Electricity markets with flexible consumption as nonatomic congestion games

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Abstract

This article studies nonatomic congestion games with symmetric players sending flow in arcs linking a unique source and destination. These games can represent systems where electricity retailers controlling flexible consumption minimize their energy costs. We focus on games with affine cost functions and define laminar Nash equilibrium where the constraints on the minimum and maximal flow that a player must send in a given arc are not binding. We show that the flow sent by a player at a laminar Nash equilibrium does not depend on the demand of other players. In laminar flow, we bound the price of anarchy and the ratio between the maximum and the minimum arc cost. Finally, we propose a simple method based on the property of a laminar Nash equilibrium to compute the price of flexibility to which energy flexibility should be remunerated in electric power systems.

1 Introduction

One of the most complex commodity to exchange is electricity. By its very nature, the electricity production must always be equal to the consumption. Unfortunately, current technology does not allow storing high volumes with enough economic and technical efficiency. To ensure the equality between production and consumption at any time, electricity is traded before its delivery. A large portion of the electricity is traded in the day-ahead energy market which settles unique prices for each hour of the day, which are commonly accepted as reference prices. With the ongoing trend for connected electric appliances, electricity retailers now not only retail electricity bought on the energy market but also control flexible consumption. This additional activity allows retailers to shift the consumption in cheaper hours to decrease their energy procurement cost. The starting point of this paper is a power system problem where each electricity retailer managing flexible consumption chooses independently the total consumption of its portfolio in each hour. The aggregated consumption of all retailers in one period determines a unique market price for this period. Given the prices in each hour, the retailers aim at minimizing their own retailing cost. One of our results is to show that this problem can be mapped to a nonatomic congestion game. Therefore, some concepts of the game theory literature are directly applicable to the electricity market to answer question such as: *Do the game converge to stable electricity prices? If it is the case, what are these prices, in particular what is the ratio between the maximum and the minimum price? How inefficient is this system with respect to the case where there would* *be only a single retailer that manages all the demand? Suppose that a power system operator would ask a retailer to change its load in order to solve an issue in the electric network, what would be the cost incurred by the retailer?*

This paper answers these questions by studying nonatomic congestion games with a continuous strategy space. As a reminder, congestion games are non-cooperative games usually related to traffic problems where each user tries to find the best path to its destination which minimize its travel time. The travel time is given by a cost function dependent on the total traffic of the path. In the classic congestion game, a user is seen as an infinitesimal quantity of traffic to carry in the network [1]. Nonatomic congestion games are a generalization of congestion games where players form coalitions and aim at minimizing the cost of their coalition [2]. One decision taken by the coalition on the strategy space can be discrete, e.g. assign one user to a given path, or continuous, e.g. assign some flow to a given arc.

Many theoretical results on congestion games can be found in the literature. The conditions of existence and uniqueness of the Nash equilibrium of nonatomic congestion games are known [3]. The price of anarchy of atomic congestion games with affine latency functions and positive coefficients is bounded by 4/3 [4]. Roughgarden and Tardos extend their result on the nonlinear case for nonatomic congestion games in [5], showing that the price of anarchy is bounded by a constant γ only dependent on the latencies. If the latencies are polynomials of degree p at most, $\gamma \leq p+1$. The problem of minimizing the maximum latency of flows in networks with congestion is studied in [6]. The authors define for a given flow the unfairness as the ratio of the maximum latency divided by the minimum latency. They bound the unfairness of the system optimum to by the same constant γ. Note that these bounds are independent of the number of players.

The effect of collusion in congestion games is investigated in [7]–[11]. Forming coalition in symmetric nonatomic games reduces the overall costs with respect to the Nash equilibrium [7]. Wan shows in [8] that even the individual cost decreases when the size of the coalition of the individual increases. The price of anarchy is known to be bounded by the minimum between k and a constant dependent on the number of edges [10]. Christodoulou and Koutsoupias provide a bound dependent on the number of players for symmetric and asymmetric congestion games [9]. The authors show that the price of anarchy is bounded by $\frac{5k+2}{2k+2}$ in the affine case and that this bound is tight. Cominetti, Correa, and Stier-Moses show in [12] that the price of anarchy is bounded in the affine case with symmetric players by $\frac{4k^2}{(k+1)(3k-1)}$ and that this bound is tight. In this paper, we propose an alternative proof based on the paper [4] of Roughgarden and Tardos.

