Integrator forwarding: a new recursive nonlinear robust design *

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Abstract

We consider the global stabilization of nonlinear systems in strict feedforward form. We show that these systems, while not feedback linearizable, can be (globally) transformed by feedback and diffeomorphism into a lower-triangular form. The transformation is explicit and recursive. We employ this transformation to design a globally stabilizing controller with optimality properties and an input-to-state stability property with respect to matched uncertainties.

Abbreviated title: Integrator forwarding
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1 Introduction

This paper addresses the problem of global stabilization (by state feedback) of single input nonlinear systems

$$\begin{aligned}
\dot{x}_1 &= f_1(x_2,\ldots,x_n) + g_1(x_2,\ldots,x_n)u \\
\dot{x}_2 &= f_2(x_3,\ldots,x_n) + g_2(x_3,\ldots,x_n)u \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(x_n) + g_{n-1}(x_n)u \\
\dot{x}_n &= g_n u
\end{aligned} \tag{1.1}$$

where $f = (f_1,\ldots,f_{n-1})$ and $g = (g_1,\ldots,g_{n-1})$ are smooth, $f(0) = 0$, and $g_n \neq 0$ is a constant. We refer to the upper-triangular form as the *strict feedforward* structure to distinguish it from the *strict feedback* structure in which case $f(x)$ has a lower-triangular form and $g^T(x) = (0,\ldots,0,g_n)$.

Throughout this paper it is assumed that the Jacobian linearization of (1.1) at $x = 0$ is stabilizable, which, because of the feedforward structure, also implies its controllability. Strict feedforward nonlinear systems are in general not feedback linearizable and require specific stabilization techniques. One typical such system will be considered in Section 5. In this paper we show that, starting with the *upper-triangular* form (1.1), we can recursively construct a global change of coordinates $z = T(x)$ and a feedback transformation $u = -\kappa^T(z)z + v$ which transforms (1.1) into the *lower-triangular* form

$$\dot{z} = L(z)z + \kappa(z)v \tag{1.2}$$

where $L(.)$ is a lower-triangular matrix. A strong property of (1.2) is that $L + L^T$ is negative definite so that with $v = 0$ the equilibrium $z = 0$ of (1.2) is globally asymptotically stable. This property is achieved with the feedback $u = -\kappa^T(z)z$ and can be verified with the simplest Lyapunov function $V(z) = z^T z$. The remaining freedom to select v can be used to meet additional design specifications.

The existence of a lower-triangular form reveals important structural properties which have played a crucial part in recent developments of robust and adaptive nonlinear designs (see in particular (Krstic *et al.*, 1995), (Tomei and Marino, 1995), and (Praly *et al.*, 1991)). However they have been available only for nonlinear systems that are *feedback linearizable*, that is when the form (1.2) can be rendered linear. Our result presents a similar systematic decomposition for a new class of nonlinear systems for which a *linear* form does not exist.

In addition to global asymptotic stability, it is important that our stabilizing feedback $u = -2\kappa^T(z)z$ can be written as

$$u = -\frac{\partial V}{\partial z}\kappa \tag{1.3}$$

It is well-known that such a control law minimizes a cost functional with a quadratic cost on the control and a positive nonlinear cost on the state (Jacobson, 1977). As a consequence of its optimality, the control law (1.3) has an infinite sector margin. Moreover, with an elementary redesign, it achieves an input-to-state stability (ISS) property with respect to matched disturbances. In a nonlinear framework, the control law (1.3) has thus far been used when the uncontrolled part $\dot{x} = f(x)$ of (1.1) is already stable so that $\frac{\partial V}{\partial x} f(x) \leq 0$ (Jacobson, 1977), (Jurdjevic and Quinn, 1978). For feedforward systems, we remove this restrictive assumption and allow the equilibrium of the uncontrolled system to be unstable.

Our approach departs from the earlier stabilization results initiated in (Teel,1992) which employ saturation functions $\sigma(s)$ in the feedback law. The scalar saturation function $\sigma(s)$ is linear in a neighborhood of the origin, $\sigma(s) = \delta s$, while for $\mid s \mid \geq \frac{\epsilon}{\delta}$, it is *$\epsilon$-saturated*, $\sigma(s) = \epsilon sign(s)$. Teel was the first to show (Teel,1992) that, after a linear change of coordinates $z = Tx$, the nested saturation feedback

$$u = -\sigma_n(z_n + \sigma_{n-1}(z_{n-1} + \ldots + \sigma_1(z_1))) \ldots) \tag{1.4}$$

achieves global asymptotic stability of the equilibrium $x = 0$. Teel's proof relies on input-output stability arguments rather than on a Lyapunov approach and his design is robust with respect to certain variations in the form of the nonlinearities. However, because the saturation level of the σ_i's is restricted to be sufficiently small, the saturation scheme is sensitive to additive disturbances.

