Global Adaptive Stabilization of Cascade Nonlinear Systems

Mrdjan Jankovic
Ford Scientific Research Laboratories
P.O. Box 2053, MD 2036 SRL
Dearborn, MI 48121
e-mail: mjankov1@ford.com

Rodolphe Sepulchre, Petar V. Kokotovic*
Dept. of Electrical and Computer Engineering
University of California
Santa Barbara, CA 93106

Abstract

An adaptive controller is developed for a class of nonlinear systems for which other approaches (e.g. backstepping) may fail to be applicable. The new design employs feedback passivation and an explicit construction of a Lyapunov function for the zero dynamics with respect to a relative degree 1 output. As an illustration a longstanding "nonlinear benchmark problem" is solved.

Keywords: nonlinear systems, adaptive control, feedback passivation

*This work was supported in part by the National Science Foundation under Grant ECS-9203491 and in part by the Air Force Office of Scientific Research under Grant F-49620-92-J-0495.

1 Introduction

A significant development in adaptive control of nonlinear systems is adaptive integrator backstepping [5], [6], [8], [9], [10] which applies to systems in the "pure feedback" and "strict feedback" forms. The purpose of this work is to present a systematic design procedure for adaptive stabilization of nonlinear systems which do not appear in either of the above two forms.

A motivation for this study is the sixth benchmark example in [6] which cannot be stabilized using adaptive integrator backstepping:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_3^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \tag{1.1}$$

In this system θ is a constant unknown parameter. This system is not feedback linearizable even when θ is known. A nonadaptive solution to this problem, when a bound on θ is known, was given by Teel [15]. However, when a bound on θ is unknown, this problem has remained unsolved.

Our approach employs feedback passivation [1], [7] as a design tool for adaptive control and is similar to those of [13] and [14]. The main difference is that we do not assume the knowledge of the feedback passivating input transformation, nor the knowledge of the Lyapunov function for the zero dynamics. Instead, we construct a Lyapunov function using our recent method [3] which applies to a cascade of a stable system and an exponentially stable system.

Our adaptive control design applies to nonlinear systems in the form

$$\begin{aligned} \dot{x} &= Fx + H(\xi)x + h^T(\xi)\theta + g^T(x,\xi,y)\theta\, y \\ \dot{\xi} &= A\xi + b^T(x,\xi,y)\theta\, y \\ \dot{y} &= \varphi^T(x,\xi,y)\theta + u \end{aligned} \tag{1.2}$$

where $H(\xi)$ and $h(\xi)$ satisfy $H(0) = 0$ and $h(0) = 0$. The states of the system are $x \in R^{n_x}$, $\xi \in R^{n_\xi}$, $y \in R$, input $u \in R$, and the unknown parameters are $\theta \in R^p$. We assume that the right hand side of (1.2) is C^r with $r > 1$. The assumption that y and u are scalars is introduced only for notational convenience. All the results apply when y and u are m vectors. For the system (1.2) we make the following assumption:

(H1) There exist positive definite matrices P_F, P_A, Q_A and a positive semidefinite matrix Q_F such that

$$\begin{aligned} F^T P_F + P_F F &= -Q_F \\ A^T P_A + P_A A &= -Q_A \end{aligned}$$

This means that $\dot{x} = Fx$ is only stable, rather than asymptotically stable, so that even $F = 0$ is admissible.

With a simple change of variables, the nonlinear benchmark problem (1.1) can be put in the form (1.2) which proves that the nonlinear systems we consider here, in general, are not feedback linearizable.

It is important to point out that the system (1.2) may fail to be controllable or even stabilizable for every vector $\theta \in R^p$. In general, its stabilizability is restricted to $\theta \in \Omega_{\mathcal{S}} \subset I\!R^p$. Even when $\theta \notin \Omega_{\mathcal{S}}$ our adaptive controller achieves boundedness of all the signals. When θ belongs to a set $\Pi \subset \Omega_S$, which will be characterized later, this adaptive controller achieves regulation of the states to 0.

Throughout the paper we say that a system is Lyapunov stable (asymptotically stable) to mean that its equilibrium at the origin is Lyapunov stable (asymptotically stable). We say that a system $\dot{z} = f(z)$ is globally stable if there exists a positive definite, radially unbounded, function $V(z)$ such that $\frac{\partial V}{\partial z} f \leq 0$ for all z. The solution of $\dot{z} = f(z)$ with the initial condition $z(s_0) = z_0$ is denoted by $\bar{z}(s; z_0, s_0)$. If the initial condition and initial time are clear we just write $\bar{z}(s)$.