Several power systems problems are related to game theory as attested in the literature survey [13]. Ibars, Navarro, and Giupponi propose a distributed load management in smart grid infrastructure to control the power demand at peak hours using dynamic pricing strategies based on a network congestion game [14]. One close but different problem of flexible consumption management to the one considered in this paper is addressed in [15]. They study the energy transaction between a single retailer and multiple consumers with a total energy constraint. The authors map the problem to an atomic separable flow game. They prove the existence of a Nash equilibrium using the communication network paper

[16]. They establish the link with the congestion games literature in paper [17].

This paper focuses on a particular regime of nonatomic congestion games with affine arc cost functions that we name *laminar* flow. This name is motivated by the analogy with the fluid mechanics regime where the flow is organized in layers without interactions. One of our results is that, if the Nash equilibrium of a nonatomic congestion game is laminar, i.e. the demand of all players are within specific bounds, the strategy of each player depends only on its own demand. The motivation for studying laminar flows is that it leads to a clean analysis. Assuming laminar flow, we provide four contributions stated in the following theorems.

Theorem 1. *Consider a nonatomic congestion game with affine cost functions. If the Nash equilibrium is laminar then the flow of each player is independent of other players.*

Two byproducts of this theorem is that the flow and arc costs are only dependent on the total demand and that a game can be checked to have a laminar equilibrium only based on the individual demands of the players.

Theorem 2. *Consider a k-players nonatomic congestion game with affine cost functions. If the Nash equilibrium is laminar then the ratio between the maximum and the minimum arc cost is bounded by $(k+1)/k$.*

A tighter bound is also proposed which is dependent on the y-intercepts of the affine cost functions.

Theorem 3. *Consider a nonatomic congestion game with a laminar Nash equilibrium, the price of flexibility for an imposed small deviation in an arc is at least two times the first derivative of the corresponding arc cost at the Nash equilibrium without deviation.*

The *price of flexibility* is a concept introduced in this paper which in our power system problem corresponds to the price of shifting the electric consumption from one period to others.

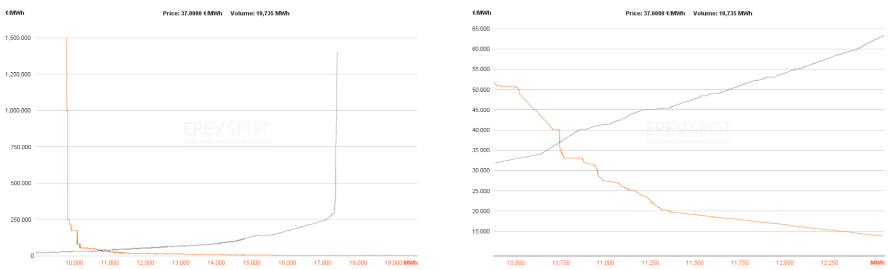
Theorem 4. *The price of anarchy of a k-player nonatomic congestion game with a laminar Nash equilibrium and affine cost functions with positive coefficients is at most*

$$\frac{4k^2}{(k+1)(3k-1)} \tag{1}$$

These contributions are disseminated in the paper as follows. Section 2 links nonatomic congestion games with retailers managing flexible electric consumption. Section 3 sets our notations and describes laminar flows in nonatomic congestion games. In laminar flow, we obtain a bound on the ratio between the minimum and maximum arc cost in Section 4. Section 5 details a method to obtain the price of flexibility. Finally, a bound on the price of anarchy of nonatomic congestion games with laminar Nash equilibrium is proven in Section 6.