For feedforward systems which include the strict form (1.1), a Lyapunov approach has been developed in (Mazenc and Praly, 1994), where the saturation still appears in the control law. In (Jankovic *et al.*, 1995), we have proposed a new Lyapunov design for an enlarged class of feedforward systems. Our approach eliminates the need for saturation functions and offers a more flexible construction of the control law. The price to be paid is an increase in computations, which may involve numerical approximations. In this paper, the new method of (Jankovic *et al.*, 1995) is employed to construct the change of coordinates $z = T(x)$.

The paper is organized as follows: in Section 2, we describe the recursive stabilization design of (1.1). In Section 3 we show that this recursive method can be used to obtain a global triangular form for the closed-loop system. The robustness properties are discussed in Section 4. Finally, in Section 5, we apply our results to the nonlinear benchmark problem introduced in (Kokotovic *et al.*, 1991).

2 Integrator forwarding design

Our stabilization method presented in (Jankovic *et al.*, 1995) is a bottom-up procedure which starts with the stabilization of the scalar system $\dot{x}_n = u$ and then "forwards an integrator" $(n-1)$ times. At each step, the stabilization of the augmented subsystem is achieved by constructing a new term in the Lyapunov function and adding a new term in the control law.

We will present the first step of the procedure in detail and then show how the construction is continued recursively. Since $g_n \neq 0$ is a constant, we assume without loss of generality that $g_n = 1$. For the bottom subsystem $\dot{x}_n = u$, the stabilization is immediate: we choose the Lyapunov function $V_n(x_n) = x_n^2$ and any control law $u_n(x_n)$ such that $u_n(0) = 0$, $u_n(x_n)x_n < 0$ in $I\!R\backslash\{0\}$ and $\frac{\partial u_n}{\partial x_n}(0) \neq 0$. This ensures global asymptotic stability (GAS) and local exponential stability (LES) of the equilibrium $x_n = 0$ of $\dot{x}_n = u_n(x_n)$. Then we augment the control law $u_n(x_n)$ by v_{n-1},

$$u_{n-1}(x_{n-1}, x_n) = u_n(x_n) + v_{n-1} \tag{2.5}$$

and design the new term v_{n-1} to stabilize the augmented subsystem

$$\begin{aligned} \dot{x}_{n-1} &= f_{n-1}(x_n) + g_{n-1}(x_n)u_n(x_n) + g_{n-1}(x_n)v_{n-1} \\ \dot{x}_n &= u_n(x_n) + v_{n-1} \end{aligned} \tag{2.6}$$

Our design of the feedback law v_{n-1} is based on a global stability property of the system (2.6) when $v_{n-1}=0$. By *global stability*, we mean that the equilibrium $(x_{n-1},x_n)=(0,0)$ is stable and all the solutions are bounded. To prove global stability of (2.6) and construct a Lyapunov function, we construct a change of coordinates. For notational convenience we let $z_n=x_n$. We start with the auxiliary system

$$\begin{array}{rcl}\dot{\xi}_{n-1} & = & f_{n-1}(\xi_n)+g_{n-1}(\xi_n)u_n(\xi_n) =: h_{n-1}(\xi_n)\\ \dot{\xi}_n & = & u_n(\xi_n), \qquad\qquad (\xi_{n-1}(0),\xi_n(0))=(x_{n-1},x_n)\end{array} \tag{2.7}$$

where the state x of (2.6) plays the role of the initial condition. Writing the solution $\xi_{n-1}(s)$ at time $s\geq 0$ in the explicit form

$$\xi_{n-1}(s)=x_{n-1}+\int_0^s h_{n-1}(\xi_n(\tau))d\tau$$

we note that the integral is uniformly bounded for all $s\geq 0$ because $h_{n-1}(0)=0$ and, after a finite initial period, the convergence of $\xi_n(s)$ to zero is exponential. As a consequence, the limit as $s\to\infty$ of $\xi_{n-1}(s)$ is well-defined and continuous, and we take it as our new state z_{n-1}:

$$z_{n-1}(x_{n-1},x_n)=x_{n-1}+\int_0^\infty h_{n-1}(\xi_n(\tau))d\tau$$

The partial derivatives of z_{n-1} satisfy $\frac{\partial z_{n-1}}{\partial x_{n-1}}=1$ and

$$\frac{\partial z_{n-1}}{\partial x_n}=\int_0^\infty \frac{\partial}{\partial z_n}(h_{n-1}(\xi_n(\tau))d\tau=\int_0^\infty \frac{\partial h_{n-1}}{\partial x_n}(\xi_n(\tau))\eta(\tau)d\tau$$

where $\eta(\tau):=\frac{\partial \xi_n}{\partial x_n}(\tau)$ is the solution of

$$\dot{\eta}=\frac{\partial u_n}{\partial x_n}(\xi_n(\tau))\eta, \qquad \eta(0)=1$$