2 Lyapunov Function for the Zero Dynamics

Let us first show that the zero dynamics of (1.2) with respect to the output y, which is given by

$$\begin{aligned} \dot{x} &= Fx + H(\xi)x + h^T(\xi)\theta \\ \dot{\xi} &= A\xi \end{aligned} \tag{2.1}$$

is globally stable. We prove this by constructing a Lyapunov function $W(x, \xi)$ for (2.1) using the results of [3] (see also [4]) which apply to more general nonlinear systems than (2.1):

$$\begin{aligned} \dot{x} &= f(x) + h(x, \xi) \\ \dot{\xi} &= A\xi \end{aligned} \tag{2.2}$$

where $\dot{x} = f(x)$ is stable, $h(x, \xi)$ is globally Lipschitz in x for any fixed ξ, and A is Hurwitz. To achieve linear parameterization of the control law for the system (1.2) the x-dynamics in (2.1) is restricted to be a linear function of x.

Following [3], we define a function $\Psi(x, \xi)$ as a line integral along the solution $(\bar{x}, \bar{\xi})$ of (2.1) starting at (x, ξ) at time 0:

$$\Psi(x, \xi) = \int_0^\infty 2\bar{x}^T(s; (x, \xi), 0) P_F [H(\bar{\xi}(s; \xi, 0))\ \bar{x}(s; (x, \xi), 0) + h^T(\bar{\xi}(s; \xi, 0))\theta]\ ds \tag{2.3}$$

The time derivative of the line integral (2.3) is

$$\dot{\Psi} = -2x^T P_F [H(\xi)x + h^T(\xi)\theta]$$

This expression shows that $\Psi(x, \xi)$ is a convenient nonquadratic cross-term in the following Lyapunov function candidate:

$$W(x, \xi) = x^T P_F x + \Psi(x, \xi) + \xi^T P_A \xi \tag{2.4}$$

Thanks to this term the derivative of W along the solutions of (2.1) satisfies

$$\dot{W} = -x^T Q_F x - \xi^T Q_A \xi \leq 0 \tag{2.5}$$

To show that $W(x,\xi)$ is indeed a valid Lyapunov function, we employ Theorems 1 and 2, and Corollary 1 of [3] and prove the following theorem.

Theorem 2.1: If (H1) is satisfied then:

1. Ψ is a well defined, C^r function of x and ξ,
2. $W(x,\xi)$ is positive definite,
3. $W(x,\xi)$ is radially unbounded,
4. The system (2.1) is globally stable.

Our design of the adaptive controller requires that, possibly after reparameterization, the control law be a linear function of the unknown parameters. This will be satisfied if the Lyapunov function for the zero dynamics is a polynomial function of the parameters. The above construction satisfies this requirement because, as we show below, the cross term Ψ is a polynomial of degree 2 in θ.

The trajectory of the system (2.1), with the initial condition (x,ξ) at time $s=0$, is given by

$$\begin{aligned} \bar{x}(s) &= \Phi_\xi(s,0)x + \int_0^s \Phi_\xi(s,\tau)h^T(\bar{\xi}(\tau))\theta \, d\tau \triangleq \Phi_\xi(s,0)x + J^T(\xi,s)\theta \\ \bar{\xi}(s) &= e^{As}\xi \end{aligned} \tag{2.6}$$

where $\Phi_\xi(s,t)$ satisfies

$$\dot{\Phi}_\xi(s,t) = [F + H(\bar{\xi}(s))]\Phi_\xi(s,t), \quad \Phi_\xi(t,t) = I$$

Substituting $\bar{x}$ and $\bar{\xi}$ into (2.3) we obtain the cross term as a quadratic polynomial in θ:

$$\begin{aligned} \Psi(x,\xi) &= \int_0^\infty 2(\theta^T J(\xi,s) + x^T\Phi_\xi^T(s,0))P_F[H(\bar{\xi}(s))(\Phi_\xi(s,0)x + J^T(\xi,s)\theta) + h^T(\bar{\xi}(s))\theta] \, ds \\ &\triangleq x^T\Psi_0(\xi)x + x^T\Psi_1(\xi)\theta + \theta^T\Psi_2(\xi)\theta \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \Psi_0(\xi) &= 2\int_0^\infty \Phi_\xi^T P_F H \Phi_\xi \, ds \\ \Psi_1(\xi) &= 2\int_0^\infty \Phi_\xi^T[(P_F H + H^T P_F)J^T + P_F h^T] \, ds \\ \Psi_2(\xi) &= 2\int_0^\infty J P_F[H J^T + h^T] \, ds \end{aligned} \tag{2.8}$$

Substituting (2.7) into (2.4) we get

$$W(x,\xi) = x^T P_F x + x^T\Psi_0(\xi)x + x^T\Psi_1(\xi)\theta + \theta^T\Psi_2(\xi)\theta + \xi^T P_A \xi \tag{2.9}$$

Ψ_i's are globally defined functions which are differentiable as many times as the functions $H(\xi)$ and $h(\xi)$ in (1.2); Ψ_i's are analytic if these functions are analytic. In general, functions

Ψ_i will have to be evaluated numerically using standard approximation methods. In a number of special cases, including the nonlinear benchmark problem (1.1), the functions Ψ_i can be explicitly computed. In fact, closed form solutions for Ψ_i's can always be computed for an important subclass of (2.1) which is obtained when $h(\xi)$ is polynomial in ξ and $H(\xi) \equiv 0$.

