Asymptotic stability for time-variant systems and observability: uniform and nonuniform criteria

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Abstract

This paper presents some new criteria for uniform and non uniform asymptotic stability of equilibria for time-variant differential equations and this within a Liapunov approach. The stability criteria are formulated in terms of certain observability conditions with the output derived from the Liapunov function. For some classes of systems, this system theoretic interpretation proves to be fruitful since—after establishing the invariance of observability under output injection—this enables us to check the stability criteria on a simpler system. This procedure is illustrated for some classical examples.

Key Words: control systems, differential equations, time-variance, observability, asymptotic stability, circle criterion

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1 Introduction

Stability of an equilibrium $x = 0$ of a differential equation $\dot{x} = f(x,t)$ is typically established by means of a Lyapunov function, that is, a positive definite function with negative semidefinite time-derivative. For *asymptotic* stability, extra conditions are needed. Negative definiteness of the time-derivative would suffice, but this property is not always possessed by the natural candidate for the Lyapunov function – a typical example being the energy for the damped oscillator.

Pursuing results initiated in [1, 2, 17], this paper presents new criteria for attractivity in such situations. Negative definiteness of $\dot{V}$ is not assumed but particular integral conditions on $\dot{V}$ are imposed to guarantee that $V(x,t)$ decreases "from time to time" and eventually converge to zero. Defining an output $y = h(x,t)$ such that $\|y\|^2 = -\dot{V}$, our integral conditions are of the form

$$\int_{t_0}^{t_0+T} \|h(x(t,x_0,t_0))\|^2 dt \geq \epsilon) \tag{1}$$

(with subtle variants in the choice of t_0, T, and ϵ) which is a (zero)-observability condition for the system $\dot{x} = f(x,t)$, $y = h(x,t)$.

The link between attractivity and observability is not new in the literature. In [10, 5] a link was established between exponential stability of *linear* time-varying differential equations and some form of observability. Also Miller and Michel [12] proved a relationship between observability and asymptotic stability for linear time-invariant differential equations and for time-invariant hamiltonian systems with added damping. Narendra and Annaswamy [15] established that uniform observability of the pair (f,h) implies uniform attractivity and therefore also uniform asymptotic stability. The main novelty of the criteria presented in this paper is that they provide sufficient conditions not only for uniform attractivity but also for nonuniform attractivity, which is known to be a considerably more difficult question.

The weakness of integral conditions like (1) is that, in principle, they require the knowledge of the solutions, at least on some time interval. For autonomous systems, and particular classes of nonautonomous systems, LaSalle Invariance Principle drastically reduces this difficulty because the invariance of the set where the output identically vanishes is usually considerably simpler to check. When LaSalle Invariance Principle is not applicable, an alternative is to develop techniques which allow to check the integral conditions not along the solutions of the original system but along the solutions of a "simpler" system. In the context of uniform asymptotic stability, Artstein [7] reduced the problem of checking uniform noticeability – a notion slightly stronger than uniform observability – to checking mere observability for the family of all *time-invariant* limiting equations of the original equation. The approach proposed in the present paper, which is called output injection and was originally used in [5], is totally different. Output injection defines a class of output feedback transformations which leave the observability properties unchanged, but, in some classical problems, transforms the original system into a system whose flow can be easily calculated. In contrast to the approach of limiting equation, the output injection approach is applicable to nonuniform results as well.

The paper is organised as follows. After formulating the problem in Section 2, we discuss several observability notions in Section 3, which differ by the uniformity requirements of the conditions. The stability criteria are presented in Section 4 and reformulated in Section 5 with the help of the output injection tool. Some classical applications are considered in Section 6.

2 General assumptions and problem statement

General assumptions. Let U be an open subset of $\mathbb{R}^n$ containing the origin. Consider the differential equation

$$\dot{x} = f(x,t) \tag{S}$$

where $f : U \times \mathbb{R} \to \mathbb{R}^n$ is continuous in x, measurable in t, and $f(0,t) = 0$ for all t. For each initial state $x_0 \in U$ and for each initial time $t_0 \geq 0$, we assume existence and unicity of the solutions $x(t; x_0, t_0)$ of (S) over $[t_0, +\infty)$.

Uniform stability assumption. Assume there exists a Liapunov function $V \in C^1(U \times \mathbb{R}; \mathbb{R})$ with the following standard properties:

(i) V is **positive definite**: there exists a function γ_1 of class $\mathcal{K}$ such that

$$\forall (x,t) \in U \times \mathbb{R} : V(x,t) \geq \gamma_1(\| x \|) \text{ and } V(0,t) = 0 \quad \forall t \in \mathbb{R}$$

(ii) V is **decrescent**: there exists a function γ_2 of class $\mathcal{K}$ such that

$$\forall (x,t) \in U \times \mathbb{R} : V(x,t) \leq \gamma_2(\| x \|)$$

(iii) $\dot{V}$ is **negative semi-definite**:

$$\forall (x,t) \in U \times \mathbb{R} : \dot{V}(x,t) := (\frac{\partial V}{\partial t} + \nabla V.f)(x,t) \leq 0$$

Uniform stability of the null solution follows from (i), (ii), and (iii) (see e.g. [18]). We will assume that V satisfies the following additional assumption:

(iv) V is continuous in x, *uniformly* with respect to time:

$$\forall x \in U,\ \forall \epsilon > 0,\ \exists \text{ an open ball } B(x,\delta) \subset U \text{ such that}$$

$$\forall x' \in B(x,\delta), \forall t \in \mathbb{R} \mid V(x,t) - V(x',t) \mid < \epsilon$$

Notice that (ii) implies that the continuity of V in $x = 0$ is uniform (with respect to t) but that (iv) requires uniform continuity for each $x \in U$.

Problem statement. Under the above assumptions (i)-(iv) to find sufficient conditions for *asymptotic stability* of the null solution of (S).

3 Observability

We introduce in this section several notions of observability, different from standard notions of observability in that they each involve a particular type of uniformity (with respect to time). The uniformity requirements are weaker than those considered in the classical definition of uniform observability ([5]). Simple examples are included in order to illustrate several aspects of the definitions.

Definition 1 An *output* for the system (S) is a mapping $h : U \times I\!R \rightarrow I\!R^m$, continuous in x and measurable in t, such that $h(0,t) = 0$ for all t. By the system (f,h) or the pair (f,h) is meant a differential equation (S) with output h i.e.

$$\begin{aligned} \dot{x} &= f(x,t) \\ y &= h(x,t) \end{aligned}$$

with $h(0,t) = 0 \; \forall t$

3.1 Observability

Let t_0 be an initial time for (S). Observability of the pair (f,h) is related to the problem of reconstructing $x(t;x_0,t_0)$ from the output $y(t) = h(x(t;x_0,t_0),t)$ defined on $[t_0, t_0+T]$. Here we require this property only for the output $y(t) \equiv 0$, i.e. the null solution $x(t) \equiv 0$ of (S) is the unique solution leading to the output $y(t) \equiv 0$:

Definition 2 The pair (f,h) is *observable* if $\forall\, t_0 \geq 0$, $\forall x_0 \neq 0$ in the neighborhood of the origin, $\exists\, T(t_0,x_0) > 0$, $\epsilon(t_0,x_0) > 0$ such that

$$W(t_0, t_0+T)(x_0) := \int_{t_0}^{t_0+T} \| \, h(x(t;x_0,t_0),t) \, \|^2 \, dt \geq \epsilon \tag{2}$$

where $\| \, . \, \|$ is the Euclidean norm. T may be interpreted as the observation period and ϵ as an energy level of the output. The system is observable if one can guarantee a nonzero level of output energy ϵ by observing the system over a finite period of time T.

Remark 1 *(linear case)* Let $f(x,t) = A(t)x$ and $h(x,t) = C(t)x$ Let $\Phi(t,t_0)$ the transition matrix associated with $\dot{x} = A(t)x$. Then

$$W(t_0, t_0+T)(x_0) = x_0^T G(t_0, t_0+T) x_0$$

where $G(t_0, t_0+T)$ is the "observability Gramian" defined by

$$G(t_0, t_0+T) := \int_{t_0}^{t_0+T} \Phi^T(t,t_0) C^T(t) C(t) \Phi(t,t_0) \, dt \tag{3}$$

The pair (A,C) is observable if $\forall t_0 \geq 0, \exists T(t_0) > 0$ such that $G(t_0, t_0+T)$ is a positive definite matrix.

3.2 Uniformity requirements

Three additional types of observability will be introduced, corresponding to three different uniformity requirements (with respect to t_0) for the constants T and ϵ introduced above.

Definition 3 The pair (f,h) is called *uniformly observable* if the constants T and ϵ in (2) can be chosen independent of t_0 [1], i.e. $\forall\, x_0 \neq 0$ in the neighborhood of the origin ,$\exists\, T(x_0) > 0, \exists\, \epsilon(x_0) > 0$ such that $W(t_0, t_0+T)(x_0) \geq \epsilon \, \forall\, t_0$

[1] For linear systems, a particular type of uniform observability was introduced by Kalman [10] in the study of the optimal regulator problem. The uniformity requirements of Definition 3 are part of the uniformity conditions of the Kalman definition. The Kalman conditon implies $\| \, \Phi(t,t_0) \, \| \leq \gamma(|t-t_0|) \forall (t,t_0)$. This property is too strong for our purposes but will play a role later.

