that they are compatible, and global stabilization can still be achieved. In essence, this amounts to the feedback interconnections being local, that is they depend only on the state of the considered integrator, bounded, and with bounded derivative, and the feedforward connections being rate-limiting instead of gain-limiting. We have proposed a globally stabilizing controller using saturations for a class of systems exhibiting both those types of interconnections.

In Chapter 3, the main concern was the lack of performance of low-gain designs for the stabilization of linear systems with inputs constraints. When large regions of attraction are required, the value of the parameter of the low-gain design $u = -K(\lambda)x$ is taken very small. This results in slow rate of convergence of the closed-loop solutions to the origin. To remedy that drawback, we consider the case where we have a fixed low-gain stabilizing controller ensuring stability of the closed-loop system in a desired region of attraction, and a local controller, ensuring satisfying local performance, but which does not achieve a large region of attraction while satisfying the constraints. We then propose a gain-scheduling between those two controllers aiming at having the fastest transfer from the stabilizing controller to the local controller. This result in a hybrid scheme, which is then exposed for general affine constraints $Fx + Gu \leq H$, and is developed, in details, for the particular cases of linear systems with input magnitude and rate constraints.

Finally, in Chapter 4, a Model Predictive Control approach is taken to the performance requirement for linear systems with input magnitude constraint. It is proposed to solve online the time-optimal problem with $x(t)$ as initial condition and the origin as end-point. This control law is bang-bang so that the only information required by the receding horizon controller is the “next” switching instant. We have proposed a continuous-time algorithm that computes the switching times as the unique attractor of a system of differential equations. Euler discretization of these ordinary differential equations yields an algorithm with fast convergence.

Further research can be pursued in all three directions that have been exposed in this thesis:

- The construction of control laws for systems presenting both feedback and feedforward nonlinearities in general is not solved. Other structures than the one introduced in Chapter 2, for which the stabilization can be achieved, should be developed. Also, control laws achieving better performance need to be built.

- Existing stabilizing control laws for linear systems with bounded input are not satisfying, because they do not generically achieve good performance. In this thesis, we have proposed a new control law, but a lot of effort can still be dedicated to the search of efficient and easily computable control laws for linear systems with bounded input.

- The algorithm that is presented in Chapter 4 for the computation of time-optimal control for linear systems has been proven to be semiglobally converging when the eigenvalues of A are real. It would be of interest to extend this convergence result to the case of matrix A with complex eigenvalues. We have shown that a semiglobal result could not be hoped for, but a local stability result and an evaluation of the region of attraction would be of interest.

Appendix

A. Proof of Lemma 1 p. 42

Proof: The equation in e coordinates is written as

$$\left\{\begin{array}{rcl}
\dot{e}_1 & = & e_2 - \mathrm{sat}_{\epsilon_1}(K_1 e_1) + g_1(\dot{x}_2, \dot{x}_3, \cdots, \dot{x}_n) \\
& \vdots & \\
\dot{e}_k & = & e_{k+1} - \mathrm{sat}_{\epsilon_k}(K_k e_k) \\
& & + \frac{df_{k-1}}{dx_{k-1}}(e_k - \mathrm{sat}_{\epsilon_{k-1}}(K_{k-1}e_{k-1}) + g_{k-1}(\dot{x}_k, \cdots, \dot{x}_n)) \\
& & + g_k(\dot{x}_{k+1}, \cdots, \dot{x}_n) + \frac{d\ \mathrm{sat}_{\epsilon_{k-1}}(K_{k-1}e_{k-1})}{de_{k-1}}\dot{e}_{k-1} \\
& \vdots & \\
\dot{e}_n & = & -\mathrm{sat}_{\epsilon_n}(K_n e_n) + \frac{df_{n-1}}{dx_{n-1}}(e_n - \mathrm{sat}_{\epsilon_{n-1}}(K_{n-1}e_{n-1}) + g_{n-1}(\dot{x}_n)) \\
& & + \frac{d\ \mathrm{sat}_{\epsilon_{n-1}}(K_{n-1}e_{n-1})}{de_{n-1}}\dot{e}_{n-1}
\end{array}\right. \tag{4.25}$$

where $\frac{d\ \mathrm{sat}_{\epsilon}(x)}{dx}$ at $x = \pm\epsilon$ is defined as an arbitrary element of the interval $[0, 1]$.