3 Feedback Passivating Control

Let us first assume that the parameter vector θ is known and design a controller which achieves global stability of the system (1.2). Our design employs feedback passivation [1] which we briefly review. With $\sigma \triangleq [x\ \xi]^T$ the system (1.2) can be rewritten as

$$\begin{aligned} \dot{\sigma} &= q(\sigma) + p(\sigma, y)y \\ \dot{y} &= \varphi'(\sigma, y) + u \end{aligned} \tag{3.1}$$

where the vector fields q, p and φ' are obtained from (1.2). With y as the output, the system (3.1) is relative degree 1 and weakly minimum phase; i.e. the Lyapunov function W given by (2.9) satisfies $\frac{\partial W}{\partial \sigma} q(\sigma) \leq 0$. Then the feedback transformation

$$u = -\varphi'(\sigma, y) - \frac{\partial W}{\partial \sigma} p(\sigma, y) + v \tag{3.2}$$

renders the system passive from the new input v to the output y. Indeed, it is straightforward to check that the Lyapunov function defined by

$$V(x, \xi, y) = W(x, \xi) + \frac{1}{2} y^2 \tag{3.3}$$

satisfies:

$$\dot{V} = \frac{\partial W}{\partial \sigma} q + yv \leq yv \tag{3.4}$$

By closing the loop from y to v with any strictly passive system we achieve global stability, and, under an additional assumption, global asymptotic stability. For this, we can choose the simplest strictly passive feedback

$$v = -y \tag{3.5}$$

In the (x, ξ, y)-coordinates the control law (3.2), (3.5) takes the form

$$\begin{aligned} u(x, \xi, y, \theta) &= -y - \varphi(x, \xi, y)^T \theta - \frac{\partial W}{\partial x} g^T(x, \xi, y)\theta - \frac{\partial W}{\partial \xi} b^T(x, \xi, y)\theta \\ &= -y - [\varphi^T + 2x^T(P_F + \Psi_0)g^T + 2\xi^T P_A b^T + x^T \sum_{i=1}^{n_x} x_i \frac{\partial \Psi_{0(i)}}{\partial \xi} b^T]\theta \\ &\quad -\theta^T [\Psi_1^T g^T + \sum_{i=1}^{n_x} x_i \frac{\partial (\Psi_1^T)_{(i)}}{\partial \xi} b^T]\theta - \theta^T \sum_{i=1}^{p} \theta_i \frac{\partial \Psi_{2(i)}}{\partial \xi} b^T \theta \end{aligned} \tag{3.6}$$

where the subscript (i) denotes the i-the column of the corresponding matrix. Without any restriction on θ this control law guarantees boundedness and output regulation.

Proposition 3.1: The states of the closed loop system (1.2), (3.6) are uniformly bounded and ξ and y converge to zero for all $\theta \in R^p$.

Proof: Because $\frac{\partial W}{\partial \sigma} q = -x^T Q_F x - \xi^T Q_A \xi$, by substituting (3.5) into (3.4) we obtain

$$\dot{V} = -x^T Q_F x - \xi^T Q_A \xi - y^2 \leq 0 \tag{3.7}$$

which proves that the states of the closed loop system (1.2), (3.6) are uniformly bounded. By LaSalle-Yoshizawa Theorem ([8], Theorem 2.1), $\dot{V} \to 0$ and, because each of the three terms in $\dot{V}$ has the same sign, we conclude that it must converge to 0. It follows that $y \to 0$ and, since Q_A is positive definite, $\xi \to 0$.

□

The above result holds for all θ, i.e. regardless of the stabilizability of (1.2). To prove the global asymptotic stability of the closed loop system (1.2), (3.6), we need one additional assumption constraining θ. The set $\Omega_{\mathcal{S}}$ of θ for which (1.2) is stabilizable, is, in general, very difficult to characterize. Instead, we define a set $\mathcal{P} \subset \Omega_{\mathcal{S}}$ as follows:

Definition 3.2: $\mathcal{P}$ is the set of all $\theta \in R^p$ for which the following condition holds:

(H2) $x = 0$ is the only solution of the set of equations

$$\begin{aligned} x^T(Q_F F^k)x = 0, &\quad k = 0, \ldots, n_x - 1 \\ L^i_{(Fx)}\chi(x,\theta) = 0, &\quad i = 0, 1, \ldots \end{aligned} \tag{3.8}$$

where $\chi(x,\theta) = \varphi^T(x,0,0)\theta + u(x,0,0,\theta)$, $L_{(Fx)}\chi \triangleq \frac{\partial \chi}{\partial x} Fx$, and $L^{i+1}_{(Fx)}\chi \triangleq L_{(Fx)}(L^i_{(Fx)}\chi)$.