The following definition considers a similar uniformity condition with uniformity imposed only for a particular sequence of initial times tending to infinity.

Definition 4 The pair (f, h) is called *recurrently uniformly observable* if $\exists\, (t_i)_{i\geq 1} \to \infty$ such that for each nonzero x_0 in a neighborhood of the origin, there exist two constants $T = T(x_0) > 0$ and $\epsilon = \epsilon(x_0) > 0$, and an integer $I(x_0) \geq 1$ such that

$$\forall i \geq I : W(t_i, t_i + T)(x_0) \geq \epsilon \tag{4}$$

The next definition only imposes a uniformity condition for the minimal level of output energy. The observation interval T may increase with t_0.

Definition 5 The pair (f, h) is called *weakly uniformly observable* if ϵ in (2) is independent of t_0 : for each x_0 in a neighborhood of the origin, for each t_0, there exist two constants $T = T(x_0, t_0) > 0$ and $\epsilon = \epsilon(x_0) > 0$ such that

$$W(t_0, t_0 + T)(x_0) \geq \epsilon \tag{5}$$

For convenience, by 'recurrent observability' and 'weak observability' we mean 'recurrent uniform observability' and 'weak uniform observability' respectively.

Proposition 1 Uniform observability implies recurrent observability which in turn implies weak observability.

Proof:

(i) Definition 4 is obviously weaker than Definition 3 since the uniformity is not required for *all* t_0 but only for a particular sequence $(t_i)_{i\geq 1} \to \infty$

(ii) Definition 5 is weaker than Definition 4: let x_0 be in a neighborhood of the origin and consider t_0. Recurrent observability implies the existence of an (increasing) sequence $(t_i)_{i\geq 1}$ such that for some constants $T > 0$ and $\epsilon > 0$, and for some integer $I \geq 1$, $W(t_i, t_i + T)(x_0) \geq \epsilon$ for $i \geq I$. Let $k \geq I$ be sufficiently large such that $t_k > t_0$. Then $W(t_0, t_k + T)(x_0) \geq W(t_k, t_k + T)(x_0) \geq \epsilon$ which proves weak observability since $t_k + T = t_0 + T'$ for some particular T'. $\square$

The following example illustrates the different definitions of observability:

Example 1 Let x be scalar and $f(x, t) = -\frac{x}{t}$, $t_0 \geq 1$.

(i) let $h(x, t) = x$; then $G(t_0, t_0 + T) = \frac{t_0 T}{t_0 + T}$. Observability is uniform: choosing $T = 1$, the observabililty Gramian $G(t_0, t_0 + T) > \frac{1}{2}$ for each $t_0 \geq 1$.

(ii) let $h(x, t) = h(t)x$ with $h(t) = \max(0, \sin(\ln\ t))$; then $h(x, t)$ vanishes on increasing intervals of time as time increases. As a consequence, for each $T > 0$, there exists a t_0 sufficiently large such that $G(t_0, t_0 + T) = 0$. Hence observability is not uniform. Next define an increasing sequence $(t_i)_{i\geq 1}$ such that $\sin(\ln\ t_i) = 1$ for each $i \geq 1$. Then $G(t_i, t_i + T) > 0$ for any choice of $T > 0$. This establishes recurrent observability.

(iii) let $h(x, t) = \frac{x}{\sqrt{t}}$; then $G(t_0, t_0 + T) = \frac{1}{2}(1 - \frac{t_0^2}{(t_0 + T)^2})$. For every constant choice of T, $G(t_0, t_0 + T)$ tends to 0 as t_0 tends to infinity. Hence observability is not uniform nor recurrent. On the contrary, choosing for instance $T = t_0$, $W(t_0, t_0 + T) = 3/8$ for all t_0. This establishes weak observability.

(iv) let $h(x,t) = \frac{x}{t}$; then $G(t_0, t_0+T) = \frac{1}{3}(\frac{1}{t_0} - \frac{t_0^2}{(t_0+T)^3})$. This quantity tends to 0 as t_0 tends to infinity whatever be the choice of $T(t_0) > 0$. The system is just observable, without any type of uniformity.

Remark 2 'Recurrent observability' is obviously equivalent to the following: there exists a sequence $(t_i)_{i\geq 1} \to \infty$ such that for each x_0 in a neighborhood of the origin, there exists a constant $T = T(x_0) > 0$ such that

$$\liminf_{t_i\to\infty} W(t_i, t_i+T)(x_0) > 0 \tag{6}$$

We stress that the choice of the sequence $(t_i)_{i\geq 1}$ in this definition is *independent of* x_0. If dependence on x_0 is allowed, the definition becomes equivalent to the following: for each x_0 in a neighborhood of the origin, there exists a constant $T = T(x_0) > 0$ and a sequence t_i such that

$$\limsup_{t_i\to\infty} W(t_i, t_i+T)(x_0) > 0 \tag{7}$$

In general, (7) does not imply any kind of observability as is illustrated by the following example.

Example 2 Consider the following perturbed oscillator:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - a(t,x_2)x_2\\ y &= a(t,x_2)x_2\end{aligned}$$

with

$$\begin{aligned}a(t,x_2) &= \cos t, \quad x_2\cos t \geq 0\\ &= 0, \qquad\ \ x_2\cos t \leq 0\end{aligned}$$

Let t_0 an initial time such that $\cos t_0 > 0$. Then the initial condition $x_0 = (0, -\cos t_0)$ leads to the solution $x(t;x_0,t_0) = (-\cos t_0 \sin t, -\cos t_0 \cos t)^T$ and to the output $y(t) \equiv 0$. As a consequence, the system is not observable. However it can be shown that condition (7) is satisfied with $T > 0$ arbitrary small (for instance by using the results of Section 3.3). * □

4 Attractivity criteria

4.1 Observability-like conditions

We start by recalling a classical integral condition, proposed by Narendra and Annaswamy [15], which guarantees that the Liapunov function V decreases along non-trivial solutions, not at each time instant, but in the average:

Theorem 1 Suppose that (S) admits a Liapunov function with properties (i)-(iii). Then *uniform* asymptotic stability of the null solution of (S) follows from the following condition: given x_0 nonzero and sufficiently close to the origin, there exists a finite $T(x_0) > 0$ and a function ξ of class $\mathcal{K}$ such that for t_0 sufficiently large

$$\forall t \geq t_0 : \int_t^{t+T} -\dot{V}(x(\tau;x_0,t_0),\tau)d\tau \geq \xi(\| x(t;x_0,t_0) \|). \tag{8}$$

□

Condition (8) can be interpreted as an observability condition on the time derivative of V. More precisely, with $h : U \times I\!R \to I\!R^m$ any output function satisfying

$$\| h(x,t) \|^2 = -\dot{V}(x,t)\ \forall (x,t) \in U \times I\!R \tag{OUT}$$

then condition (8) implies in particular that the pair (f,h) is eventually uniformly observable. Indeed both T and ϵ featuring in the definition of uniform observability may be taken independent of t_0; this is obvious for T and for ϵ one takes $\xi(\| x_0 \|)$.

Theorem 1, which is established by Narendra and Annaswamy [15], is related to the result proved by Artstein [7] where uniform asymptotic stability is obtained when the system is uniformly noticeable using a nonincreasing Liapunov function. This uniform noticeability used by Artstein can be seen as a stronger version of the uniform observability which we use. The integral length of condition (8) i.e. $T(x_0)$ may depend on x_0 whereas the uniform noticeability condition of Artstein requires an interval length T independent of x_0. The next sections of the present paper provide alternative convergence criteria, which are not formulated by Narandra and Annasamy [15] neither by Artstein [7]. Our conditions are formulated as observability conditions for the pair (f,h). They are weaker than (8); in turn they only guarantee attractivity rather than uniform attractivity. This will be discussed in more detail later on.

4.2 Recurrent observability and attractivity

We introduce an additional assumption on the solutions of (S). We assume that (for t_0 sufficiently large) the solutions satisfy the following continuity assumption with respect to the initial condition x_0:

Continuity Assumption on Solutions: Let x_0 be sufficiently close to the origin. Then there exists a $\delta > 0$ and a function $\gamma_{x_0}(t)$ independent of t_0 such that

$$\forall t \geq 0 : \| x(t_0+t; y_0, t_0) - x(t_0+t; x_0, t_0) \| \leq \gamma_{x_0}(t) \| y_0 - x_0 \| \tag{9}$$

$\forall y_0 \in B(x_0, \delta)$ and $\forall t_0$ is sufficiently large.

The following lemma is a sufficient condition for (9) to be true:

Lemma 1 Suppose that for each $x_0 \in U$ and for large t, f is locally Lipschitz in x with bounded local Lipschitz function $l_{x_0}(t)$. Suppose that the null solution of (S) is uniformly stable. Then the solutions of (S) satisfy the Continuity Assumption.