We note that this is the variational equation of $\dot{x}_n=u_n(x_n)$ for the variations of the initial condition $x_n(0)$. Because $\xi_n(\tau)$ converges to zero and $\frac{\partial u_n}{\partial x_n}(0)$ is a negative constant, the solution $\eta(s)$ converges to zero and, after a finite time, the convergence is exponential. As a consequence, the partial derivative $\frac{\partial z_{n-1}}{\partial z_n}$ exists for all x_n and is continuous. The same argument, repeated for higher order partial derivatives, proves that z_{n-1} is a *smooth* function of its arguments. The change of coordinates

$$\begin{pmatrix} z_{n-1}\\ z_n\end{pmatrix}=T(x_{n-1},x_n)$$

is a global diffeomorphism because

$$\frac{\partial T}{\partial x}=\begin{pmatrix}1 & \frac{\partial z_{n-1}}{\partial z_n}\\ 0 & 1\end{pmatrix}$$

is nonsingular for all (x_{n-1},x_n).

The new state z_{n-1} has the important property to be constant along the solutions of (2.6) when $v \equiv 0$. This fact is established by observing that

$$\forall t \geq 0: \quad \frac{\partial h_{n-1}}{\partial t}(\xi_n(\tau)) = \frac{dh_{n-1}}{d\tau}(\xi_n(\tau)), \tag{2.8}$$

which follows from the chain rule and from the fact that $\frac{\partial \xi_n}{\partial t}(\tau)$ and $\frac{d\xi_n}{d\tau}(\tau)$ are both solutions of the same differential equation

$$\dot{\eta} = \frac{\partial u_n}{\partial x_n}(\xi_n(\tau))\eta, \qquad \eta(0) = u_n(x_n)$$

Using (2.8), we obtain the explicit calculation:

$$\begin{array}{rcl} \dot{z}_{n-1} & = & \dot{x}_{n-1} + \int_0^\infty \frac{\partial h_{n-1}}{\partial t}(\xi_n(\tau))d\tau \\ & = & h_{n-1}(x_n) + \int_0^\infty \frac{dh_{n-1}}{d\tau}(\xi_n(\tau))d\tau \\ & = & h_{n-1}(x_n) + \lim_{s\to\infty} h_{n-1}(\xi_n(s)) - h_{n-1}(\xi_n(0)) \\ & = & h_{n-1}(x_n) - h_{n-1}(\xi_n(0)) = 0 \end{array}$$

Thus, in the new coordinates, with $v = 0$, the system (2.6) becomes

$$\begin{array}{rcl} \dot{z}_{n-1} & = & 0 \\ \dot{z}_n & = & u_n(z_n), \end{array} \tag{2.9}$$

The positive definite function

$$V_{n-1}(z_{n-1}, z_n) = V_n(z_n) + \frac{1}{2}z_{n-1}^2$$

is radially unbounded and nonincreasing along the trajectories of (2.9) since

$$\dot{V}_{n-1} = \frac{\partial V_n}{\partial z_n}u_n(z_n) \leq 0$$

This proves global stability of the equilibrium $(x_{n-1}, x_n) = (z_{n-1}, z_n) = (0, 0)$ of (2.9) and (2.6) with $v_{n-1} = 0$. When we reintroduce the control v_{n-1} in (2.6), we obtain

$$\begin{array}{rcl} \dot{z}_{n-1} & = & \kappa_{n-1}(z_n)v_{n-1}, \\ \dot{z}_n & = & u_n(z_n) + v_{n-1}, \end{array} \tag{2.10}$$

where

$$\kappa_{n-1} := \frac{\partial z_{n-1}}{\partial x}g(x) = \frac{\partial z_{n-1}}{\partial x_n} + g_{n-1} \tag{2.11}$$

Along the trajectories of (2.10), the time-derivative of V_{n-1} becomes

$$\dot{V}_{n-1} \leq v_{n-1}(\frac{\partial V_n}{\partial z_n} + z_{n-1}\kappa_{n-1})$$

Hence its negativity is enhanced with the control law

$$v_{n-1}(z_{n-1}, z_n) = -\frac{\partial V_n}{\partial z_n} - z_{n-1}\kappa_{n-1}(z_n) \tag{2.12}$$

which implies

$$\dot{V}_{n-1} = \frac{\partial V_n}{\partial z_n} u_n(z_n) - (\frac{\partial V_n}{\partial z_n} - z_{n-1}\kappa_{n-1}(z_n))^2 \leq 0$$