The condition (H2) can be related to the controllability of the system (1.2) using the results of [11], [12]. Indeed, it can be shown that if the controllability distribution is nonsingular at every point (x, ξ, y) where $\xi = 0, y = 0$, then the condition (H2) is satisfied. For our purpose it is more significant that, under this condition, the largest invariant set $E \subset I\!R^{n_x + n_\xi + 1}$ of the closed loop system (1.2), (3.6) where $\dot{V} = 0$ is just the origin. Another important point is that even when the closed form solution for the control law cannot be obtained, the equations (3.8) are given in the closed form as illustated in Example 4.

With the characterization of the set of feasible parameter vectors θ given by Definition 3.2, we can prove the following result:

Proposition 3.3: If $\theta \in \mathcal{P}$ then the closed loop system (1.2), (3.6) is globally asymptotically stable.

Proof: Boundedness of the states follows from Proposition 3.1. By LaSalle's Invariance Principle ([8], Theorem 2.2), the states converge to E, the largest invariant set of the closed loop system (1.2), (3.6) where $\dot{V} = -x^T Q_F x - \xi^T Q_A \xi - y^2 = 0$. We immediately obtain that, for $(x, \xi, y) \in E$,

1. $\xi = 0$, $y = 0$
2. $\dot{x} = Fx$
3. $x^T Q_F x \equiv 0$
4. $\dot{y} = \chi(x, \theta) \equiv 0$

To prove the proposition, it suffices to show that the above four conditions can be satisfied only for $x = 0$. From 2 and 3, we conclude that, in E, $\frac{d^k}{dt^k}(x^T Q_F x) = x^T (Q_F F^k) x = 0, \quad k = 0, \ldots, n_x - 1$. From 2 and 4 we have that $\frac{d^i}{dt^i}\chi(x,\theta) = L^i_{(Fx)}\chi(x,\theta) = 0, \quad i = 0, 1, \ldots$. Thus, (3.8) holds in E and, if $\theta \in \mathcal{P}$, x must be equal to 0. This proves the global asymptotic stability of the closed loop system.

$\square$

Example 1: Using the results of this section we design a controller which achieves global asymptotic stability for the benchmark system (1.1).

We first transform (1.1) into the form (1.2) with the change of coordinates $y = x_2 + x_3$:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_2^2 + \theta(y - 2x_2)y \\ \dot{x}_2 &= -x_2 + y \\ \dot{y} &= -x_2 + y + u \end{aligned} \tag{3.9}$$

The zero dynamics with respect to the output y is in the form (2.1):

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

To construct the Lyapunov function (2.4) we let $P_F = 1$ and compute

$$\Psi(x_1, x_2) = \int_0^\infty \bar{x}_1(s)(\bar{x}_2(s) + \theta \bar{x}_2^2(s)) \, ds$$

where

$$\begin{aligned} \bar{x}_1(s) &= x_1 + \int_0^s (x_2 e^{-\tau} + \theta x_2^2 e^{-2\tau}) \, d\tau \\ \bar{x}_2(s) &= x_2 e^{-s} \end{aligned}$$

Integrating we obtain

$$\Psi(x_1, x_2) = -x_1^2 + (x_1 + x_2 + \frac{1}{2}\theta x_2^2)^2$$

Thus, our Lyapunov function for the zero dynamics is

$$W(x_1, x_2) = (x_1 + x_2 + \frac{1}{2}\theta x_2^2)^2 + x_2^2$$

When θ is known, the control law (3.6) is implementable and is given by

$$u = -2y - x_1 - x_2 - \theta[(x_1 + x_2)(y - x_2) + \frac{1}{2}x_2^2] - \frac{1}{2}\theta^2(y - x_2)x_2^2 \tag{3.10}$$

By Proposition 3.1 the control law (3.10) achieves boundedness of the states and convergence of x_2 and y to 0. To characterize the set $\mathcal{P}$ we check (H2). For this system $F=0$, $Q_F=0$, $\varphi=0$, so that the only nontrivial equation in (3.8) is

$$\chi(x_1,\theta)=u(x_1,0,0,\theta)=-x_1=0$$

Because, $x_1=0$ is the only solution to this equation independently of θ, we have that $\mathcal{P}=I\!R$ and the global asymptotic stability is achieved for all θ.