Proof: Let $\epsilon > 0$ such that $\overline{B}_\epsilon(0) \subset U$. Uniform stability of the origin implies that for some $\delta > 0$, all solutions of (S) with initial condition $x_0 \in B_\delta(0)$ remain in the compact set $\overline{B}_\epsilon(0)$. Since the stability is uniform δ is independent of t_0. Define the constant K as the maximum of the bounded Lipschitz functions $l_x(t)$ associated with a finite covering of $\overline{B}_\epsilon(0)$. By continuity of solutions of differential equations with respect to initial conditions, we have for t_0 large enough:

$$\forall t \geq 0 : \| x(t_0+t; y_0, t_0) - x(t_0+t; x_0, t_0) \| \leq \| y_0 - x_0 \| e^{K.t}$$

provided that x_0 and y_0 belong to $B_\delta(0)$. □

Proposition 2 Let $f(x,t) = A(t)x$ and suppose uniform stability of the origin. Then the Continuity Assumption is satisfied.

Proof: Let $\Phi(t, t_0)$ is transition matrix asociated to $\dot{x} = A(t)x$. Then the Continuity Assumption can be reformulated as follows: there exists a function γ such that for t_0 sufficiently large and for $t \geq t_0$

$$\| \phi(t + t_0, t_0) \| \leq \gamma(t) \tag{10}$$

This is implied by uniform stability since uniform stability of the origin is equivalent to the existence of a constant γ such that (10) holds. □

We are now in a position to state one of the main theorems of the paper.

Theorem 2 Suppose that (S) satisfies the Continuity Assumption and admits a Liapunov function with properties (i)-(iv). Let $h(x, t)$ be an output function satisfying (OUT). Then recurrent observability of the pair (f, h) is a sufficient condition for asymptotic stability of the null solution of (S).

Proof Assume that the pair (f, h) is recurrently observable. Then there exists a sequence $(t_j)_{j\geq 1} \to \infty$ as $j \to \infty$ such that for each x in a neighborhood $\mathcal{N}$ of the origin, there exists a finite time $T = T(x) > 0$ such that

$$\liminf_{t_j \to \infty} \int_{t_j}^{t_j+T} -\dot{V}(x(t; x, t_j), t)dt > 0 \tag{11}$$

Choose $\epsilon > 0$ small enough such that $\overline{B}_\epsilon(0) \subset \mathcal{N}$. Uniform stability of the null solution implies the existence of a positive δ such that if $x_0 \in B_\delta(0)$, t_0 arbitrary, the state $x(t; x_0, t_0)$ belongs to $B_\epsilon(0)$ for $t \geq t_0$. Consider this trajectory at times $t_j > t_0$ defined by the above sequence, i.e. $x(t_j; x_0, t_0)$. By compactness of $\overline{B}_\epsilon(0)$ there exists an increasing subsequence of times $t_i > t_0$, with $t_i \to \infty$ as $i \to \infty$ such that (denoting $x(t_i; x_0, t_0)$ by x_i)

$$\lim_{i\to\infty} x_i = p$$

with $p \in \overline{B}_\epsilon(0)$. We will show that p is necessarily the origin. Since the argument can be repeated for each x_0 in a neighborhood of the origin, this establishes weak attractivity of the origin. But weak attractivity implies attractivity under the assumption of uniform stability (see for instance ex. 6.8. p. 28 in [16]). The proof is therefore complete if we can show $p = 0$. The proof goes by contradiction. Assume that $p \neq 0$.

I. Since $V(x(t; x_0, t_0), t)$ is nonincreasing by Property (iii), and positive, its limit exists:

$$\bar{V} := \lim_{t\to\infty} V(x(t; x_0, t_0), t) \tag{12}$$

Since $\lim_{i\to\infty} x_i = p$, assumption (iv) on V implies

$$\lim_{i\to\infty} \mid V(x_i, t_i) - V(p, t_i) \mid = 0 \tag{13}$$

and by (12)

$$\lim_{i\to\infty} V(p, t_i) = \lim_{i\to\infty} V(x_i, t_i) = \bar{V} \tag{14}$$

II. Consider now the sequence $x(t_i + T; p, t_i), t_i \to \infty$ (with $T = T(p)$ chosen as stipulated in the definition of recurrent observability). By the Continuity Assumption, there exists $K^* > 0$ and $\nu > 0$, such that for t_i large enough and for $q \in B_\nu(p)$:

$$\| x(t_i + T; q, t_i) - x(t_i + T; p, t_i) \| \leq \| q - p \| K^*$$

Then

$$\lim_{i\to\infty} \parallel x(t_i+T;x_i,t_i) - x(t_i+T;p,t_i) \parallel = 0 \tag{15}$$

since $\lim_{i\to\infty} x_i = p$. Then by assumption (iv) on V, (15) implies that

$$\lim_{i\to\infty} |V(x(t_i+T;x_i,t_i),t_i+T) - V(x(t_i+T;p,t_i),t_i+T)| = 0 \tag{16}$$

Consider the sequence of elements $V(x(t_i+T;p,t_i),t_i+T)$ with $t_i \to \infty$ .These elements belong to the interval $[0, \gamma_2(\parallel p \parallel)]$ by Property (i) and Property (ii). Then there exists a subsequence $t_{i_{sub}} \to \infty$ of the sequence (t_i) corresponding to a subsequence $i_{sub} \to \infty$ such that

$$\lim_{i_{sub}\to\infty} V(x(t_{i_{sub}}+T;p,t_{i_{sub}}),t_{i_{sub}}+T) = a \tag{17}$$

for some $a \geq 0$ Noting that for each $t_{i_{sub}}$ we have

$$\begin{aligned} &V(x(t_{i_{sub}}+T;p,t_{i_{sub}}),t_{i_{sub}}+T) = \\ &= V(p,t_{i_{sub}}) + \int_{t_{i_{sub}}}^{t_{i_{sub}}+T} \dot{V}(x(t;p,t_{i_{sub}}),t)dt \end{aligned} \tag{18}$$

we conclude from taking the limit in (18), taking into account (17), (11) and (14) that

$$a < \bar{V} \tag{19}$$

$x_{i_{sub}} := x(t_{i_{sub}};x_0,t_0)$

$$\begin{aligned} &|V(x(t_{i_{sub}}+T;x_{i_{sub}},t_{i_{sub}}),t_{i_{sub}}+T) - a| \leq \\ &|V(x(t_{i_{sub}}+T;x_{i_{sub}},t_{i_{sub}}),t_{i_{sub}}+T) - V(x(t_{i_{sub}}+T;p,t_{i_{sub}}),t_{i_{sub}}+T)| + \\ &+|V(x(t_{i_{sub}}+T;p,t_{i_{sub}}),t_{i_{sub}}+T) - a| \end{aligned} \tag{20}$$

and by (16) and (17) we obtain from (20)

$$\lim_{i_{sub}\to\infty} V(x(t_{i_{sub}}+T;x_{i_{sub}},t_{i_{sub}}),t_{i_{sub}}+T) = a < \bar{V} \tag{21}$$

III. By definition of $\bar{V}$,

$$V(x(t;x_0,t_0),t) \geq \bar{V},\ \forall t \tag{22}$$

But (21) states that $V(x(t_{i_{sub}}+T;x_{i_{sub}},t_{i_{sub}})) < \bar{V}$ for i_{sub} large enough. This is in contradiction with (22) since all states

$$x(t_{i_{sub}}+T;x_{i_{sub}},t_{i_{sub}})$$

belong to the trajectory $x(t;x_0,t_0)$. Therefore our original assumption that $p \neq 0$ is impossible. □

Although both Theorem 1 and Theorem 2 require a decay of the Liapunov function over a finite period of time T, there are important differences. Recall that Theorem 1 implies *uniform* asymptotic stability, while Theorem 2 only guarantees asymptotic stability.

We illustrate these differences further by discussing a few examples. A first obvious difference between the two theorems is that the decay of V over some time period featuring in Theorem 2 is imposed for a sequence of times rather than for all t:

Example 3 Let $f(t,x) = -k^2(t)x$ with $x \in I\!R^n$ and $k^2(t)$ a scalar function. The Liapunov function $V(x) = x^T x/2$ has a semi-definite time derivative

$$\dot{V}(t,x) = -k^2(t)x^T x$$

which implies uniform stability of the origin. Attractivity of the origin depends on the "decaying" effect of $k^2(t)$. Let $k^2(t) = \max(0, \sin(\ln t))$. Then (8) cannot be satisfied for reasons similar to those explained in Example 2.1. Defining the output $h(x,t) = k(t)\sqrt{x^T x}$, one verifies (as in Example 1) that the pair (f,h) is recurrently observable as in Example 2. As a consequence, asymptotic stability follows from Theorem 2.