(a) (By induction) Because $\dot{x}_n = -\mathrm{sat}_{\epsilon_n}(K_n e_n)$ (a) holds for $i = n$

Suppose now that it is true for $j = k+1 \cdots n$, we show that it is true for $j = k$. We know that $|\dot{x}_j| < 2\epsilon_j$ for $j > k$ inside Ω_k. We have:

$$\dot{x}_k = e_{k+1} - \mathrm{sat}_{\epsilon_k}(K_k e_k) + g_k(\dot{x}_{k+1}, \cdots, \dot{x}_n)$$

Using the fact that x is in Ω_k and $|\dot{x}_j| < 2\epsilon_j$ for $j > k$, we obtain:

$$|\dot{x}_k| < \frac{\epsilon_{k+1}}{K_{k+1}} + \epsilon_k + \mathcal{O}(\epsilon_{i+1}^2, \cdots, \epsilon_n^2)$$

By (2.16), we conclude that $|\dot{x}_k| < 2\epsilon_k$ up to μ-scaling (that is, by scaling all the ϵ_i's with μ sufficiently small, $\mathcal{O}(\epsilon_m^2)$ becomes negligeable in front of the other terms and $|\dot{x}_k| < 2\epsilon_k$ follows from (2.16))

To prove **(b)** and **(c)** we consider the system in the e coordinates (eq. 4.25).

We show (b) and (c) by induction.

For $k = 1$, we have:

$$\dot{e}_1 = e_2 - \text{ sat}_{\epsilon_1}(K_1 e_1) + g_1(\dot{x}_2, \dot{x}_3, \cdots, \dot{x}_n)$$

which becomes, when $x \in \Omega$,

$$\dot{e}_1 = e_2 - \epsilon_1 \text{ sign}(e_1) + g_1(\dot{x}_2, \dot{x}_3, \cdots, \dot{x}_n)$$

where $g_1(\dot{x}_2, \dot{x}_3, \cdots, \dot{x}_n) = \mathcal{O}(\epsilon_2^2, \cdots, \epsilon_n^2)$ by using (a). From (2.16) and after μ-scaling, we conclude that the derivative of e_1 has the sign of $-\epsilon_1 \text{ sign}(e_1)$ i.e. $e_1\dot{e}_1 < 0$ when $|e_1| = \frac{\epsilon_1}{K_1}$. We also see that $|\dot{e}_1|$ is bounded by $\epsilon_1 + \frac{\epsilon_2}{K_2} + \mathcal{O}(\epsilon_2^2, \cdots, \epsilon_n^2) < 2\epsilon_1$ inside Ω_1 (after μ-scaling). Hence (b) and (c) hold for $i = 1$.

Induction step: Suppose that (b) and (c) hold for $i = k-1$. We will show that (b) and (c) also hold for $i = k$.

Consider the e_k equation:

$$\begin{aligned} \dot{e}_k \;\; = \;\; & e_{k+1} - \text{ sat}_{\epsilon_k}(K_k e_k) \\ & + \tfrac{df_{k-1}}{dx_{k-1}}(e_k - \text{ sat}_{\epsilon_{k-1}}(K_{k-1}e_{k-1}) + g_{k-1}(\dot{x}_k, \cdots, \dot{x}_n)) \\ & + g_k(\dot{x}_{k+1}, \cdots, \dot{x}_n) + \frac{d \text{ sat}_{\epsilon_{k-1}}(K_{k-1}e_{k-1})}{de_{k-1}}\dot{e}_{k-1} \end{aligned} \tag{4.26}$$

When $x \in \Omega_k$, the last term of (4.26)

$$\frac{d \text{ sat}_{\epsilon_{k-1}}(K_{k-1}e_{k-1})}{de_{k-1}}\dot{e}_{k-1}$$

is zero when $x \notin \Omega_{k-1}$ and it is $K_{k-1}\dot{e}_{k-1}$ when $x \in \Omega_{k-1}$, which is bounded by $2K_{k-1}\epsilon_{k-1}$ in this part o Ω_{k-1}. Evaluating (4.26) when $|e_k| = \frac{\epsilon_k}{K_k}$, we obtain that $\dot{e}_k e_k < 0$ if:

$$\begin{aligned} \epsilon_k \;\; > \;\; & \tfrac{\epsilon_{k+1}}{K_{k+1}} + D_{k-1}(\tfrac{\epsilon_k}{K_k} + \epsilon_{k-1} + \mathcal{O}(\epsilon_k^2, \cdots, \epsilon_n^2)) \\ & + \mathcal{O}(\epsilon_{k+1}^2, \cdots, \epsilon_n^2) + 2K_{k-1}\epsilon_{k-1} \end{aligned} \tag{4.27}$$

By (2.16), $\frac{\epsilon_k}{K_k} < \epsilon_{k-1}$ and $\frac{\epsilon_{k+1}}{K_{k+1}} + 2(D_{k-1} + K_{k-1})\epsilon_{k-1} < \epsilon_k$, so that (4.27) is verified, i.e. $\dot{e}_k e_k < 0$

Using the bound (2.16), we analogously see that the bound

$$|\dot{e}_k| < 2\epsilon_k$$

is verified.