2 Flexible consumption retailing as a nonatomic congestion game

One of the most complex commodity to exchange is electricity. By its very nature, the electricity production must always be equal to the consumption. Unfortunately, current technology does not allow storing high volumes with enough economic and technical efficiency. To ensure the equality between production and consumption at any time, electricity is traded before its delivery. Part of the trade are conducted years or months ahead in long term contracts while the rest is cleared on energy spot markets. The most common is the day-ahead energy market whose prices are taken as reference. This market divides the day into periods, typically twenty-four, and provide a unique price for each of these periods. Participants to these markets submit bids to supply or consume a certain amount of electric energy at a given price for the period under consideration. The bids are ranked to form the demand and the offer curves. The intersection of the two curves defines the system marginal price. This unique price for each hour is the price paid by every accepted participant whatever the price they submitted. This scheme gives incentive for the participants to bid at their marginal costs and therefore discourage gaming on the price they require [18]. Figure 1 shows the production and consumption aggregated curves of the French spot market for the first hour of the 1st April 2014 [19]. The intersection of the non-decreasing offer curve with the demand curve leads to the system marginal price of 37€/MWh.



(a) 0h-1h

(b) Zoom on the market clearing price

Figure 1: Aggregated curves of the market clearing of the 1st April 2014 [19].

With the ongoing trend for connected electric appliances, electricity retailers now not only retail electricity but also control flexible consumption. This additional activity allows retailers to shift the consumption in cheaper periods to decrease their energy procurement cost. As a result, one retailer can decrease its cost at the expense of the others resulting in higher total costs. The least global energy procurement cost would be obtained if only one entity was controlling the entire flexibility. In practice, the electrical system is composed of more than one retailer which game to minimize their own cost to buy electricity.

The mapping to a nonatomic congestion game is as follows. To each market period, e.g. each hour, corresponds one arc between a single source and a single destination. The cost function of an arc is the offer curve of the market at the corresponding period. The system marginal price of one period is the cost of the arc at the Nash Equilibrium. Each retailer is a player with a total flow equal to the energy needs of its clients. The retailers minimize their own energy procurement cost which is the sum over the periods of the electricity price times the energy consumed in the corresponding period. One could wonder the implications of this game on the electricity prices. Do the game converge to stable electricity prices? If it is the case, what are these prices, in particular what is the ratio between the maximum and the minimum price? How inefficient is this system with respect to the case where there would be only a single retailer that manages all the demand? In the gaming theory literature, the first question is equivalent to showing that there exists a Nash equilibrium. The second can be found under the term *unfairness* [6] while the third is obtained by the price of anarchy [20]. Note that the results obtained in this paper are valid for general nonatomic congestion games with single source and sink.

In addition to the reduction of energy procurement costs, demand side flexibility can also be used to provide services to other actors of the electric system. These services may target the relief of a congestion of a line or cover the unexpected loss of a production unit. The first case is related to active network management which is studied intensively in the literature [21]–[24]. The second is handled through reserve markets and the methods to provide reserve services by the demand side flexibility has also been broadly investigated [25]–[27]. One important unknown in these works and others is the price at which demand side flexibility can be sold. The price of the flexibility from production units is easy to obtain as a function of the fuel cost and the unit maintenance cost. The provision of flexibility from production unit is also easier to handle as increasing the production in an hour has barely no impact on what can be done in the following hours. Conversely, changing the consumption of an electric appliance in an hour impacts its consumption in the following one. For instance, a retailer controlling supermarket fridges can interrupt them for one hour. The internal temperature of the fridges increases and consequently the fridges consume more later. The consumption is therefore shifted from one hour to another. Following these thoughts, the cost associated to this shifting is related to the difference of prices between the market prices in the periods where the energy consumption is modified. In this paper, we provide a simple method to compute the price at which the flexibility of the demand side should be remunerated. This method depends only on public data of the clearing of the day-ahead energy market. The result of the method is supported by its link to the Nash equilibrium of the corresponding congestion game.

3 Laminar flow in congestion game

Consider k players sending flow in a set $\mathcal{T}$ of arcs with the same source and destination. The goal of a player $i \in \mathcal{K}$ is to minimize its total cost by choosing which quantity to send in each arc $t \in \mathcal{T}$, $x_{i,t}$ such that the flow sent is equal to the demand of the player D_i. The total flow is $D = \sum_{i\in\mathcal{K}} D_i$. The aggregated flow in one arc is given by $x_t = \sum_{i\in\mathcal{K}} x_{i,t}$ and define the price in one arc using the cost function $c_t(x_t) : \mathbb{R}_+ \to \mathbb{R}$. Therefore, the prices are independent of the identity of the player. The total cost incurred by player i, $C_i(\mathbf{x}_i)$ is given by