□

Example 2: The benchmark system (1.1) is stabilizable for any value of the parameter θ. That this is not always the case is illustrated by

$$\begin{aligned}\dot{x}&=\xi+\theta_1 y\\ \dot{\xi}&=-\xi+\theta_2 y\\ \dot{y}&=u\end{aligned}\tag{3.11}$$

This linear system with two parameters is in the form (1.2). From $\frac{d}{dt}(x+\xi)=(\theta_1+\theta_2)y$ it is obvious that (3.11) is not stabilizable for $\theta_1+\theta_2=0$. Nevertheless, the boundedness and regulation of ξ and y is achieved for any θ_1 and θ_2.

The zero dynamics with respect to y is

$$\begin{aligned}\dot{x}&=\xi\\ \dot{\xi}&=-\xi\end{aligned}$$

With $P_F=1$ the Lyapunov function (2.4) is

$$W(x,\xi)=x^2+2x\xi+2\xi^2$$

and the control law (3.6) becomes

$$u=-2(\theta_1+\theta_2)x-2(\theta_1+2\theta_2)\xi-y$$

According to Proposition 3.1, this control law achieves the boundedness of the states and convergence of ξ and y to 0.

In this example the only nontrivial equation in (H2) is

$$\chi(x,\theta)=u(x,0,0,\theta)=-2x(\theta_1+\theta_2)=0\tag{3.12}$$

If $x=0$ is to be the only solution of (3.12) we must restrict θ to belong to $\mathcal{P}=\{(\theta_1,\theta_2)\in I\!R^2:\theta_1+\theta_2\neq 0\}$. Thus, according to Proposition 3.3, if $\theta\in\mathcal{P}$ the global asymptotic stability of the closed loop system is achieved.

□

4 Adaptive Controller Design

The control law (3.6) is a cubic polynomial in θ. To design an adaptive version of (3.6) we resort to overparameterization by introducing a new parameter ϑ_l for every product of θ_i's which appears in (3.6). Thus: $\vartheta_{l_1} \triangleq \theta_i\theta_j$, $\vartheta_{l_2} \triangleq \theta_i\theta_j\theta_k$, etc. In this way we have defined the augmented vector $\Theta^T = [\theta^T \ \vartheta^T] \in IR^q$ where $q \leq \frac{1}{6}(p^3 + 6p^2 + 11p)$.

We rewrite the control law (3.6) as

$$u(x, \xi, y, \Theta) = -y - w^T(x, \xi, y)\Theta \tag{4.1}$$

where the function w can be obtained from (3.6). Because the parameter vector is not known, we replace it with our estimate $\hat{\Theta}^T \triangleq [\hat{\theta}^T \ \hat{\vartheta}^T]$:

$$u(x, \xi, y, \hat{\Theta}) = -y - w^T(x, \xi, y)\hat{\Theta} \tag{4.2}$$

Next we modify the Lyapunov function (3.3) to include the parameter estimation error $\tilde{\Theta} \triangleq \Theta - \hat{\Theta}$:

$$V_e(x, \xi, y, \tilde{\Theta}) = W(x, \xi) + \frac{1}{2}y^2 + \frac{1}{2}\tilde{\Theta}^T\tilde{\Theta} \tag{4.3}$$

Its time derivative is

$$\dot{V}_e = -x^T Q_F x - \xi^T Q_A \xi - y^2 + y w^T(x, \xi, y)\tilde{\Theta} + \tilde{\Theta}^T\dot{\tilde{\Theta}}$$

Thus, by choosing the parameter update law

$$\dot{\hat{\Theta}} = w(x, \xi, y)y \tag{4.4}$$

we achieve

$$\dot{V}_e = -x^T Q_F x - \xi^T Q_A \xi - y^2 \leq 0 \tag{4.5}$$

Using (4.5) it is easy to prove that the designed adaptive controller achieves boundedness of all the signals and regulation of ξ and y to 0. Again it is not required that the system (1.2) be stabilizable.

Proposition 4.1: For any $\theta \in R^p$ the system (1.2) with the adaptive controller (4.2), (4.4) is globally stable and ξ and y converge to 0 as $t \to \infty$.

When the parameters are unknown the adaptive version of (3.6), given by (4.2), (4.4), achieves the same basic properties as (3.6). Additional properties can be deduced by analyzing E', the largest invariant set of the closed loop system (1.2), (4.2), (4.4) where $\dot{V}_e = 0$. In general, E' is different from E and the analysis is more difficult than in Proposition 3.3.

We still want to examine whether the condition (H2) can guarantee the regulation of x to 0, possibly with a modified adaptive controller. We will do it in two steps. First we remove the dependence of E' on $\hat{\vartheta}$ by introducing the following assumption:

(H3) $\varphi(x,0,0)=0$ and either $\dfrac{\partial h}{\partial \xi}(0)=0$ or $b(x,0,0)=0$.

It will be clear in the sequel that this assumption is needed only for the uncertain parts of φ, h, and b. For example, if instead of $h^T(\xi)\theta$ we had $h_0(\xi)+h_1(\xi)^T\theta$, then (H3) applies only to h_1.