The induction step proves (b) and (c) for $k \in \{2, \cdots, n-1\}$. For $k = n$, (2.16) is slightly different. Inequality (4.27) becomes:

$$\epsilon_n > D_{n-1}(\mathcal{O}(\epsilon_n^2) + \frac{\epsilon_n}{K_n} + \epsilon_{n-1}) + 2K_{n-1}\epsilon_{n-1} \tag{4.28}$$

which follows from (2.16) up to μ-scaling. This ends the proof. □

Bibliography

Anderson, B.D.O., Moore, J.B. (1971), *Linear optimal control*, Prentice-Hall International.

Athans, M., Falb, P.L. (1966), *Optimal Control. An introduction to the theory and its applications*, Mc Graw-Hill, New York.

Bellman, R. (1957), *Dynamic Programming*, Princeton University Press.

Bellman, R., Glicksberg, I., Gross, O. (1956), "On the bang-bang control problem," *Quarterly of Applied Mathematics*, **14**, 11-18.

Bemporad A., Morari M., Dua V., Pistikopoulos E.N.(2001), "The Explicit Linear Quadratic Regulator for Constrained Systems", Tech. Report AUT99-16, ETH, submitted to *Automatica.*

Coron J.M., Praly L., Teel A. (1995), "Feedback stabilization of nonlinear systems: sufficient conditions and Lyapunov and Input-Output techniques." In *Trends in Control,* A. Isidori (ed.). Springer Verlag, pp. 293-347.

De Doná, J.A. (2000), *Input Constrained Linear Control*, PhD. thesis, The University of Newcastle, Australia.

Desoer, C.A. (1959), "The bang-bang servo problem treated by variational techniques," *Information and Control*, **2**, pp. 333-348.

Dornheim M.A. (1992), "Report pinpoints factors leading to YF-22 crash," *Aviation Week Space Technol.*, pp. 53-54, no. 9, 1992.

Eaton, J.H.. (1962), "An iterative solution to time-optimal control," *Journal of Mathematical Analysis and Applications*, **5**, pp. 329-344.

Fadden, E.J. (1965), "Computational aspects of a class of optimal control problems," Technical Report ORA Project 06181, University of Michigan, Ann Arbor.

Fadden, E.J., Gilbert E.G. (1964), "Computational aspects of the time optimal control problem," *in* A.V. Balakrishnan & L.W. Neustadt, eds, "Computing Methods in Optimization Problems," Academic Press, New York.

Fancher S. P. (1966), "Iterative computation procedure for an optimum control problem," *IEEE Trans. on Automatic Control*, **10**, pp. 346-348.

Freeman R.A., Kokotović P.V. (1996), *Robust Control of Nonlinear Systems.* Birkhäuser, Boston.

Freeman R., Praly L. (1998), "Integrator backstepping for bounded controls and control rates",*IEEE Trans. on Automatic Control*, **43**, pp. 258-262.

Fuller A.V. (1969), In-the-large stability of relay and saturating control systems with linear controller, *Internat. J. Control.*

Gilbert E.G. (1964), "The application of hybrid computers to the iterative solution of optimal control problems, *in* A.V. Balakrishnan & L.W. Neustadt, eds, "Computing Methods in Optimization Problems," Academic Press, New York, pp. 261-284.

Grognard F., Sepulchre R., Bastin G., Praly L. (1998), " Nested linear low-gain design for semiglobal stabilization of feedforward systems," *IFAC NOLCOS'98 Symposium*, Enschede.

Grognard F., Sepulchre R., Bastin G. (1999), "Global stabilization of feedforward systems with exponentially unstable Jacobian linearization," *Systems and Control Letters*, **37**, pp. 107-115.

Grognard F., Sepulchre R., Bastin G. (2000a), "Improving the performance of low-gain designs for bounded control of linear systems", submitted to *Automatica.*

Grognard F., Sepulchre R., Bastin G. (2000b), "Control of linear systems with affine constraints: a gain-scheduling approach", *39th IEEE Conference on Decision and Control*, Sydney, pp. 4785-4790.