$$C_i(\mathbf{x}_i) = \sum_{t\in\mathcal{T}} c_t(x_t)x_{i,t}. \tag{2}$$

where $\mathbf{x}_i = \{x_{i,t}|\forall t \in \mathcal{T}\}$. Note that this cost depends on the actions of the other players through the term x_t. The total system cost, $C(\mathbf{x})$, is only dependent on the aggregated flows

$$C(\mathbf{x}) = \sum_{t\in\mathcal{T}} c_t(x_t)x_t = \sum_{i\in\mathcal{K}} C_i(\mathbf{x}_i) \tag{3}$$

where $\mathbf{x} = \{x_t | \forall t \in \mathcal{T}\}$. Note that to one $\mathbf{x}$ can correspond more than one solution of same total system cost in terms of $\mathbf{x}_i$. Such a game is depicted in Figure 2. Additional constrains that could have been added to the model is

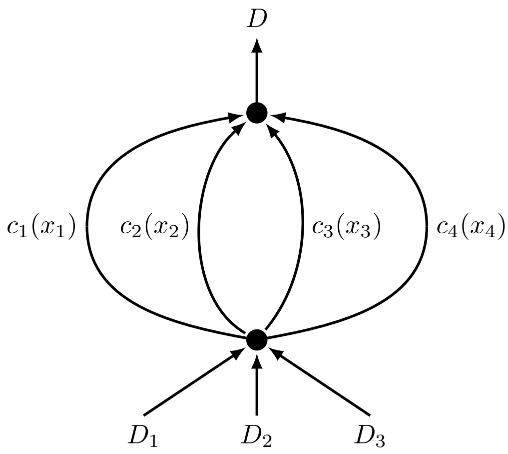


Figure 2: Visual representation of a nonatomic congestion game with three players and four arcs.

bounds on the flows such that

$$x_{i,t}^{\min} \leq x_{i,t} \leq x_{i,t}^{\max}. \tag{4}$$

The classic version of nonatomic congestion games, which is studied in this paper, takes $x_{i,t}^{\min} = 0$ and $x_{i,t}^{\max} = +\infty$. This paper focuses on Nash equilibriums of nonatomic congestions games where this constraint is either not existing or not active.

Definition 1. *A congestion game is laminar if, for each player $i \in \mathcal{K}$ and each arc $t \in \mathcal{T}$,*

$$x_{i,t}^{\min} < x_{i,t} < x_{i,t}^{\max}. \tag{5}$$

In fluid mechanics, a streamline is an imaginary line with no flow normal to it, only along it. When the flow is laminar, the streamlines are parallel and for flow between two parallel surfaces we may consider the flow as made up of parallel laminar layers. In laminar flow, no mixing occurs between adjacent layers [28]. As we see later in Theorem 1, if a congestion game is laminar, the strategy of a player at the Nash equilibrium does not depend on the demand of the others. We first consider these classic bounds for a small example to highlight the change from laminar flow to another regime where the flow in one arc is zero for one player.

We denote $\mathbf{x}^*$ the optimal flow which minimizes the total cost: $\forall \mathbf{x} \in X, C(\mathbf{x}^*) \leq C(x)$. Note that if there is more than one retailer, the solution in terms of $\mathbf{x}_i$ is not unique. The Nash equilibrium denoted by $\mathbf{x}_i^N$ $\forall i \in \mathcal{K}$ and the resulting aggregated flows by $\mathbf{x}^N$. At the Nash equilibrium, no player has incentive to change its flows given the flows of the others. Note that if there is only one retailer, $\mathbf{x}^N = \mathbf{x}^*$. To shorten the notation, we define $c_t^N = c_t(x_t^N)$. The system is at its Nash equilibrium $\mathbf{x}_i^N$ if no retailer i can improve its strategy given the strategy of the others. As a result, the strategy $\mathbf{x}_i^N$ is a solution of the following optimization problem.