A second and more fundamental difference is that, for a particular initial condition x_0, our observability condition deals with the trajectory $x(t; x_0, t_0)$ over the finite time interval $[t_0, t_0 + T]$ for *different* initial times t_0. The Narendra condition deals with the trajectory $x(t; x_0, t_0)$ over the infinite interval $[t_0, +\infty)$. In particular, our condition does not guarantee that the decay of V remains uniform or even recurrent along the *whole* trajectory. This is illustrated in the following example:

Example 4 Let x be scalar, $t > 0$ and

$$\begin{aligned} f(x,t) &= -x, \quad tx^2 \geq 1 \\ &= 0, \quad\;\; tx^2 < 1 \end{aligned} \tag{23}$$

Contrary to the general assumptions, this time f is not continuous in x. The notions of solutions of (23) and of asymptotic stability of the null solution are considered in the sense of Filippov (see e.g. [9]). Theorem 2 still applies provided that the solutions of (23) satisfy the Continuity Assumption.

The solutions of (23) can easily be described in the plane (x,t) (i.e. the phase plane of the system $\dot{x} = f(x,t)$, $\dot{t} = 1$). The curve $C \equiv \{(x,t) \mid tx^2 = 1\}$ is invariant in the following sense: if $(x_0, t_0) \in C$ then the only Filippov solution is given by $x(t; x_0, t_0) = \sqrt{t_0 x_0^2 / t}$ and hence $(x(t; x_0, t_0), t) \in C$ for all $t \geq t_0$. On the other hand, this invariant curve is reached in finite time with the solution $x(t; x_0, t_0) = x_0$ if $t_0 x_0^2 < 1$ or $x(t; x_0, t_0) = x_0 e^{-(t-t_0)}$ if $t_0 x_0^2 > 1$. As a consequence, every solution in the sense of Filippov eventually reaches the curve C and then converges to $x = 0$.

Clearly attractivity is not uniform (w.r.to time) and (8) does not apply. The Liapunov function $V(x) = x^2/2$ has a negative semi-definite time derivative $\dot{V}(t,x) = xf(t,x)$. This implies stability. Defining the output $h(x,t) = -f(x,t)$, one easily calculates

$$\liminf_{t_0 \to \infty} W(t_0, t_0 + T)(x_0) = \frac{x_0^2}{2}(1 - e^{-2T}) \tag{24}$$

Observability of the pair (f,h) is hence recurrent (in fact for every sequence $(t_i)_{i \geq 1}$ converging to infinity). Asymptotic stability of the origin follows from Theorem 2. Notice that along any particular trajectory of the system, V eventually decreases as $1/t$, i.e. does **not** decay uniformly over a fixed period of time T.

4.3 Weak observability and attractivity

Theorem 2 requires a decay of the Liapunov function over a *finite* period T. In this section we provide a weaker criterion (i.e. the period T may increase with the initial time) at the expense of a stronger Continuity Assumption:

Strong Continuity Assumption on Solutions: the solutions of (S) satisfy the Strong Continuity Assumption if Continuity Assumption (9) is satisfied for each x_0 in a neighborhood of the origin, with a function γ_{x_0} *bounded* over $[0, +\infty)$

Theorem 3 Suppose that (S) satisfies the Strong Continuity Assumption and admits a Liapunov function with properties (i)-(iv). Let $h(x,t)$ be an output function satisfying (OUT). Then *weak observability* of the pair (f,h) is a sufficient condition for asymptotic stability of the null solution of (S).

Proof: We omit the proof since it follows the same lines as in Theorem 2. By using the Strong Continuity Assumption, the relation (15) can be replaced by

$$\lim_{i\to\infty} (\limsup_{T\to\infty} (\| x(t_i+T; x_i, t_i) - x(t_i+T; p, t_i) \|)) = 0 \tag{25}$$

and every subsequent relation involving the period T is adapted accordingly. $\square$

We end this section with a version of Theorem 3 in the linear case. For $f(x,t) = A(t)x$ we have noticed in the proof of Proposition 2 that uniform stability of the origin is equivalent to the Strong Continuity Assumption.

Next suppose that a *quadratic* Liapunov function exists with a negative semidefinite derivative. More precisely, assume that there exists a Liapunov function $V(x,t) = x^T P(t) x$, with P(t) symmetric and continuously differentiable, with the following properties:

(i') there exists constant α and β such that $\alpha I \leq P(t) \leq \beta I$;

(ii') $\dot{V} = -x^T C(t) C^T(t) x$ for some matrix $C(t) : I\!R \to I\!R^m \times I\!R^n$.

Notice that in this case condition (iv) is satisfied. Indeed, if x and y belong to $B_\epsilon(0)$, we have

$$| x^T P(t) x - y^T P(t) y | = | (x-y)^T P(t)(x+y) | \leq 4\epsilon\beta$$

which establishes condition (iv).

One immediately obtains the following theorem

Theorem 4 Consider the linear system $\dot{x} = A(t)x$. Suppose that $V(x,t) = x^T P(t) x$ is a Liapunov function satisfying conditions (i')-(ii'). Then $\dot{x} = A(t)x$ is asymptotically stable if the pair (A, C) is weakly observable, i.e.

$$\liminf_{t_0\to\infty} \int_{t_0}^{\infty} \Phi^T(t,t_0) C(t) C^T(t) \Phi(t,t_0)\, dt > \delta I \tag{26}$$

with δ an arbitrary positive constant[2] .

[2]When a linear pair (A, C) is considered, it is easy to prove the equivalence between Definition 5 for weak observability and (26) by taking $\epsilon(x_0) = \epsilon x_0^T x_0$.

5 Output injection

5.1 Invariance of observability under output injection

In the previous section we have reduced a particular problem of asymptotic stability to an observability question. Checking the observability of a pair (f, h) may in general be quite hard; it may require an explicit knowledge of the trajectories of the system. However, the property that observability is invariant under a large class of transformations of the system may be used to arrive at a solvable problem in particular situations.

Consider a feedback gain $k(x,t) : U \times I\!R \rightarrow I\!R^{n\times m}$ and define $k(t) := \sup_{x\in U} \parallel k(x,t) \parallel$. Assume that $k(t)$ has a finite $L^2(t_0, t_0 + T)$-norm $\parallel k \parallel$ uniform with respect to each finite T and each initial time t_0 under consideration (this will depend on the notion of observability).

The transformations we will consider in this section are called *output injections* and are a generalization of the notion of output injection defined for linear systems.(see e.g. [5, 12])

Definition 6 The pair (f, h) can be transformed into the pair (g, h) by *output injection* if there exists a mapping $k(x,t)$ introduced above such that

$$\forall (x,t) \in U \times I\!R : \; g(x,t) = f(x,t) + k(x,t)h(x,t) \tag{27}$$

The mapping k is called the feedback gain of the output injection.

The goal of this section is to characterize both the class of systems and the class of output injections leaving the different observability properties introduced in Section 3.1 invariant.

Theorem 5 Observability is invariant under output injection.

Proof. Suppose that (f, h) is observable at time t_0. Then there exists a period T such that $h(x(t; x_0, t_0), t) \stackrel{\text{a.e.}}{\equiv} 0$ over $[t_0, t_0 + T]$ if and only if $x_0 = 0$. Next consider any other system (g, h) related to (f, h) by (27). Assume that for some $\bar{x}_0 \neq 0$, the output $h(x^*(t; \bar{x}_0, t_0), t) \stackrel{\text{a.e.}}{\equiv} 0$ over $[t_0, t_0 + T]$. Here $x^*(t; \bar{x}_0, t_0)$ is a solution of $g(x,t)$ The relation (27) implies that $f(x,t) = g(x,t)$ when $h(x,t) = 0$; this means that $\dot{x} = f(t,x)$ and $\dot{x} = g(t,x)$ have the same solution over $[t_0, t_0 + T]$ for the initial condition $\bar{x}_0$. Since (f, h) is observable, this implies that $\bar{x}_0 = 0$. This establishes a contradiction. □

The next theorem covers the situations in which the period of observation is constant, i.e. independent of t_0.

Theorem 6 Consider the pair (f, h) with

- f Lipschitz in $U \times I\!R$
- h continuous in x, uniformly with respect to t

Then uniform observability and recurrent observability are invariant under output injection.

Proof. We offer a proof for recurrent observability. A similar argument can be used for uniform observability. Let $g = f + kh$ with a feedback gain $\| k \|$. Let $x^*(t; x_0, t_0)$ denote the solution of $\dot{x} = g(x, t)$ and $x(t; x_0, t_0)$ denote the solution of $\dot{x} = f(x, t)$.