The right-hand side vanishes only if

$$z_n = 0 \text{ and } v_{n-1}(z_{n-1}, 0) = \kappa_{n-1}(0) z_{n-1} = 0 \tag{2.13}$$

Because the Jacobian linearization of (2.6) is controllable, this property is not lost in the new coordinates, and from (2.10), it is apparent that $\kappa_{n-1}(0) \neq 0$. As a consequence, (2.13) is satisfied only at the origin $(z_{n-1}, z_n) = (0, 0)$. This proves that $\dot{V}_{n-1}$ is negative definite.

Hence, by exploiting the GAS and LES of the equilibrium $x_n = 0$ of $\dot{x}_n = u_n(x_n)$, we have augmented the control law by (2.12) to achieve GAS and LES of the equilibrium $(x_{n-1}, x_n) = (0, 0)$ of the augmented system (2.6). This procedure is now repeated recursively for $i = 1, \ldots, n-1$. We define

$$u_i(x_i, \ldots, x_n) = u_{i+1}(x_{i+1}, \ldots, x_n) + v_i, \tag{2.14}$$

and show how to design the additional control v_i, given that the feedback $u_{i+1}(x_{i+1}, \ldots, x_n)$ has achieved global asymptotic stability (GAS) and local exponential stability (LES) of the equilibrium $(x_{i+1}, \ldots, x_n) = (0, \ldots, 0)$ of the subsystem

$$\left\{ \begin{array}{rcl} \dot{x}_{i+1} & = & f_{i+1}(x_{i+2}, \ldots, x_n) + g_{i+1}(x_{i+2}, \ldots, x_n) u_{i+1}(x_{i+1}, \ldots, x_n) \\ & \vdots & \\ \dot{x}_n & = & u_{i+1}(x_{i+1}, \ldots, x_n) \end{array} \right. \tag{2.15}$$

denoted in vector form by

$$\dot{X}^{i+1} = F^{i+1}(X^{i+1}) \tag{2.16}$$

The additional control v_i is used to stabilize the augmented system

$$\left\{ \begin{array}{rcl} \dot{x}_i & = & f_i(X^{i+1}) + g_i(X^{i+1})(u_{i+1}(X^{i+1}) + v_i) \\ \dot{X}^{i+1} & = & F^{i+1}(X^{i+1}) + G^{i+1}(X^{i+1}) v_i \end{array} \right. \tag{2.17}$$

To design v_i, we denote $\Xi^{i+1} = (\xi_{i+1}, \ldots, \xi_n)$ and consider the auxiliary system

$$\left\{ \begin{array}{rcl} \dot{\xi}_i & = & f_i(\Xi^{i+1}) + g_i(\Xi^{i+1}) u_{i+1}(\Xi_{i+1}) =: h_i(\Xi^i) \\ \dot{\Xi}^{i+1} & = & F^{i+1}(\Xi^{i+1}), \qquad\qquad (\xi_i(0), \Xi^{i+1}(0)) = (x_i, X^{i+1}) \end{array} \right. \tag{2.18}$$

where the state of (2.17) plays the role of the initial condition. The solution for $\xi_i(s)$ is explicitely

$$\xi_i(s) = x_i + \int_0^s h_i(\Xi^i(\tau)) d\tau$$

After a initial period, the integrand converges exponentially to zero, so the limit of the integral as $s \to \infty$ is well-defined and we have the following result:

Theorem 1 *The function*

$$z_i(x_i, X^{i+1}) = \lim_{s\to\infty} \xi_i(s) \tag{2.19}$$

is smooth and the change of coordinates $(x_i, X^{i+1}) \to (z_i, X^{i+1})$ *is a global diffeomorphism.*

Proof: Crucial to the proof is the existence and smoothness of the integral

$$\int_0^\infty h_i(\Xi^i(\tau))d\tau$$

The proof is an extension of the arguments used for the subsytem (2.6) and can be found in (Jankovic *et al.*, 1995) (Theorem 1, Theorem 2, and Corollary 1). The transformation $(z_i, X^{i+1}) = T(X^i)$ is a global diffeomorphism because the matrix $\frac{\partial T}{\partial x}$ is upper-triangular with ones on the diagonal. □

With the help of the change of coordinates (2.19), the stability analysis of (2.17) with $v_i = 0$ becomes transparent, because in the new coordinates the system takes the form

$$\left\{ \begin{array}{rcl} \dot{z}_i & = & 0 \\ \dot{X}^{i+1} & = & F^{i+1}(X^{i+1}) \end{array} \right. \tag{2.20}$$

If $V^{i+1}(X^{i+1})$ is a Lyapunov function for (2.16), then $V^i = V^{i+1} + z_i^2$ is a Lyapunov function for (2.17) because it is smooth, radially unbounded, positive definite, and nonincreasing along the solutions of (2.20).