$$\min_{\mathbf{x}_i} \quad \sum_{t\in\mathcal{T}} c_t(x_t)x_{i,t} \tag{6a}$$

$$\text{s.t.} \quad \sum_{t\in\mathcal{T}} x_{i,t} = D_i \qquad : \lambda_i \tag{6b}$$

$$x_{i,t} \geq x_{i,t}^{min} \qquad : \kappa_{i,t} \geq 0 \qquad \forall t \in \mathcal{T} \tag{6c}$$

$$x_{i,t} \leq x_{i,t}^{max} \qquad : \gamma_{i,t} \geq 0 \qquad \forall t \in \mathcal{T} \tag{6d}$$

Taking the Karush-Kuhn-Tucker conditions of (6) provide the following necessary optimality condition

$$\lambda_i^N + \kappa_{i,t} - \gamma_{i,t} = c_t^N + \frac{\partial c_t^N}{\partial x_{i,t}} x_{i,t}^N = c_t^N + \frac{\partial c_t^N}{\partial x_t} x_{i,t}^N \qquad \forall t \in \mathcal{T} \tag{7}$$

As the prices does not depend on the identity of the buyer, $\frac{\partial c_t}{\partial x_{i,t}} = \frac{\partial c_t}{\partial x_t}$. By complementarity slackness, either $\kappa_{i,t} = 0$ or $x_{i,t} = x_{i,t}^{min}$ and either $\gamma_{i,t} = 0$ or $x_{i,t} = x_{i,t}^{max}$.

Observation 1. *Given two periods $t, u \in \mathcal{T}$, $x_{i,t}^N = x_{i,t}^{min}$ and $x_{i,t}^{min} < x_{i,u}^N < x_{i,t}^{max}$ implies $c_t^N \geq c_u^N$ where the equality only holds if $\frac{\partial c_u}{\partial x_u} = 0$.*

Proof of Observation 1. Condition (7) yields, by complementarity slackness of $\kappa_{i,t}, \gamma_{i,t}$ and $x_{i,t}$,

$$\lambda_i^N = c_t^N - \kappa_{i,t} = c_u^N + \frac{\partial c_u^N}{\partial x_u} x_{i,u} \tag{8}$$

with $\kappa_{i,t} \geq 0$ and $\frac{\partial c_u^N}{\partial x_u} x_{i,u} \geq 0$ which implies the inequality $c_t^N \geq c_u^N$. □

In most of the paper, we consider affine cost functions $c_t(x_t) = a_t x_t + b_t$ with $a_t > 0$. The optimality conditions (7) are in this case

$$\lambda_i^N + \kappa_{i,t} - \gamma_{i,t} = (a_t x_t^N + b_t) + a_t x_{i,t} = 2a_t x_{i,t}^N + a_t \sum_{j\in\mathcal{K}\setminus\{i\}} x_{j,t}^N + b_t \tag{9}$$

We now focus on the affine game represented in Figure 3 with three players and three arcs. We fix the total demand of the two last players and analyses how the equilibrium change with respect to the total demand of the first player. The last two players have an identical total demand and therefore plays an identical strategy. In the following we write only the results of player two. The first arc is the most expensive which incentivize the first player to avoid sending flow in this arc. The bound on the flow are $x_{i,t} \in [0, +\infty[$ for all players. Computations are performed using the symbolic capabilities of the open-source software Maxima [29].

First, we consider that D_1 is such that the game is laminar, $x_{i,t} > 0 \ \forall i \in \mathcal{K}, t \in \mathcal{T}$. The analytical solution of the Nash equilibrium is

$$\mathbf{x}_1 = \left(\frac{8D_1 - 5}{44}, \frac{6D_1 - 1}{22}, \frac{24D_1 + 7}{44}\right) \tag{10}$$

$$\mathbf{x}_2 = (35/44, 29/22, 127/44)\,. \tag{11}$$

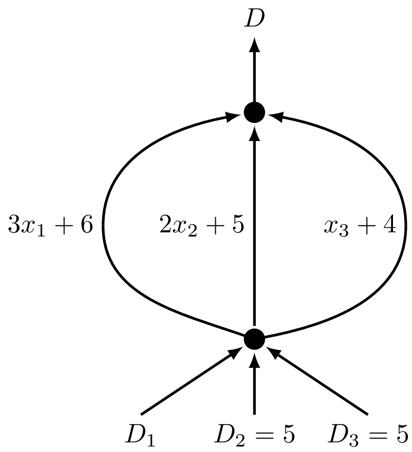


Figure 3: Example of affine game.