If (f, h) is recurrently observable, then there exists a sequence $(t_i)_{i \geq 1}$ such that for each $x_0 \neq 0$ there is a finite $T > 0$ such that

$$\liminf_{t_i \to \infty} \int_0^T \| h(x(t_i + t; x_0, t_i), t_i + t) \|^2 \, dt > 0 \tag{28}$$

Next assume that the pair (g, h) is not recurrently observable; this implies that for some $\bar{x}_0 \neq 0$ the following holds:

$$\liminf_{t_i \to \infty} \int_0^T \| h(x^*(t_i + t; \bar{x}_0, t_i), t_i + t) \|^2 \, dt = 0 \tag{29}$$

By definition of g, we have for $t \in [0, T]$

$$\| \int_0^t \dot{x}^*(t_i + t; \bar{x}_0, t_i) - \int_0^t f(x^*(t_i + t; \bar{x}_0, t_i), t_i + t) \| =$$
$$= \| \int_0^t k(x^*(t_i + t; \bar{x}_0, t_i), t_i + t) h(x^*(t_i + t; \bar{x}_0, t_i), t_i + t) dt \| \leq$$
$$\leq \| k \| \left(\int_0^t (\| h(x^*(t_i + t; \bar{x}_0, t_i), t_i + t) \|)^2 dt \right)^{1/2}$$

which implies by (29), for some subsequence (t_i), also denoted by (t_i)

$$\| \dot{x}^*(t_i + t; \bar{x}_0, t_i) - f(x^*(t_i + t; \bar{x}_0, t_i), t_i + t) \| \stackrel{\text{a.e.}}{\to} 0 \text{ as } t_i \to \infty$$

for all $t \in [0, T]$. Using a standard theorem on approximations of solutions (see [8], p.285) we have for all $t \in [0, T]$:

$$\lim_{t_i \to \infty} \| x(t_i + t; \bar{x}_0, t_i) - x^*(t_i + t; \bar{x}_0, t_i) \| = 0$$

First notice that since h is continuous it is also uniformly continuous on compact spaces. This and the fact that by assumption h is also continuous in x uniformly continuous w.r.t. to the time imply that for all $t \in [0, T]$

$$\lim_{t_i \to \infty} \| h(x(t_i + t; \bar{x}_0, t_i), t_i + t) - h(x^*(t_i + t; \bar{x}_0, t_i), t_i + t) \| = 0 \tag{30}$$

But (29) implies that

$$\| h(x^*(t_i + t; x_0, t_i), t_i + t) \| \stackrel{\text{a.e.}}{\to} 0 \text{ for } t_i \to \infty \tag{31}$$

This and (30) allow to conclude that for all $t \in [0, T]$

$$\lim_{t_i \to \infty} \| h(x(t_i + t; \bar{x}_0, t_i), t_i + t) \|^2 \stackrel{\text{a.e.}}{=} 0$$

For large i, the function $\| h(x(t_i + t; \bar{x}_0, t_i), t_i + t) \|^2$ is bounded above by an integrable function over $[0, T]$. Then by Lebesgue's Dominated Convergence Theorem one concludes that

$$\lim_{t_i \to \infty} \int_0^T \| h(x(t_i + t; \bar{x}_0, t_i), t_i + t) \|^2 \, dt = 0 \tag{32}$$

This contradicts (28) only if $\bar{x}_0 = 0$. Therefore (29) cannot be true which implies that observability of (g, h) is recurrent. □

Example 5 Consider the scalar equation

$$\dot{x} = f(t,x) = -\gamma^2(x,t)x \tag{33}$$

where γ is a scalar continuous function of x, uniformly w.r.t. t, and with finite L^2-norm in the neighborhood of the origin . Motivated by the previous section we consider the output $h(x,t) = \gamma(x,t)x$, associated to the Liapunov function $V(x,t) = x^2/2$. Since $0 = f(x,t) + \gamma(x,t)h(x,t)$, the pair (f,h) can be transformed by output injection into the pair $(0,h)$. By Theorem 6, observability of the pair (f,h) is recurrent provided that there exists a sequence $(t_i)_{i\geq 1} \to \infty$ such that for each $x_0 \neq 0$ in a neighborhood of the origin

$$\liminf_{t_i\to\infty} \int_{t_i}^{t_i+T} \gamma^2(x_0,t) > 0 \tag{34}$$

for some finite $T > 0$.

The next theorem is essentially restricted to the linear case (although an immediate nonlinear extension is given in Remark 3). Unlike Theorem 6, it allows variable periods of integration $T(t_0)$.

Theorem 7 Consider the pair (A,C) and let $\Phi(t,s)$ be the transition matrix of $\dot{x} = A(t)x$. Suppose that $\Phi(t,s)$ is bounded for all (t,s) (i.e the system is uniformly stable). Then weakly uniform observability of the pair (A,C) is invariant under linear output injection provided that the feedback gain matrix $K(t)$ and the output matrix $C(t)$ have a finite norm in $L^2(t_0, t_0 + T(t_0))$ for each t_0.

Proof. Define $F(t) := A(t) + K(t)C(t)$. Let $\Psi(t_0+t, t_0)$ denote the transition matrix of $\dot{x} = F(t)x$. Assume that the pair (F,C) is not weakly observable i.e. for some $x_0 \neq 0$ and some sequence $t_i \to \infty$

$$\lim_{t_i\to\infty} \int_{t_i}^{t_i+T(t_i)} \| C(t)\Psi(t,t_i)x_0 \|^2 \, dt = 0 \tag{35}$$

By the variation of constants formula we know that for each i and each $t \in [t_i, t_i + T(t_i)]$

$$\Psi(t,t_i)x_0 = \Phi(t,t_i)x_0 + \int_{t_i}^{t} \Phi(t,s)K(s)C(s)\Psi(s,t_i)x_0 ds \tag{36}$$

By assumption $\Phi(t,s)$ is bounded for all (t,s) and therefore (36) implies for each i

$$\sup_{t\in[t_i,t_i+T(t_i)]} \| \Psi(t,t_i)x_0 - \Phi(t,t_i)x_0 \| \leq C_1 \int_{t_i}^{t_i+T(t_i)} \| K(s)C(s)\Psi(s,t_i)x_0 \| \, ds \tag{37}$$

for some constant C_1. By the Cauchy-Schwartz inequality we have for each i

$$\int_{t_i}^{t_i+T(t_i)} \| K(s)C(s)\Psi(s,t_i)x_0 \| \, ds \leq$$
$$\leq (\int_{t_i}^{t_i+T(t_i)} \| K(s) \|^2 \, ds)^{1/2} (\int_{t_i}^{t_i+T(t_i)} \| C(s)\Psi(s,t_i)x_0 \|^2 \, ds)^{1/2}$$

and since the feedback gain $K(t)$ has a finite norm $L^2(t_i, t_i + T(t_i))$ for each i

$$\int_{t_i}^{t_i+T(t_i)} \| K(s)C(s)\Psi(s,t_i)x_0 \| \, ds \leq C_2 (\int_{t_i}^{t_i+T(t_i)} \| C(s)\Psi(s,t_i)x_0 \|^2 \, ds)^{1/2} \tag{38}$$

for some constant C_2. Taking the limit of (37) for t_i tending to infinity we obtain by (35) and (38)

$$\lim_{t_i \to \infty} (\sup_{t \in [t_i, t_i+T(t_i)]} \| \Psi(t,t_i)x_0 - \Phi(t,t_i)x_0 \|) = 0 \tag{39}$$

Now consider for each i the quantity

$$\int_{t_i}^{t_i+T(t_i)} \| C(t)\Phi(t,t_i)x_0 \|^2 \, dt \tag{40}$$

which is equal to

$$\int_{t_i}^{t_i+T(t_i)} \| C(t)\Psi(t,t_i)x_0 \|^2 \, dt + \int_{t_i}^{t_i+T(t_i)} \| C(t)(\Phi(t,t_i) - \Psi(t,t_i))x_0 \|^2 \, dt + \\ + 2 \int_{t_i}^{t_i+T(t_i)} \| C(t)(\Phi(t,t_i) - \Psi(t,t_i))x_0 \| \| C(t)\Psi(t,t_i)x_0 \| \, dt \tag{41}$$

By (35) the first term of (41) tends to zero as i tends to infinity. Since the output matrix $C(t)$ has a finite norm $L^2(t_i, t_i + T(t_i))$ for each i, the second term of (41) satisfies for some constant C_3

$$\int_{t_i}^{t_i+T(t_i)} \| C^T(t)(\Phi(t,t_i) - \Psi(t,t_i))x_0 \|^2 \, dt \leq C_3 \sup_{t \in [t_i, t_i+T(t_i)]} \| (\Phi(t,t_i) - \Psi(t,t_i))x_0 \|^2$$

and therefore also tends to zero as i tends to infinity by (39). Finally invoking the Cauchy-Schwartz inequality, we obtain that the last term of (41) tends to zero as $i \to \infty$ We conclude that

$$\lim_{t_i \to \infty} \int_{t_i}^{t_i+T(t_i)} \| C(t)\Phi(t,t_i)x_0 \|^2 \, dt = 0 \tag{42}$$

which is in contradiction with weak uniform observability of the pair $A(t), C(t)$. □

Remark 3 With a minor adaptation of the above proof, it is possible to show that Theorem 7 remains true if the output $h(x,t)$ and the feedback gain $k(x,t)$ are Lipschitz mappings with a Lipschitz function that has finite norm $L^2(t_0, t_0 + T(t_0))$ for each t_0.