Reintroducing the control v_i in (2.20), we obtain

$$\begin{pmatrix} \dot{z}_i \\ \dot{X}^{i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ F^{i+1}(X^{i+1}) \end{pmatrix} + \begin{pmatrix} \kappa_i(X^{i+1}) \\ G^{i+1}(X^{i+1}) \end{pmatrix} v_i$$

where

$$\kappa_i(X^{i+1}) = \frac{\partial z_i}{\partial x} g(x) = g_i(X^{i+1}) + \frac{\partial z_i}{\partial X^{i+1}}(X^{i+1})G^{i+1}(X^{i+1})$$

Hence, the negativity of $\dot{V}^i$ is enhanced with the control law

$$v_i = -2\kappa_i z_i - \frac{\partial V^{i+1}}{\partial x_n} \tag{2.21}$$

which yields

$$\dot{V}^i = \frac{\partial V^{i+1}}{\partial X^{i+1}}.F^{i+1} - v_i^2 \leq 0$$

By LaSalle's Invariance Principle, the solutions converge to an invariant set where $\frac{\partial V^{i+1}}{\partial X^{i+1}}F^{i+1} = v_i = 0$. In this set $X^{i+1} = 0$ and $v_i = -\kappa_i(0)z_i = 0$. Because the Jacobian linearization of (2.17) is controllable, this property is not lost in the new coordinates, and therefore $\kappa_i(0) \neq 0$. Hence $v_i = -\kappa_i(0)z_i = 0$ implies $z_i = 0$ and global asymptotic stability of the equilibrium $X^i = 0$ is established. We summarize this discussion as follows:

Theorem 2 *The origin* $X^i = 0$ *of (2.17) is globally stable. With the feedback (2.21), the origin is globally asymptotically stable and locally exponentially stable.*

Starting from the system $\dot{x}_n = u$, the conclusions of Theorem 2 can be recursively employed to achieve global stabilization of (1.1). An accompanying Lyapunov function is recursively build by starting with $V^n = x_n^2 = z_n^2$ and adding z_i^2 at each step. At the last step, this yields the simple quadratic form $V = z^T z$.

The change of coordinates of Theorem 1 has an important geometric interpretation. By construction, the manifold $z_i = 0$ is the *stable manifold* of the system (2.17) when $v_i = 0$. To achieve *asymptotic stability* of the equilibrium $X^i = 0$, our control laws render this manifold attractive. This requires the determination of the stable manifold. Its global existence as a consequence of the triangular structure of the system was proven in (Yang, 1993) and the C^1-differentiability of (2.19) was proven in (Mazenc and Praly, 1994). However, instead of pursuing this exact transformation, Mazenc and Praly have proposed a change of coordinates which only approximates the stable manifold but is algebraically solvable. In this case, because the approximation is valid only for $\|X^{i+1}\|$ small, the control law must be saturated when $\|X^{i+1}\|$ is large. For the feedforward structure considered in this paper, the approximation of the manifold coincides with its tangent approximation at the origin, and the corresponding change of coordinates is the same as in Teel's design.

3 A global lower triangular form

The recursive methodology described in the preceding section offers a considerable flexibility in the choice of feedback. At each step, it is sufficient to choose the additional feedback v_i such that the augmented Lyapunov function has a negative definite time derivative. To obtain the main result of this paper we choose at each step, instead of (2.21), the simplified feedback

$$v_i = -\kappa_i(z_{i+1}, \ldots z_n) z_i \tag{3.22}$$

With this choice we obtain the following decomposition:

Theorem 3 *There exists a global diffeomorphism $z = T(x)$ and a feedback transformation*

$$u = -\kappa^T(z) z + v \tag{3.23}$$

with $\kappa^T(z) = [\kappa_1(z) \ldots \kappa_n(z)]$, which transform the feedforward system $\dot{x} = f(x) + g(x)u$, with f and g in the upper-triangular form (1.1), into the lower-triangular form

$$\dot{z} = L(z) z + \kappa(z) v \tag{3.24}$$

where

$$L(z) = \begin{pmatrix} -\kappa_1^2 & 0 & \ldots & & 0 \\ -\kappa_1\kappa_2 & -\kappa_2^2 & 0 & \ldots & 0 \\ \vdots & -\kappa_2\kappa_3 & -\kappa_3^2 & \ddots & \\ & \vdots & & \ddots & \vdots \\ -\kappa_n\kappa_1 & -\kappa_n\kappa_2 & \ldots & & -\kappa_n^2 \end{pmatrix}$$