We see that only the flows of the first player are dependent on D_1 at the Nash equilibrium in laminar flow. This observation is the object of Theorem 1. The cost of the first player is $(24D_1^2 + 444D_1 - 3)/44$ and the one of players 2 and 3 is $(120D_1 + 2217)/44$. The Nash equilibrium of the game is laminar if $D_1 \geq 5/8$. If $D_1 < 5/8$, the flow is not laminar anymore and $x_{1,1} = 0$. The new equilibrium is

$$\mathbf{x}_1 = \left(0, \frac{4D_1 - 1}{12}, \frac{8D_1 + 1}{12}\right) \tag{12}$$

$$\mathbf{x}_2 = \left(\frac{2D_1 + 25}{33}, \frac{527 - 8D_1}{396}, \frac{1153 - 16D_1}{396}\right) \tag{13}$$

where $\mathbf{x}_2$ now depends on D_1. This equilibrium is valid for $1/4 \leq D_1 \leq 5/8$. The cost of the first player is given by $(928D_1^2 + 15824D_1 - 33)/1584$ and the one of the other players by $(-64D_1^2 + 13280D_1 + 239261)/4752$. Figure 4 shows the equilibrium as a function of D_1. The total cost and player's costs is represented in Figure 4a. Prices and flows in each arcs are given respectively in Figure 4b and 4c. The individual flows of players one and two in the two first arcs is plotted in Figure 4d.

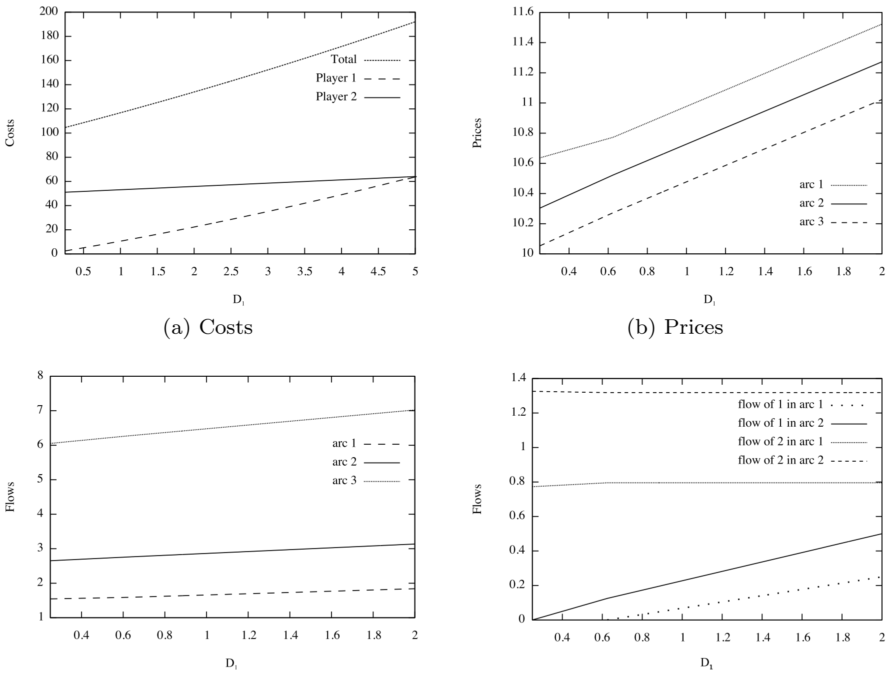
As highlighted by the example, we have the following result for a laminar nonatomic congestion game:

Theorem 1. *Consider a nonatomic congestion game with affine cost functions. If the Nash equilibrium is laminar then the flow of each player is independent of other players.*

Proof. In the case of affine prices and laminar flow, the equilibrium point of the game can be computed by solving the following system of equations:

$$\sum_{t\in\mathcal{T}} x_{i,t} = D_i \qquad \forall i \in \mathcal{K} \tag{14a}$$

$$2a_t x_{i,t} + a_t \sum_{j\in\mathcal{K}\setminus\{i\}} x_{j,t} - \lambda_i = -b_t \qquad \forall i \in \mathcal{K}, t \in \mathcal{T} \tag{14b}$$



(a) Costs (b) Prices

(c) Flows (d) Player's flows

Figure 4: Equilibrium of a three players and three arcs game.