Grognard F., Sepulchre R. (2001a), "Stability of dynamical systems that compute time-optimal switchings," to appear at *40th IEEE Conference on Decision and Control*, Orlando.

Grognard F., Sepulchre R., De Dona J. (2001b), "Dynamical systems that compute time-optimal switchings," *NOLCOS 2001 Symposium*, St Petersburg.

Grognard F., Jadot J., Magni L., Bastin G., Sepulchre R., Wertz V. (2001c), "Robust stabilization of a nonlinear cement mill model," *IEEE Trans. on Automatic Control*, **46**, no. 4, pp. 618-623.

Hanus R., Kinnaert M., Henrotte J.L. (1987), "Conditioning technique, a general anti-windup and bumpless transfer method", *Automatica*, **23**, no. 6, pp. 723-739.

Ho, Y.C. (1962), "A successive approximation technique for optimal control systems subject to input saturation," *Journal of Basic Engineering, Transactions of ASME*, **84D**, 33-40.

Isidori A. (1995), *Nonlinear Control Systems.* 3rd. ed., Springer-Verlag, London.

Jankovic M., Sepulchre R., Kokotović P.V. (1996), "Global stabilization of nonlinear cascade systems," *IEEE Trans. on Automatic Control*, **41**, pp. 1723-1735.

Kapoor N., Teel A., Daoutidis P. (1996), "On anti-integrator-windup and global asymptotic stability", *Proceedings 13th IFAC World Congress*, San Francisco, **D**, pp. 67-72.

Khalil H.K. (1996), *Nonlinear Systems*, 2nd ed., Prentice-Hall, Upper Saddle River, NJ.

Kothare M., Campo P., Morari M., Nett C. (1994), "A unified framework for the study of anti-windup designs", *Automatica*, **30**, no. 12, pp. 1869-1883.

Krstić M., Kanellakopoulos I., Kokotović P.V. (1995), *Nonlinear and Adaptive Control Design.* Wiley, New York.

Lee, E.B., Markus, L. (1967), *Foundations of Optimal Control Theory*, John Wiley & Sons, Inc., New York.

Lenorowitz J.M. (1990), "Gripen control problem resolved through in-flight ground simulations," *Aviation Week Space Technol.*, pp. 74-75, June 18, 1990.

Lin Z. (1997), "Semi-global stabilization of linear systems with position and rate-limited actuators", *Systems & Control Letters*, **30**, no. 1, pp. 1-11.

Lin Z. (1998), "Global Control of Linear Systems with Saturating Actuators", *Automatica*, **34**, no. 7, pp. 897-905.

Lin Z., Saberi A. (1993), “Semiglobal exponential stabilization of linear systems subject to input saturation via linear feedbacks”, *Systems & Control Letters*, **21**, no. 3, pp. 225-239.

Lin Z., Stoorvogel A., Saberi A. (1996), “Output Regulation for Linear Systems Subject to Input Saturation”, *Automatica*, **32**, no. 1, pp. 29-47, 1996.

Marconi L., Isidori A. (2000), “Robust global stabilization of a class of uncertain feedforward nonlinear systems,” *Systems and Control Letters*, **41**, pp. 281-290.

Mayne D.Q., Michalska H. (1990), “Receding horizon control of nonlinear systems”, *IEEE Trans. on Automatic Control*, **35**, pp. 814-824.

Mayne D.Q., Rawlings J.B., Rao C.V., Schockaert P.O.M. (2000), “Constrained model predictive control: stability and optimality,” *Automatica*, **36**, pp. 789-814.

Mazenc F., Praly L.(1996), “Adding an integration and global asymptotic stabilization of feedforward systems,” *IEEE Trans. on Automatic Control*, **41**, pp. 1559-1578.

Megretski A. (1996), “L_2 BIBO output feedback stabilization with saturated control”, *Proceedings 13th IFAC World Congress*, San Francisco, **D**, pp. 435-440.

Neustadt, L.W. (1960), “Synthesizing time-optimal control systems,” *Journal of Mathematical Analysis and Applications*, **1**, pp. 464-493.

Neustadt, L.W. (1961), “Time-optimal control systems with position and integral limits,” *Journal of Mathematical Analysis and Applications*, **3**, pp. 402-427.

, Paiewonski, B. et al. (1964), “A study of synthesis techniques for Optimal Plants,” *Technical documentary report*, ASD-TDR-63-239.