Proposition 4.2: If (H3) is satisfied then

$$\frac{\partial u}{\partial \hat{\vartheta}}(x,0,0,\hat{\Theta})=0$$

that is, the certainty equivalence control law (4.2) is independent of $\hat{\vartheta}$ when $\xi=0$, $y=0$.

Proof: Recall that ϑ stands for the terms quadratic and cubic in θ in the control law (3.6). To prove the proposition it suffices to show that the functions multiplying these non-linear terms vanish when $\xi=0$, $y=0$. The portion of the control law (3.6) which is nonlinear in θ is

$$\theta^T[\Psi_1^T g^T+\sum_{i=1}^{n_x} x_i \frac{\partial(\Psi_1^T)_{(i)}}{\partial \xi} b^T]\theta+\theta^T\sum_{i=1}^{p}\theta_i\frac{\partial \Psi_{2(i)}}{\partial \xi} b^T\theta \tag{4.6}$$

Under assumption (H3), expression (4.6) vanishes when $\xi=0$, $y=0$. To see this, note that $\Psi_1(\xi)$ is at least linear in ξ because J and h are both at least linear in ξ. Also $\frac{\partial \Psi_{2(i)}}{\partial \xi}(0)=0$ because Ψ_2 is at least quadratic in ξ. Finally, from (H3), either $\frac{\partial \Psi_{1(i)}}{\partial \xi}(0)=0$ (when $\frac{\partial h}{\partial \xi}(0)=0$) or $b^T(x,0,0)=0$. Therefore, when $\xi=0$ and $y=0$, the terms nonlinear in θ vanish and the adaptive control law (4.2) depends only on $\hat{\theta}$.

□

Proposition 4.2 shows that the set E' is independent of $\hat{\vartheta}$. Note that, in E', $y=0$ in which case the estimate $\hat{\theta}$ is a constant vector denoted by $\bar{\theta}$. To achieve the regulation of x to 0 under (H2), we will make sure that $\bar{\theta}\in\mathcal{P}$. To this end we introduce a projection in the parameter update law which will keep $\hat{\theta}\in\Pi$, a closed and convex subset of $\mathcal{P}$. We note that the set Π need not be bounded.

Conformal with the partition of Θ into θ and ϑ, we let $w^T(x,\xi,y)=[w_1^T(x,\xi,y)\; w_2^T(x,\xi,y)]$. Then we modify the update law (4.4) as

$$\begin{aligned}\dot{\hat{\theta}}&=Proj_{\Pi}\{w_1(x,\xi,y)y\}\\ \dot{\hat{\vartheta}}&=w_2(x,\xi,y)y\end{aligned} \tag{4.7}$$

where $\mathrm{Proj}_{\Pi}\{\cdot\}$ is the standard projection operator (c.f. Section 4.4 in [2]) which guarantees that the vector $\hat{\theta}(t)$ remains in the set Π.

Now we are ready to state the theorem which connects the condition (H2) with the adaptive regulation of x to 0.

Theorem 4.3: If the assumption (H3) is satisfied and if the closed and convex set $\Pi\subset\mathcal{P}$

contains θ, then the system (1.2) with the adaptive controller (4.2), (4.7) is globally stable and x, ξ, and y converge to 0 as $t \to \infty$.

Proof: An important property of the projection used in the parameter update law is that if the set Π contains the true parameter vector θ then $\tilde{\theta}^T Proj_\Pi\{w_1(x,\xi,y)y\} \geq \tilde{\theta}^T w_1(x,\xi,y)y$. Using this inequality we obtain

$$\begin{aligned} \dot{V}_e &= -x^T Q_F x - \xi^T Q_A \xi - y^2 + y w_1^T \tilde{\theta} - \tilde{\theta}^T Proj_\Pi\{w_1(x,\xi,y)y\} \\ &\leq -x^T Q_F x - \xi^T Q_A \xi - y^2 \leq 0 \end{aligned}$$

By the LaSalle-Yoshizawa Theorem the states of the system are uniformly bounded and y and ξ converge to 0.

Now we examine the largest invariant set E' where the following must hold:

1. $\xi = 0$, $y = 0$, $\dot{\hat{\Theta}} = 0$
2. $\hat{\theta}(t) = \bar{\theta} \in \Pi \subset \mathcal{P}$
3. $\dot{x} = Fx$
4. $x^T Q_F F^k x = 0,\ k = 0, \ldots, n_x - 1$
5. $0 \equiv \dot{y} = u(x, 0, 0, \Theta)$.