Example 6 Consider the scalar equation

$$\dot{x} = f(t,x) = -a^2(t)b(x) \tag{43}$$

with 'a' measurable and 'b' Lipschitz. Suppose that $b(x)x > 0$ for $x \neq 0$. We consider the output $h(x,t) = a(t)\sqrt{b(x)x}$, according to the Liapunov function $V(x,t) = x^2/2$. Define $k(x,t) := h(x,t)/x$. A Lipschitz function for $h(x,t)$ is given by $Ka(t)$ where K is a Lipschitz constant for $\sqrt{b(x)x}$. We claim that a necessary and sufficient condition for weakly uniform observability of the pair (f,h) is weakly uniform observability of the pair $(0,h)$, i.e.

$$\forall t_0 \geq 0 : \int_{t_0}^{\infty} a^2(t) dt = \infty \tag{44}$$

Indeed it is necessary and sufficient that (44) holds in order that for each t_0 there exists a $T(t_0)$ such that

$$\int_{t_0}^{t_0+T(t_0)} a^2(t)dt = 1 \tag{45}$$

Therefore we have for all t_0

$$\int_{t_0}^{t_0+T(t_0)} Ka^2(t)dt = K \tag{46}$$

which shows that the output and the gain have finite energy over the period of integration. Since I is the transition matrix of $\dot{x} = 0$ and since $f(x,t) = 0 - k(x,t)h(x,t)$, Theorem 7 and Remark 3 apply and weakly uniform observability is unchanged via the considered output injection.

Since weakly uniform observability is weaker than recurrent (or uniform) observability, the assumptions of Theorem 7 provide an alternative set of assumptions for Theorem 6 to be true; in particular, we obtain the following corollary.

Corollary 1 Consider the pair $(A(t), C(t))$ and let $\Phi(t,s)$ the transition matrix of $\dot{x} = A(t)x$. Suppose that for some function γ of class $\mathcal{K}$

(a) $\| \Phi(t,s) \| \leq \gamma(| t-s |)\ \forall(t,s)$

(b) $C(t)$ has a finite $L^2(t_0, t_0+T)$-norm for each t_0 and each finite T.

Then uniform observability of the pair $(A(t), C(t))$ is invariant under linear output injection provided that the feedback gain matrix $K(t)$ has a finite norm $L^2(t_0, t_0+T)$ for each t_0 and each finite T. □

It is easily verified that under the additional assumptions (a) and (b), our definition of uniform observability is equivalent to the Kalman definition. As a consequence, Corollary 1 can be considered as a restatement of the following result of Anderson and Moore [5]: uniform observability (in the sense of Kalman) is invariant under output injection provided that the gain $K(t)$ has a finite norm $L^2(t_0, t_0+T)$ for each t_0 and each finite T. In particular, Corollary 1 illustrates the role of the additional assumptions in the Kalman definition of uniform observability.

5.2 Attractivity criteria revisited

A limitation of the stability criteria derived in Section 4 is in the verification of the observability condition and of the (Strong) Continuity Assumption on the original system. Using the ideas of the previous section, we will show that in particular situations, it is sufficient to verify the criteria on an associated system rather than on the original system. This will be helpful in the applications, in particular when the trajectories of the associated system are explicitly known.

Theorem 8 Suppose that (S) admits a Liapunov function with properties (i)-(iv). Let $h(x,t)$ be an output function defined by (OUT). Suppose that $g(x,t) = f(x,t)+k(x,t)h(x,t)$ for some mapping $k(x,t)$ where g, k and h satisfy the assumptions of Theorem 6. Then recurrent observability of (g,h) is a sufficient condition for asymptotic stability of the null solution of (S).

Proof. Let $x(t;x_0,t_0)$ denote the solution of $\dot{x} = f(t,x)$ and $x^*(t;x_0,t_0)$ denote the solution of $\dot{x} = g(t,x)$. The proof is an adaptation of the proof of Theorem 2. First expressing recurrent observability x is replaced by x^* in (11). Then only Part II (which uses the Continuity Assumption of $f(x,t)$) is replaced by the following:

II'. Since

$$\lim_{t\to\infty} V(x(t;x_0,t_0),t) = \lim_{t_i\to\infty} V(x_i,t_i) \quad (= \bar{V})$$

it follows immediately that

$$\lim_{t_i\to\infty} \int_{t_i}^{\infty} -\dot{V}(x(t;x_0,t_0),t)dt = 0$$

or equivalently

$$\lim_{t_i\to\infty} \int_{t_i}^{\infty} \| h(x(t;x_0,t_0),t) \|^2 \, dt = 0$$

In particular, we have

$$\lim_{t_i\to\infty} \int_{t_i}^{t_i+T} \| h(x(t;x_i,t_i),t) \|^2 \, dt = 0$$

which implies for a.e. $t \in [0,T]$

$$\lim_{t_i\to\infty} \| h(x(t_i+t;x_i,t_i),t_i+t) \| \, dt = 0$$

Using a similar argument as in the proof of Theorem 6 we conclude that for all $t \in [0,T]$

$$\lim_{t_i\to\infty} \| x(t_i+t;x_i,t_i) - x^*(t_i+t;x_i,t_i) \| = 0 \tag{47}$$

On the other hand, by the Lipschitz assumption on $g(x,t)$, the continuity of solutions with respect to the inital conditions implies

$$\| x^*(t_i+T;x_i,t_i) - x^*(t_i+T;p,t_i) \| \leq e^{kT} \, \| x_i - p \|$$

and therefore

$$\lim_{t_i\to\infty} \| x^*(t_i+T;x_i,t_i) - x^*(t_i+T;p,t_i) \| = 0 \tag{48}$$

By the triangle inequality, we conclude from (47) and (48) that

$$\lim_{t_i\to\infty} \| x(t_i+t;x_i,t_i) - x^*(t_i+t;p,t_i) \| = 0 \tag{49}$$

This implies that

$$\lim_{t_i\to\infty} |V(x(t_i+T;x_i,t_i),t_i+T) - V(x^*(t_i+T;p,t_i),t_i+T)| = 0 \tag{50}$$

Consider the sequence of elements $V(x^*(t_i+T;p,t_i),t_i+T)$ with $t_i \to \infty$. These elements each belong to the interval $[0,\gamma_2(\| x^*(t_i+T;p,t_i) \|)]$ by Property (ii). By taking t_i large enough and because of (49) and of the boundedness of $\| x(t_i+T;x_i,t_i) \|$ these elements belong to a finite interval. Then there exists a subsequence $t_{i_{sub}} \to \infty$ of the sequence (t_i) such that for some $a \geq 0$

$$\lim_{t_{i_{sub}}\to\infty} V(x^*(t_{i_{sub}}+T;p,t_{i_{sub}}),t_{i_{sub}}+T) = a \tag{51}$$

Using (11) (with x replaced by x^*)and (14) we conclude that this limit satisfies the inequality

$$a < \bar{V} \tag{52}$$

From the triangle inequality we obtain with $x_{i_{sub}} := x(t_{i_{sub}}; x_0, t_0)$

$$\begin{aligned} |V(x(t_{i_{sub}} + T; x_{i_{sub}}, t_{i_{sub}}), t_{i_{sub}} + T) - a| \leq \\ |V(x(t_{i_{sub}} + T; x_{i_{sub}}, t_{i_{sub}}), t_{i_{sub}} + T) - V(x^*(t_{i_{sub}} + T; p, t_{i_{sub}}), t_{i_{sub}} + T)| + \\ +|V(x^*(t_{i_{sub}} + T; p, t_{i_{sub}}), t_{i_{sub}} + T) - a| \end{aligned} \tag{53}$$

and by (50) and (51) we obtain from (53)

$$\lim_{i_{sub} \to \infty} V(x(t_{i_{sub}} + T; x_{i_{sub}}, t_{i_{sub}}), t_{i_{sub}} + T) = a < \bar{V} \tag{54}$$

This ends the proof of part II'. □

As an illustration of the above theorem, consider the system discussed in Example 5: it follows from Theorem 8 that (34) is a sufficient condition for asymptotic stability of the null solution of (33).

Mutatis mutandis, we obtain a similar reformulation of Theorem 3 using the result of Theorem 7 and Remark 3.

Theorem 9 Suppose that (S) admits a Liapunov function with properties (i)-(iv). Let $h(x,t)$ an output function defined by (OUT). Suppose that $f(x,t) = A(t)x + k(x,t)h(x,t)$ for some mappings $k(x,t)$ and $h(x,t)$. Define $k(t) := \sup_{x \in U} k(t,x)$ and similarly define $h(t)$ Denote by $\Phi(t,s)$ the transition matrix of $\dot{x} = A(t)x$.

Then the null solution of (S) is asymptotically stable if the following conditions hold:

(i) $h(x,t)$ and $k(x,t)$ are Lipschitz in x with Lipschitz function $k(t)$ and $l(t)$

(ii) $\Phi(t,s)$ is bounded for all (t,s).

(iii) (A,h) is weakly uniformly observable and the functions $h(t)$ and $k(t)$ have a finite norm $L^2(t_0, t_0 + T(t_0))$ for each t_0.

Proof. The proof is an adaptation of the proof of Theorem 3, similar to the proof of Theorem 8. □

As an illustration of the above theorem, consider the system discussed in Example 6: it follows from Theorem 9 that (44) is a sufficient condition for asymptotic stability of the null solution of (43).