Plant, J.B. (1968), *Some iterative solutions in optimal control*, Research Monograph No 44, The M.I.T. Press, Cambridge, Massachussets.

Pontryagin, L.S., Boltyanskii, V., Gamkrelidze, R., Mischenko, E. (1962), *The Mathematical Theory of Optimal Processes*, Interscience Publishers, Inc., New York.

Saberi A., Han J., Stoorvogel A.A. (2001), "Constrained stabilization problems for linear plants," submitted.

Saberi A., Kokotović P.V., Sussmann H.J. (1990), "Global stabilization of partially linear composite systems," *SIAM J. Control and Optimization,* **28**, pp. 1491-1503.

Saberi A., Lin Z., Teel A. (1996), "Control of Linear Systems with Saturating Actuators", *IEEE Trans. on Automatic Control*, **41**, no. 3, pp. 368-378.

Schmittendorf, W.E., Barmish B.R. (1980), "Null controllability of linear systems with constrained controls," *SIAM J. Control and Opt.*, **18**, pp. 327-345.

Sepulchre R. (2000), "Slow peaking and global stabilization of a perturbed chain of integrators," *IEEE Trans. on Automatic Control*, **45**.

Sepulchre R., Jankovic M., Kokotović P.V.(1996), *Constructive Nonlinear Control*, Springer-Verlag.

Smith, F.B. (1961), "Time-optimal control of higher-order systems," *IRE Trans. on Automatic Control*, **6**, pp. 16-21.

Sontag E.D. (1983), "A Lyapunov-like characterization of asymptotic controllability," *SIAM J. on Control and Optimization*, **21**, pp. 462-471.

Sontag, E.D. (1998), *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, second edition, Springer, New York.

Sontag E.D., Sussmann H.J. (1990), "Nonlinear output feedback design for linear systems with saturating controls", *Proceedings 29th Conference on Decision and Control*, Honolulu, pp. 3414-3416.

Stoorvogel A., Saberi A. (1999), "Output regulation of linear plants with actuators subject to amplitude and rate constraints", *Int. Journal of Rob. and Nonlinear Control*, **9**, no. 10, pp. 631-657.

Sussmann H.J., Kokotović P.V. (1991), "The peaking phenomenon and the global stabilization of nonlinear systems," *IEEE Trans. on Automatic Control*, **36**, pp. 424-439.

Teel A. (1992a), " Global stabilization and restricted tracking for multiple integrators with bounded controls", *Systems & Control Letters*, **18**, pp. 165-171.

Teel A. (1992b), "Using saturation to stabilize a class of single-input partially linear composite systems", *IFAC NOLCOS'92 Symposium*, Bordeaux.

Teel A. (1995), "Semiglobal stabilizability of linear null controllable systems with input nonlinearities", *IEEE Trans. on Automatic Control*, **40**, no. 1, pp. 96-100.

Teel A. (1996), "A nonlinear small gain theorem for the analysis of control systems with saturation," *IEEE Trans. on Automatic Control*, **41**, no. 9, pp. 1256-1270.

Teel A. (1998a), "A nonlinear control viewpoint on anti-windup and related problems," *IFAC NOLCOS'98 Symposium*, Enschede.

Teel A. (1999), "Anti-windup for exponentially unstable linear systems," *International Journal of Robust and Nonlinear Control*, **9**, no. 10, pp. 701-716.

Teel A.R., Praly L. (1995), "Tools for semiglobal stabilization by partial state feedback and output feedback," *SIAM J. Control and Optimization*, **33**, pp. 1443-1488.

Tsinias J. (1997), "Input to state stability properties of nonlinear systems and applications. to bounded feedback stabilization using saturation." *ESSAIM: Control, Optimization and Calculus of Variations,* vol.2, pp 57-85.

Utkin V.I. (1992), *Sliding Modes in Control and Optimization*, Springer-Verlag, New York.

Vincent, T.L., Grantham, W.J. (1997), *Nonlinear and Optimal Control Systems*, John Wiley and Sons, New York.

Willems J.C.(1971), "Least squares stationary optimal control and the algebraic Riccati equation", *IEEE Trans. on Automatic Control*, vol 16,, no. 6, pp. 621-634.

Yastreboff, M.M. (1969a), *Investigation of Several Synthesis Procedures for On-Line Time Optimal Control*, PhD. thesis, The University of New South Wales, Australia.

Yastreboff, M.M. (1969b), "Synthesis of time-optimal control by time interval adjustment," *IEEE Trans. on Automatic Control*, **14**, pp. 707-710 .