The last item follows from (H3) because $\dot{y} = \varphi^T\theta + u$ and φ^T vanishes when $\xi = 0$, $y = 0$. By Proposition 4.2, $u(x,0,0,\Theta)$ is independent of $\hat{\vartheta}$. Thus, $u(x,0,0,\Theta) = \chi(x,\bar{\theta})$ and the items 2, 3, and 5 imply that $L^i_{(Fx)}\chi(x,\bar{\theta}) = 0$ for $i = 0, 1, \ldots$. Since $\bar{\theta} \in \mathcal{P}$, condition (H2) must hold and the only point which satisfies 3 through 5 is $x = 0$. By LaSalle's Invariance Principle, the regulation of x to 0 is achieved.

□

Example 3: Returning to the benchmark system (1.1) we now allow that the parameter θ be unknown. Our adaptive control law is a certainty equivalence version of the control law (3.10) with θ replaced by $\hat{\theta}$ and θ^2 replaced by an independent new estimate $\hat{\vartheta}$:

$$u = -2y - x_1 - x_2 - \hat{\theta}[(x_1 + x_2)(y - x_2) + \frac{1}{2}x_2^2] - \frac{1}{2}\hat{\vartheta}(y - x_2)x_2^2 \tag{4.8}$$

In Section 3 we have shown that for this problem $\mathcal{P} = IR$. Because the purpose of the projection in (4.7) was to keep $\hat{\theta}$ in $\mathcal{P}$, we conclude that in this case it is not needed. Thus, the parameter update law is given by (4.7) with the projection removed

$$\begin{aligned} \dot{\hat{\theta}} &= y[(x_1 + x_2)(y - x_2) + \tfrac{1}{2}x_2^2] \\ \dot{\hat{\vartheta}} &= y(y - x_2)x_2^2 \end{aligned} \tag{4.9}$$

By Proposition 4.1 the adaptive controller (4.8), (4.9) achieves boundedness of x_1, x_2, y, $\hat{\theta}$, $\hat{\vartheta}$ and the regulation of x_2 and y to 0. By comparing (1.2) and (3.9) we conclude that assumption (H3) holds because $\varphi = -x_2 + y$ and $b = 1$ do not include parametric uncertainties. Since $\mathcal{P} = I\!R$, Theorem 4.3 establishes that the regulation of x_1 to 0 is also achieved. □

Example 4: To illustrate several prominent features of this adaptive design let us consider the following nonlinear system

$$\begin{aligned} \dot{x} &= x\xi + \theta_1 \sin^2 \xi + x^2 y^2 \\ \dot{\xi} &= -\xi + \theta_2 y \\ \dot{y} &= \theta_3 a(y) x^2 + u \end{aligned} \tag{4.10}$$

Selecting $W(x) = x^2$, the cross term Ψ in the Lyapunov function for the zero dynamics

$$\begin{aligned} \dot{x} &= x\xi + \theta_1 \sin^2 \xi \\ \dot{\xi} &= -\xi \end{aligned} \tag{4.11}$$

becomes

$$\Psi(x,\xi) = \int_0^\infty 2\theta_1 \bar{x} \ \sin^2 \bar{\xi} \ ds$$

After substituting the solution $(\bar{x}, \bar{\xi})$ of (4.11) into the integral, rearanging terms, and defining $\mu = \xi e^{-s}$, we obtain

$$\Psi(x,\xi) = -x^2 + \left(x e^{\xi} + \theta_1 \rho(\xi) \right)^2 \tag{4.12}$$

where

$$\rho(\xi) = \int_0^\xi \frac{e^\mu}{\mu} \ \sin^2 \mu \ d\mu$$

Even though this integral cannot be evaluated in the closed form, it globally defines an analytic function which can be either precomputed and approximated by one of the standard curve fitting techniques or computed on-line by integration.

From (4.12) we obtain the Lyapunov function for the zero dynamics (4.11)

$$V(x,\xi) = \left(x e^{\xi} + \theta_1 \rho(\xi) \right)^2 + \xi^2 \tag{4.13}$$

and the control law for the system (4.10)

$$\begin{aligned} u(x,\xi,\theta) \ &= -\theta_3 a(y) x^2 - y - \tfrac{\partial V}{\partial x} x^2 y - \tfrac{\partial V}{\partial \xi} \theta_2 \\ &= -\theta_3 a(y) x^2 - y - 2e^{\xi} \left(x e^{\xi} + \theta_1 \rho(\xi) \right) \left(x^2 y + \theta_2 x + \theta_1 \theta_2 \tfrac{\sin^2 \xi}{\xi} \right) \end{aligned} \tag{4.14}$$

The algebraic conditions (H2) reduce to

$$\chi(x,0,0,\theta) = -2\theta_2 x^2$$

providing the set

$$\mathcal{P} = \{ \theta \in I\!R^3 : \theta_2 \neq 0 \} \tag{4.15}$$

Note that when $\theta_2 = 0$ the system (4.10) is not stabilizable.