Before proving the theorem, we show an immediate property of (3.24):

Proposition 1 *With $v \equiv 0$, the equilibrium $z = 0$ of (3.24) is GAS and LES. The Lyapunov function $V = z^T z$ has a negative definite time derivative*

$$\dot{V} = -\phi^T(z) Q \phi(z) < 0, \ z \neq 0 \tag{3.25}$$

with Q a positive definite constant matrix and $\phi(z) = [\kappa_1(z)z_1 \dots \kappa_n(z)z_n]^T$.

Proof: Denote by C a lower triangular matrix filled with ones and define $K := \mathrm{diag}\{\kappa_1, \dots, \kappa_n\}$. Then, using $v = 0$ in (3.24) we get

$$\dot{z} = -KCK^T z$$

and, hence,

$$\dot{V} = -(z^T K)(C + C^T)(Kz) = -\phi^T(z)(C + C^T)\phi(z)$$

The symmetric matrix $Q = C + C^T$ is positive definite because it has $n-1$ eigenvalues equal to one and one eigenvalue equal to $n+1$. To prove that $\dot{V}$ is negative definite, we show

$$\| \phi \| = \sum_{i=1}^{n} \kappa_i^2(z) z_i^2 = 0 \Rightarrow \| z \| = 0 \tag{3.26}$$

Suppose $\| \phi \| = 0$. Because $\kappa_n = 1$, this implies $z_n = 0$. But κ_{n-1} is function of z_n only and $\kappa_{n-1}(0) \neq 0$. Thus $z_{n-1} = 0$. Proceeding with the same argument, we show that $z_i = 0$ for each i, i.e. $\| z \| = 0$, which proves (3.26).

The local exponential stability is established if $\dot{V}$ is locally quadratic. By construction,

$$\frac{\partial^2 \dot{V}}{\partial z^2}(0) = K(0) Q K^T(0)$$

which is not singular because none of the κ_i's vanishes at the origin. □

Proof of Theorem 3: The proof is by induction: starting with the bottom subsystem $\dot{x}_n = u$, we recursively apply the change of coordinates (2.19) and the control (3.22). At each step, we augment the system by one equation and obtain the lower triangular form (3.24) in the z-coordinates.

Initial step: We let $z_n = x_n$, $\kappa_n = 1$ and $v_n = -z_n$. The diffeormorphism $z_n = x_n$ and the feedback transformation $u = -z_n + v$ yields (3.24) for this scalar case:

$$\dot{z}_n = -\kappa_n^2 z_n + \kappa_n v \tag{3.27}$$

Induction step: Assume that the subsystem (2.16) has the form (3.24) in the z-coordinates. By Proposition 1, when $v = 0$ the equilibrium $X^{i+1} = 0$ is GAS and LES. Applying the change of coordinates (2.19) of Theorem 1, we obtain the augmented system

$$\begin{aligned} \dot{z}_i &= \kappa_i v_i \\ \dot{Z}^{i+1} &= L^{i+1}(Z^{i+1}) Z^{i+1} + \kappa^{i+1}(Z^{i+1}) v_i \end{aligned} \tag{3.28}$$

The choice of $v_i = -\kappa_i z_i + v$ corresponds to the feedback transformation $u = -\sum_{k=1}^{i} \kappa_k z_k + v$ and yields the form (3.24) for the augmented system. $\square$

The choices (2.21) and (3.22) can be considered as two implementations of the same recursive design technique: at each step, the new control v_i is designed to achieve asymptotic stability of the corresponding equilibrium which is only stable when $v_i = 0$; a Lyapunov function V_i is constructed to be nonincreasing along the trajectories of the "uncontrolled system" ($v_i = 0$); finally, a feedback law is chosen to dissipate the "energy" represented by the Lyapunov function. The choice in the preceding section is standard:

$$v_i = -\frac{\partial V_i}{\partial x} g(x) \tag{3.29}$$

Instead, the choice (3.22) in this section corresponds to

$$v_i = -\frac{1}{2} \frac{\partial (V_i - V_{i+1})}{\partial x} g(x) \tag{3.30}$$

and results in two distinct features of the closed-loop system. First, it renders the system lower-triangular in the z-coordinates. As a consequence, the dynamics of the closed-loop system can be decomposed into a succession of nested invariant manifolds: the trajectories are attracted to the manifold $z_1 = 0$, which is invariant. In this manifold, the trajectories are attracted to the manifold $z_1 = z_2 = 0$, which is also invariant. Eventually, the trajectories are attracted to the invariant manifold $z_1 = \ldots = z_{n-1} = 0$, and on which the dynamics reduces to $\dot{z}_n = -z_n$.