We denote this system $Ay = d$ where $y = [\mathbf{x}_1 \dots \mathbf{x}_k \ \lambda_1 \dots \lambda_k]^T$. The sketch of the proof is as follows: we provide the analytical formula of A^{-1}. The inverse is used to obtain $y = A^{-1}d$ which leads to the analytical formula of $x_{i,t}$. We introduce the following convenient notations:

$$\beta = \sum_{t \in \mathcal{T}} \prod_{v \in \mathcal{T} \setminus \{t\}} a_v \tag{15a}$$

$$\alpha_t = \frac{\prod_{v \in \mathcal{T} \setminus \{t\}} a_v}{\beta} \tag{15b}$$

$$\delta_{t,u} = \frac{\prod_{v \in \mathcal{T} \setminus \{t,u\}} a_v}{\beta(k+1)} = \delta_{u,t} \tag{15c}$$

$$\gamma_t = \sum_{u \in \mathcal{T} \setminus \{t\}} \delta_{t,u} \tag{15d}$$

Observe that $\beta, \alpha_t, \delta_{t,u}$ and γ_t only depend on k, a_t and b_t. Using the analytical form of A^{-1} provided in the appendix, we obtain

$$x_{i,t} = D_i \alpha_t - b_t \gamma_t - \sum_{u \in \mathcal{T} \setminus \{t\}} b_u \delta_{t,u} \tag{16}$$

$$= \frac{D_i(k+1) \prod_{v \in \mathcal{T} \setminus \{t\}} a_v - b_t \sum_{u \in \mathcal{T} \setminus \{t\}} \prod_{v \in \mathcal{T} \setminus \{t,u\}} + \sum_{u \in \mathcal{T} \setminus \{t\}} b_u \prod_{v \in \mathcal{T} \setminus \{t,u\}} a_v}{(k+1) \sum_{t \in \mathcal{T}} \prod_{v \in \mathcal{T} \setminus \{t\}} a_v} \tag{17}$$

The complete proof is available in the appendix. □

The motivation for studying laminar flows is that it leads to a clean analysis. The following results are consequences from the previous theorem.

Corollary 1. *Consider a nonatomic congestion game with affine cost functions whose Nash equilibrium is laminar. At this equilibrium, the arc flows and the costs depend only on the total demand.*

Corollary 2. *If each player $i \in \mathcal{K}$ demand D_i is such that, $\forall t \in \mathcal{T}$*

$$x_{i,t}^{\min} < D_i\alpha_t - b_t\gamma_t - \sum_{u\in\mathcal{T}\setminus\{t\}} b_u\delta_{t,u} < x_{i,t}^{\max} \tag{18}$$

then the Nash equilibrium of the congestion game is laminar.

4 Ratio between the maximum and minimum arc cost

We are now interested in the ratio between the maximum cost of sending flow in one arc with respect to the minimum cost. In the following we make the hypothesis that the game is laminar with affine costs functions $c_t(x_t) = a_t x_t + b_t$ and $a_t, b_t \in \mathbb{R}_+$. Note that the following applies also for symmetric players with the additional constraints $x_{i,t} \geq 0$ by removing the edges in which $x_{i,t} = 0$.

The following theorem proves a bound on this ratio depending only on the number of players and the y-intercepts b_t which can be itself bounded by $\frac{k+1}{k}$.

Theorem 2. *Consider a k-players nonatomic congestion game with affine cost functions of the form $a_t x_t + b_t$ and $a_t, b_t \in \mathbb{R}_+$. If the Nash equilibrium is laminar then the ratio between the maximum and the minimum arc cost, occurring respectively in arcs t and u, is bounded by*

$$\frac{(k+1)b_t}{b_u + kb_t} \leq \frac{k+1}{k} \tag{19}$$

Proof of Theorem 2. If the Nash equilibrium is laminar, the equilibrium strategy of player i is obtained by solving the system

$$\sum_{t\in\mathcal{T}} x_{i,t} = D_i : \lambda_i \qquad \forall i \in \mathcal{K} \tag{20}$$

$$\lambda_i = 2a_t x_{i,t} + a_t \sum_{j\in\mathcal{K}\setminus\{i\}} x_{j,t} + b_t \qquad \forall t \in \mathcal{T}, i \in \mathcal{K} \tag{21}$$