6 Applications

6.1 Persistency of excitation

The concept of persistent excitation emerged in the context of linear identification. On the basis of the input and the output signals, one tries to identify the parameters of a linear transfer function. It is a common observation that the spectrum of the input signal must be rich enough in order to excite all the modes of the system and in turn to ensure convergence of the parameters to their true values. The "paradigm" error equation has the form

$$\dot{x} = -m(t)m^T(t)x \tag{PE}$$

where $x : I\!R^+ \to I\!R^n$ and $m : I\!R^+ \to I\!R^n$. The state represents the error between the true parameters and their estimates and m is a time-varying vector depending on the past inputs and outputs.

Stability of the zero-error is readily established with the Liapunov function $V(x) = x^T x$ which leads to the semi-definite negative time derivative

$$\dot{V}(t,x) = -(m^T(t)x(t))^2 \leq 0 \tag{55}$$

However, except for the particular case where the input signal (and in turn the vector m) is periodic, the standard extensions of Liapunov theory cannot be applied to show convergence of the parameters: the limit sets of (PE) have no invariance property and no non-trivial negative function $W(x)$ may serve as an upper bound for $\dot{V}(t,x)$. On the other hand, (PE) and (55) have geometrically appealing features and suggest that convergence of the parameters is ensured provided that m is sufficiently exciting, i.e. that within a moving fixed time interval no subspace of $I\!R^n$ stays orthogonal to m.

Theorem 10 Uniform asymptotic stability of the null solution of (PE) is equivalent to the existence of positive constants α, β, and T such that

$$\forall t : \alpha I \leq \int_t^{t+T} m(\tau)m^T(\tau)d\tau \leq \beta I \tag{56}$$

The condition (56) is usually referred in the literature as the *persistent excitation* of the vector m. Hereafter we sketch the proof proposed by Anderson [4].

Proof: The lower bound of (56) implies uniform observability of the pair $(0, m)$. The pair $(0, m)$ is related by output injection to the pair $(-mm^T, m)$. The feedback gain is given by $K(t) := -m(t)$ and has finite L_2 norm over the interval T due to the upper bound in (56). By Corollary 1, the pair $(-mm^T, m)$ is uniformly observable, which implies uniform asymptotic stability of (PE). □

By the results of the previous sections, we may along the same lines weaken condition (56) and still guarantee asymptotic stability of the null solution of (PE).

Proposition 3 A sufficient condition for asymptotic stability of the null solution of (PE) is the existence of a sequence $(t_i)_{i\geq 1}$ tending to infinity and of positive constants α, β, and T such that

$$\alpha I \leq \lim_{t_i \to \infty} \int_{t_i}^{t_i+T} m(\tau)m^T(\tau)d\tau \leq \beta I \tag{57}$$

Proof: The lower bound of (57) implies recurrent observability of the pair $(0, m)$. The upper bound (57) implies recurrent observability of the pair $(-mm^T, m)$ by Theorem 6. Finally, asymptotic stability of the null solution follows from Theorem 2. □

Proposition 4 A sufficient condition for asymptotic stability of the null solution of (PE) is the existence of positive constants α and β such that

$$\forall t:\ \exists T(t) : \alpha I \leq \int_t^{t+T(t)} m(\tau)m^T(\tau)d\tau \leq \beta I \tag{58}$$

Proof: The lower bound of (58) implies weak observability of the pair $(0, m)$. The upper bound of (58) implies weak observability of the pair $(-mm^T, m)$ by Theorem 7. Finally asymptotic stability of the null solution follows from Theorem 4. □

Notice that both conditions (56) and (57) impose in particular

$$\forall t_0 : \sup_{t \geq t_0} (m^T(t)p)^2 \geq \frac{\alpha}{T} \parallel p \parallel \tag{59}$$

which prevents situations where $\parallel m(t) \parallel$ tends to zero as t tends to infinity. On the contrary, condition (58) is for instance satisfied for the scalar equation $\dot{x} = -\frac{x}{t}$ defined for $t_0 \geq 1$. Here $m(t) = \sqrt{1/t}$ tends asymptotically to zero. Nevertheless the pair $(0, m)$ is weakly uniformly observable (choose for instance $T(t) = t$) and hence guarantees asymptotic stability of (PE).

It is important to notice that in each of the three conditions (56), (57), and (58), the upper bound β plays a similar role: it provides a sufficient condition which allows to examine observability of the pair $(-mm^T, m)$ in terms of the observability of the pair $(0, m)$. A particular consequence of this upper bound is that condition (58) is only sufficient for asymptotic stability of (PE) and imposes restrictions on the time-dependence of $m(t)$: the excitation of $m(t)$ is allowed to decrease over finite periods of time, but the loss of excitation must be uniform in the state space. The limitation of this requirement is illustrated in the following example:

Example 7 Consider

$$m(t)^T = \begin{cases} \begin{pmatrix} 1 & 0 \end{pmatrix} & \text{for } 2i \leq t < 2i+1 \\ \begin{pmatrix} 0 & 1/\sqrt{t} \end{pmatrix} & \text{for } 2i+1 \leq t < 2i+2 \end{cases}$$

with $i \in I\!N$. By reordering the time-axis we obtain two asymptotically stable decoupled equations $\dot{x}_1 = -x_1$ and $\dot{x}_2 = -x_2/t$. It is quickly verified that condition (58) can not be satisfied, basically since there is no $T(t)$ that works for *both* these equations. □

Morgan and Narendra [13] studied the stability properties of (PE) using ad hoc techniques. They obtained necessary and sufficient conditions for uniform asymptotic stability of (PE) which are equivalent with the result of Theorem 10. They also obtain a sufficient condition for nonuniform asymptotic stability of (PE). The relationship between the condition of Morgan and Narendra [13] and the conditions of Proposition 3 and Proposition 4 is not obvious but the result of [13] allows to prove asymptotic stability of the system of Example 7.

On the other hand, Condition (58) without the upper bound, i.e. weakly uniform observability of the pair $(0, m)$ is not sufficient for asymptotic stability of (PE) as illustrated by the following example:

Example 8 Define

$$\alpha_0 = \pi/2,\ \alpha_{i+1} = \alpha_i/2,\ i \geq 0,$$
$$t_0 = 0,\ t_{i+1} = t_i + 1/\sin^2\alpha_i,$$
$$m_i^T = (\sin\alpha_i, \cos\alpha_i).$$

Let $m(.) : I\!R^+ \to I\!R^2 : m(t) = m_i,\ t_i \leq t < t_{i+1}$.

Weakly uniform observability of $(0, m)$ is obviously satisfied: indeed, let $t_i \geq 0$. If $e_1 := (1, 0)$ then for each integer n, $\int_{t_n}^{t_{n+1}} (m^T(\tau)e_1)^2 d\tau = 1$ which implies that $\int_{t_i}^{\infty} (m^T(\tau)e_1)^2 d\tau$ is unbounded. The condition is thus fulfilled for $p = \epsilon e_1,\ \epsilon \neq 0$. Now if p is not parallel to e_1, then for n sufficiently large, $(m_n^T p)^2 > (m_n^T e_1)^2$ and as a consequence, $\int_{t_n}^{t_{n+1}} (m^T(\tau)p)^2 d\tau > 1$. This implies that $\int_{t_i}^{\infty} (m^T(\tau)p)^2 d\tau$ is unbounded for all $p \neq 0$.

Now it can easily be shown that the above $m(t)$ does not force the convergence of solutions of (PE) to the origin. Let $m_1^{\perp} = (\cos\alpha_1, -\sin\alpha_1)$ and consider the solution starting from $m_1^{\perp}$ at time $t = t_2$, i.e. $x(t; m_1^{\perp}, t_2)$. Then $x_1(t_2; m_1^{\perp}, t_2) = \cos\alpha_1 > 0$ and by using an obvious trigonometrical argument, it is clear that $x_1(t; m_1^{\perp}, t_2)$ can only increase for all $t > t_2$ which prevents the convergence of the solution to the origin. □

To summarize, the sufficient conditions above for parameter convergence give conclusions only if the hypotheses of $m(t)$ are uniform with respect to each direction of the state space. This suggests that further sufficient conditions are to be obtained directly from observability of the pair $(-mm^T, m)$ and not from observability of the pair $(0, m)$.

Nonlinear extensions. By the the results of the previous sections we obtain similar results for the nonlinear equation

$$\dot{x} = -m(t, x)m^T(t, x)x \qquad (NLPE)$$

where $x : I\!R^+ \to I\!R^n$ and $m : I\!R^+ \times I\!R^n \to I\!R^n$. The equation (NLPE) describes an error equation in closed-loop identification, i.e. when the past inputs depend on the past outputs and on the estimated parameters. According to Theorem 8, a sufficient condition for parameter convergence is the existence of a sequence $(t_i)_{i\geq 1}$ tending to infinity, a positive constant T and a function α of class $\mathcal{K}$ such that

$$\lim_{i\to\infty} \int_{t_i}^{t_i+T} m(\tau, p)m^T(\tau, p)d\tau \geq \alpha(\| p \|)I \qquad (60)$$

i.e. recurrent observability of the pair $(0, m)$, provided that $m(t, x)$ is bounded and continuous in x, uniformly in t. Similarly, (56), (57) and (58) provide sufficient conditions if $m(t)$ is a Lipschitz function for $m(t, x)$.