Case 1: If $a(0) = 0$, the condition (H3) is satisfied, so the adaptive controller (4.2), (4.7) achieves boundedness of the signals and regulation of (x, ξ, y) to 0 provided that we can find an appropriate closed and convex set Π for the projection. From (4.15) we conclude that θ_2 is the only parameter which requires projection for which we need to know the sign of θ_2 and a lower bound on $|\theta_2|$. The resemblance to linear adaptive systems, where the sign of the high-frequency gain is a standard assumption, is not accidental. The projection in our adaptive design serves exactly the same purpose: to avoid the set of parameter values for which the system cannot be stabilized.

Case 2: If $a(0) \neq 0$ the condition (H3) is not satisifed. Nevertheless, we can run the adaptive controller (4.2), (4.4) which, by Proposition 4.1, guarantees boundedness of the signals and regulation of ξ and y to 0. To guarantee the regulation of x we need that

$$(\theta_3 - \hat{\theta}_3)a(0) - \theta_2 \neq 0$$

This expression and some apriori boundes on θ_2 and θ_3 can still be used to find an appropriate set for projection, but because it involves the estimates of θ (in general, it will also involve $\hat{\vartheta}$) it is much more difficult to manage. Assumption (H3) allowed us to avoid this problem and use (H2) to find a set for projection. $\square$

5 Conclusion

In this paper we have developed a feedback passivation approach to the design of adaptive controllers for a class of cascade nonlinear systems which, in general, are not feedback linearizable. A key feature of this approach is a systematic construction of a parameterized Lyapunov function for the zero dynamics subsystem. This construction has allowed us to solve the longstanding nonlinear benchmark problem (1.1) which was formulated in [6]. A direction for further research is to extend the recursive design methods of [3] to the problem of adaptive control of feedforward systems which are more general than the cascade form (1.2) considered here.

References

[1] C.I. Byrnes, A. Isidori, J.C. Willems, "Passivity, Feedback Equivalence, and the Global Stabilization of Minimum Phase Systems," IEEE Trans. Automatic Control, 36, (1991), 1228-1240.

[2] P.A. Ioannou, J. Sun, *Stable and Robust Adaptive Control*, Prentice-Hall, Englewood Cliffs, 1995.

[3] M. Jankovic, R. Sepulchre, P.V. Kokotovic, "Constructive Lyapunov stabilization of nonlineare cascade systems," to appear in IEEE Transactions on Automatic Control.

[4] M. Jankovic, R. Sepulchre, P.V. Kokotovic, "Global stabilization of an enlarged class of cascade nonlinear systems," Proceedings of 34-th IEEE CDC, New Orleans, LA, (1995), 993-998.

[5] I Kanellakopoulos, P.V. Kokotovic, A. S. Morse, "Systematic Design of Adaptive Controllers for Feedback Linearizable systems," IEEE Trans. Automatic Control, vol. 36, (1991), 1241-1253.

[6] I Kanellakopoulos, P.V. Kokotovic, A. S. Morse, "Adaptive Feedback Linearization of Nonlinear Systems," 311-346, *Foundations of Adaptive Control*, P.V. Kokotovic ed., Springer-Verlag, 1991.

[7] P.V. Kokotovic, H.J. Sussmann, "A Positive Real Condition for Global Stabilization of Nonlinear Systems," Systems & Control Letters, 13, (1989), 125-133.

[8] M. Krstic, I Kanellakopoulos, P.V. Kokotovic, *Nonlinear and Adaptive Control Design*, Wiley, New York, 1995.

[9] M. Krstic, I Kanellakopoulos, P.V. Kokotovic, "Adaptive Nonlinear Control without Overparameterization," Systems and Control Letters, 19, (1992), 177-185.

[10] M. Krstic, P.V. Kokotovic, "Adaptive Nonlinear Design with Controller-Identifier Separation and Swapping," IEEE Trans. Automatic Control, vol. 40, (1995), 426-440.

[11] K.K. Lee, A. Arapostathis, "Remarks on Smooth Feedback Stabilization of Nonlinear Systems," Systems & Control Letters, 10, (1988), 41-44.

[12] W. Lin, "Global Asymptotic Stabilization of General Nonlinear Systems with Stable Free Dynamics by Using Passivity and Smooth Saturation," to appear in Automatica.

[13] A Rodriguez, R. Ortega, "Adaptive Stabilization of Nonlinear Systems: The Nonfeedback Linearizable Case," Proceedings 11th IFAC Congress, Tallinn, 1990, vol 2, 303-305.

[14] M.M. Seron, D.J. Hill, A.L. Fradkov, "Adaptive Passification of Nonlinear Systems," Proceedings 33rd CDC, Lake Buena Vista, Fl, 1994, 190-195.

[15] A.R. Teel, "Feedback Stabilization: Nonlinear Solutions to Inherently Nonlinear Problems," PhD Dissertation, University of California, Berkeley, 1992.