The second property resulting from the the choice (3.30) is that at each step,

$$u_i = -\frac{1}{2} \frac{\partial V_i}{\partial x} g(x) \tag{3.31}$$

This is a consequence of the additive structure of both V_i and v_i:

$$\begin{aligned} V_i &= V_{i+1} + z_i^2 \\ u_i &= u_{i+1} + v_i \\ &= -\frac{1}{2} \frac{\partial V_{i+1}}{\partial x} g(x) - z_i \frac{\partial z_i}{\partial x} g(x) \\ &= -\frac{1}{2} \frac{\partial V_i}{\partial x} g(x) \end{aligned}$$

In particular, the final expression of the control law is in the form (3.31):

$$u = -\frac{1}{2} \frac{\partial V}{\partial x} g(x) \tag{3.32}$$

This form achieves the optimality and robustness properties discussed in the next section. We point out that the form (3.32) can also be achieved if the standard choice (3.29) is made at each each step but this requires a modification of the Lyapunov function V.

4 Optimality and robustness properties

In the z-coordinates, for the Lyapunov function $V = z^T z$, the control (3.32) is

$$u = -\kappa^T(z)z = -\frac{1}{2}\frac{\partial V}{\partial z}(z)\kappa(z) \tag{4.33}$$

This implies the following optimality property for the modified ("doubled") control law $u = -2\kappa^T(z)z$:

Proposition 2 *The control law $u = -2\kappa^T z$ achieves GAS and LES of the equilibrium $z = 0$ and minimizes the cost functional*

$$J(z,u) = \int_0^\infty (\phi^T(z)Q\phi(z) + \frac{1}{2}u^2)dt$$

The optimal value function $J(z, u(z))$ is $V(z)$.

Proof: Using (3.25), a direct calculation following (Jacobson, 1977) shows that $V = z^T z$ is the positive definite solution of the Hamilton-Jacobi-Bellman equation

$$\phi^T(z)Q\phi(z) - \frac{\partial V}{\partial z}L(z)z - \frac{1}{2}(\frac{\partial V}{\partial z}\kappa)^2 = 0$$

in which case the optimal control has the form $u = -\frac{\partial V}{\partial z}\kappa = -2\kappa^T z$ □

We note that if the system (1.1) is linear in x, the vector $\phi(z)$ is linear in z. In this case, the cost on the state and the cost on the control are both quadratic and the forwarding design solves a particular LQR problem.

As a consequence of its optimality for the cost functional $J(x,u)$, the control law $u = -2\kappa^T z$ has the stability margins of LQ control laws (Glad, 1987), (Tsitsiklis and Athans, 1984).

Proposition 3 *The control law $u = -2\kappa^T z$ achieves global asymptotic stability of the equilibrium $z = 0$ with the following stability margins:*

- *a sector margin $(\frac{1}{2}, \infty)$, that is $u = \psi(-2\kappa^T z)$ achieves global asymptotic stability for any nonlinearity ψ satisfying the sector condition*

$$\frac{1}{2}s^2 \leq s\psi(s)$$

- *a phase margin $[0, 60°)$, that is $u = \psi(-2\kappa^T z)$ achieves global asymptotic stability for any pure phase shift ψ, smaller than $60°$ at all frequencies.*

The sector margin property can be employed to achieve the following input-to-state stability in the sense of (Sontag, 1989):

Proposition 4 *Consider the system (1.1) with the perturbed last equation*

$$\dot{x}_n = u + d(x)w \tag{4.34}$$

where $w \in L^\infty$ is an exogeneous disturbance. For a suitable function $\gamma \in \mathcal{K}_\infty$, the feedback law

$$u = -(1 + \gamma(\| z \|^2))\kappa^T z \tag{4.35}$$

achieves input-to-state stability with respect to w. □

Proof: If we consider the term $\gamma(\| z \|^2))\kappa^T z$ as the additional feedback v in the feedback transformation (3.23), we obtain from Proposition 1

$$\dot{V} \leq - \| \phi \|^2 - \gamma(\| z \|^2)(\frac{\partial V}{\partial z}\kappa)^2 + \| d(z) \| \| w \|_\infty | \frac{\partial V}{\partial z}\kappa |$$

It follows that $\dot{V}$ remains negative definite as long as

$$\| w \|_\infty \| d(z) \| | \frac{\partial V}{\partial z}\kappa | < 2\gamma^{1/2}(\| z \|) \| \phi \| | \frac{\partial V}{\partial z}\kappa | \tag{4.36}$$