These nonlinear extensions are related to the result of Artstein [6]. The approach in the present paper obtains sufficient conditions for asymptotic stability whereas Artstein [6] only considers the uniform case.

Further generalizations. The generalization of the above results to the multidimensional case ($m : I\!R^+ \to I\!R^n \times I\!R^p$) is immediate. On the other hand, (PE) has also played a major role in the stability of adaptive systems (see for instance [15]). Other equations appear in this context allowing a similar treatment. As an illustration, consider the equation

$$\dot{x} = F(t)x := \begin{bmatrix} 0 & -mB^T \\ Bm^T & A \end{bmatrix} x \qquad (61)$$

where $m : \mathbb{R}^+ \to \mathbb{R}^n \times \mathbb{R}^p$, A is a real constant $n \times n$ matrix with $A + A^T = -I$, and B a real constant $n \times p$ matrix of full rank. It is shown in [4] that the null solution of (61) is asymptotically stable if for some positive constants α, β, T, and all t,

$$\int_t^{t+T} \int_t^s m(\tau)d\tau \int_t^s m^T(\tau)d\tau \; ds \geq \alpha I \tag{62}$$

and

$$\int_t^{t+T} \| m(\tau) \|^2 \, d\tau \leq \beta. \tag{63}$$

The proof is similar to the proof given for (PE). Starting from the Liapunov function $V(x) = x^T x$, one has $\dot{V}(t,x) = -x^T C C^T x$ with $C = [0 \; I]^T$. Using the output injection $K(t) = [-Bm^T \; A^T]^T$, observability of the pair (F, C) can be studied by means of observability of the pair (G, C) with

$$G(t) := F(t) - K(t)C^T(t) = \begin{bmatrix} 0 & 0 \\ Bm^T & 0 \end{bmatrix} \tag{64}$$

The transition matrix $\Phi(t, t_0)$ of $\dot{x} = G(t)x$ is given by

$$\Phi(t, t_0) = \begin{bmatrix} I & 0 \\ \int_{t_0}^t Bm^T(\tau)d\tau & I \end{bmatrix} \tag{65}$$

and the condition $\| \Phi(t,s) \| \leq \gamma(| t - s |)$ is satisfied according to (63). By Corollary 1 we conclude that uniform observability of the pair (G, C) is equivalent to uniform observability of the pair (F, C) and therefore provides a sufficient condition for exponential stability of the null solution of (61). A final argument given in [4] shows that uniform observability of the pair (G, C) is equivalent to (62). Similarly, it follows from Corollary 1 that recurrent observability of the pair (G, C) is a equivalent to recurrent observability of the pair (F, C) and therefore a sufficient condition for asymptotic stability of the null solution of (61); the same argument shows that recurrent observability of the pair (G, C) is equivalent to the existence of a positive sequence $(t_i)_{i \geq 1}$ and of positive constants α, β, T, such that

$$\lim_{t_i \to \infty} \int_{t_i}^{t_i+T} \int_{t_i}^s m(\tau)d\tau \int_{t_i}^s m^T(\tau)d\tau ds \geq \alpha I \tag{66}$$

Notice that contrary to (PE), the results cannot be extended in the present case to a non uniform period of observation. This may be made clear as follows: if (62) is satisfied only with a non-uniform period of observation, the condition imposes a coupling between a uniform dynamics ($\dot{x}_1 = 0$) and a non uniform dynamics ($\dot{x}_2 = Bm^T x_1$) which is hard to handle as has been illustrated in Examples 7 and 8.

6.2 A relaxed circle criterion

Another well-known problem in system theory involving nonautonomous differential equations of the type studied above is concerned with the stability analysis of the feedback system where the forward element of the closed-loop system is linear and time-invariant while the feedback gain is (nonlinear and) time-variant. This type of system has received much attention in the literature (see for instance [18, chapter 6]) and can be considered a useful model for many engineering systems.

Assume that the linear transfer function $\mathcal{F} := q(s)/p(s)$ is such that the polynomials

$$p(s) = s^n + p_{n-1}s^{n-1} + \ldots + p_0$$

and

$$q(s) = q_{n-1}s^{n-1} + \ldots + q_0$$

have no common factors, and that the degree n of $p(s)$ is higher than the degree of $q(s)$. The closed-loop system equation has the form

$$\dot{x} = Ax + bk(x,t)c^T x \tag{FS}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -p_0 & -p_1 & -p_2 & \ldots & -p_{n-1} \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}$$

and

$$c^T = [q_0 \; q_1 \; \ldots \; q_{n-1}].$$

Different criteria have been obtained for the (exponential) asymptotic stability of the null solution of (FS). The proposed conditions typically allow to construct a quadratic Liapunov function whose time-derivative is negative-definite or at least negative semi-definite. The goal of this section is to show how these various criteria can be relaxed. For clarity reasons, we consider the *linear* case in the developments ($k(x,t)$ replaced by $k(t)$); straightforward nonlinear extensions can be considered according to Theorem 6 and the nonlinear extensions of Theorem 7 (Remark 3).

The next theorem, referred to in the literature as the circle criterion because of its graphical interpretation in the Nyquist plane [18], provides a sufficient condition for asymptotic stability of the null solution of (FS).

Theorem 11 [Circle criterion] The null solution of (FS) is asymptotically stable in the large if there is a positive ϵ such that

$$k_1 + \epsilon \leq k(x,t) \leq k_2 - \epsilon \tag{67}$$

holds for all x and t, and if

$$\frac{k_2\mathcal{F}(s) + 1}{k_1\mathcal{F}(s) + 1} = \mathcal{P}(s) \tag{68}$$

is a positive real function.

Proof (sketch): The theorem can be proved by means of Liapunov theory and essentially follows from the following: if $\mathcal{P}(s)$ is a positive real function, then there exists a positive definite quadratic form $V(x)$ such that its time derivative satisfies

$$\dot{V}(x,t) \leq -(k_2 - k(x,t))(k(x,t) - k_1)(c^T x)^2 \leq (c^T x)^2 \leq 0 \tag{69}$$

which implies asymptotic stability by standard techniques as explained in the introduction.
□

If (67) is replaced by

$$k_1 \leq k(x,t) \leq k_2 \tag{70}$$

then the above Liapunov function only allows to conclude uniform stability of the null solution.

We now propose an extra-condition to (70), substantially weaker than (67) but still guaranteeing asymptotic stability of the origin. As announced we restrict the analysis to the linear time-variant case. Assume $k(t)$ is measurable.

Introduce the notation

$$h^2(t) := (k_2 - k(t))(k(t) - k_1)$$

$$F(t) := A + bk(t)c^T$$

$$H(t) := h(t)c^T$$

Notice that the pair (F,c) is uniformly observable in the sense of Kalman (and therefore also in any sense defined in the previous sections), since (A, c) is uniformly observable.

Proposition 5 Observability of the pair (F, H) is recurrent if there exists a sequence $(t_i)_{i\geq 1}$ tending to infinity such that

$$k_1 < \lim_{t_i \to \infty} k(t_i) < k_2. \tag{71}$$

Proof: (by contradiction) Assume that observability of the pair (F, H) is not recurrent. Then for all sequences $t'_i \to \infty$, $\exists x_0 \neq 0$ such that $\forall T > 0$ and with $\Phi(t, s)$ the transition matrix associated with $F(t)$

$$\liminf_{t'_i \to \infty} \int_{t'_i}^{t'_i+T} h^2(t) x_0^T \Phi^T(t, t'_i) c c^T \Phi(t, t'_i) x_0 \, dt = 0$$

Take t'_i and T such that the interval $[t'_i, t'_i + T]$ is centered on t_i. This implies that a subsequence of

$$h^2(t'_i + t) x_0^T \Phi^T(t'_i + t, t'_i) c^T c \Phi(t'_i + t, t'_i) x_0$$

uniformly tends to zero over the interval $[0, T]$ as t'_i tends to infinity. Since the pair (F, c) is uniformly observable, this implies—keeping the same notation for the subsequence—that $h^2(t'_i + t)$ uniformly tends to zero over the interval $[0, T]$ as t'_i tends to infinity. By definition, this is possible only if $k(t'_i + t)$ uniformly tends to k_1 or k_2 over the interval $[0, T]$ as t'_i tends to infinity. But this contradicts (71) since $t_i \in [t'_i, t'_i + T]$ for each i. □

Theorem 12 [Relaxed circle criterion] Under the assumptions of Theorem 11, with $\epsilon = 0$, the condition (71) is sufficient for asymptotic stability of the null solution of (FS).

Proof: The proof is a direct consequence of the proof of Theorem 11, Proposition 5 and of Theorem 2. □

7 Conclusion

In this paper the problem of asymptotic stability for time-variant systems is discussed from a Liapunov point of view. We propose several notions of observability with the output derived from the Liapunov function. We develop a number of sufficient conditions for asymptotic stability, all formulated within an observability framework. This theory is then illustrated by means of some classical examples from control theory for which we provide nontrivial extensions.

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