Choose $\chi \in \mathcal{K}_\infty$ such that

$$\chi(\| z \|) \| d(z) \| \leq \gamma^{1/2}(\| z \|) \| \phi \|$$

Such a function always exists if, for instance, $\gamma(\| z \|) \geq \| d(z) \|^2 \| z \|$. Recall from Corollary 1 that $\| \phi \|$ vanishes only when $\| z \| = 0$. This means that $\chi^{-1} \in \mathcal{K}_\infty$ satisfies

$$\| z \| > \chi^{-1}(\| w \|_\infty) \Rightarrow \dot{V} \leq -\frac{1}{2} \| \phi \|^2 \tag{4.37}$$

which is sufficient (see (Sontag,1989)) to establish the ISS-property of the system. □

The modification of the control law $u = -2\kappa^T(z)z$ into (4.35) does not impair its optimality properties. This is because (4.35) is again of the form

$$u(z) = -\frac{1}{2}\frac{\partial \tilde{V}}{\partial z}(z).\kappa(z)$$

with respect to the modified Lyapunov function

$$\tilde{V}(z) = \int_0^{\|z\|^2} (1 + \gamma(s))ds$$

Following the proof of Proposition 2, this implies that the control

$$u = -2(1 + \gamma(\| z \|^2))\kappa^T z \tag{4.38}$$

is also optimal, and minimizes the modified cost functional

$$J(z,u) = \int_0^\infty ((1 + \gamma(\| z \|^2)\phi^T(z)Q\phi(z) + \frac{1}{2}u^2)dt$$

Hence the control (4.38) achieves simulatneously an ISS-property with respect to input disturbances and an optimality property with respect to a cost functional with a quadratic cost on the control.

5 A benchmark example

As an illustration of the forwarding design we consider a slight modification of the nonlinear benchmark problem proposed in (Kokotovic *et al.*, 1991)

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 + (x_2 - x_3)^2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & u + w \end{array} \tag{5.39}$$

where $w \in L^\infty$ is an exogeneous disturbance.

The unperturbed ($w = 0$) system (5.39) is not feedback linearizable. The recursive steps of our design yield

$$z_3 = x_3, \ \kappa_3 = 1, \ z_2 = x_2 + x_3, \ \kappa_2 = 1 \tag{5.40}$$

$$z_1 = x_1 + \int_0^\infty (\xi_2(s) + (\xi_2(s) - \xi_3(s))^2) ds \tag{5.41}$$

By construction, $(\xi_2(s), \xi_3(s))$ is the solution of the subsystem

$$\begin{array}{rcl} \dot{\xi}_2 & = & \xi_3, \ \xi_2(0) = x_2, \\ \dot{\xi}_3 & = & -2\xi_3 - \xi_2, \ \xi_3(0) = x_3 \end{array},$$

Substituting its solution in (5.41) and integrating we obtain

$$z_1 = x_1 + 2x_2 + x_3 + \frac{1}{2}(x_2 + x_3)^2 + 2x_2^2, \tag{5.42}$$

$$\kappa_1 = 1 + x_2 + x_3$$

The change of coordinates $z = T(x)$ and the feedback law

$$u = -\kappa^T z + v \tag{5.43}$$

transforms the benchmark system (5.39) into

$$\begin{array}{rcl} \dot{z}_1 & = & -(1 + z_2 + z_3)^2 z_1 \\ \dot{z}_2 & = & -z_2 - (1 + z_2 + z_3) z_1 \\ \dot{z}_3 & = & -z_3 - z_2 - (1 + z_2 + z_3) z_1 \end{array} + \begin{pmatrix} (1 + z_2 + z_3) \\ 1 \\ 1 \end{pmatrix} (v + w)$$

When $w = 0$, the feedback law (5.43) (with $v = 0$) achieves GAS and LES of the origin with an infinite sector margin. By Proposition 2, the doubled-gain control law $u = -2\kappa^T z$ minimizes a cost functional with a quadratic control penalty. To also achieve an ISS-property with respect to w, we employ the additional control law $v = -2 \parallel z \parallel \kappa^T(z) z$, as described in Proposition 4.

6 Conclusion

We have addressed the global stabilization problem for a class of nonlinear systems which, in general, are not feedback linearizable. With a recursive construction, we have shown that the original upper-triangular form can be transformed by a diffeormorphism and feedback into a global lower-triangular form. Our Lyapunov function is quadratic in the transformed coordinates and has a negative definite time derivative. In addition to global asymptotic stability and local exponential stability, our controller has optimality properties and also guarantees two important robustness properties: an infinite sector margin and an ISS-property with respect to matched uncertainties.

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