The set of equations given by (21) can be written on the form

$$a_t \begin{pmatrix} 2 & 1 & 1 \\ 1 & \ddots & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_{1,t} \\ \vdots \\ x_{i,t} \\ \vdots \\ x_{k,t} \end{pmatrix} = \begin{pmatrix} \lambda_1 - b_t \\ \vdots \\ \lambda_i - b_t \\ \vdots \\ \lambda_k - b_t \end{pmatrix} \tag{22}$$

which can be concisely written as

$$a_t(\mathbb{1}_k + \mathbb{I}_k)\mathbf{x}_t^K = \lambda^K - b_t \tag{23}$$

where $\mathbb{I}_k$ is an identity matrix of dimension k and $\mathbb{1}_k$ a square matrix of ones of dimension k. For $a_t > 0$,

$$\mathbf{x}_t^K = (\mathbb{1}_k + \mathbb{I}_k)^{-1} \frac{\lambda^K - b_t}{a_t} \tag{24}$$

$$= \left(\mathbb{I}_k - \frac{1}{k+1} \mathbb{1}_k \right) \frac{\lambda^K - b_t}{a_t} \tag{25}$$

using Lemma 5 available in the Appendix. In particular for player i,

$$x_{i,t} = \frac{k}{k+1} \frac{\lambda_i - b_t}{a_t} + \sum_{j \in \mathcal{K} \setminus \{i\}} \frac{-1}{k+1} \frac{\lambda_j - b_t}{a_t} \tag{26}$$

$$= \frac{k\lambda_i - \sum_{j \in \mathcal{K}} \lambda_j - b_t}{(k+1)a_t} \tag{27}$$

$$x_t = \sum_{i \in \mathcal{K}} x_{i,t} = \frac{\sum_{i \in \mathcal{K}} \lambda_i - kb_t}{(k+1)a_t} \tag{28}$$

The sums of the dual variables λ_i can be bounded independently of x_t using (28) and $a_t, x_t \geq 0$.

$$\sum_{i \in \mathcal{K}} \lambda_i = (k+1)a_t x_t + kb_t \geq kb_t \tag{29}$$

The following observation is used later to bound the ratio.

Observation 2. *Given $a, b, c, d \in \mathbb{R}_+$. If $a \geq c$ and $b \geq d$ then*

$$\frac{a+b}{c+b} \leq \frac{a+d}{c+d}. \tag{30}$$

For convenience, we define that the maximum cost is obtained in arc t and the minimum in arc u. The ratio between the maximum and the minimum arc cost in the case where $a_t, a_u > 0$ is given by

$$\frac{\max\{c_t^N | t \in \mathcal{T}\}}{\min\{c_t^N | t \in \mathcal{T}\}} = \frac{c_t^N}{c_u^N} = \frac{b_t + a_t x_t}{b_u + a_u x_u} \tag{31}$$

$$= \frac{b_t + \frac{\sum_{i \in \mathcal{K}} \lambda_i - kb_t}{k+1}}{b_u + \frac{\sum_{i \in \mathcal{K}} \lambda_i - kb_u}{k+1}} \tag{32}$$

$$= \frac{b_t + \sum_{i \in \mathcal{K}} \lambda_i}{b_u + \sum_{i \in \mathcal{K}} \lambda_i} \tag{33}$$

$$\leq \frac{(k+1)b_t}{b_u + kb_t} \leq \frac{k+1}{k} \tag{34}$$

where the last inequality is obtained using (29) and Observation 2. The previous bound is also valid for the case where $a_t = 0$. The proof is straightforward using (28) and that (21) simplifies into $\lambda_i = b_t$.

We now focus on the case where $a_u = 0$. In this period, (21) simplifies into $\lambda_i = b_u$ and we also have $b_t \leq b_u$ as $c_t^N \geq c_u^N$. The ratio between the maximum and the minimum arc cost can be bounded by

$$\frac{\max\{c_t^N | t \in \mathcal{T}\}}{\min\{c_t^N | t \in \mathcal{T}\}} = \frac{c_t^N}{c_u^N} = \frac{b_t + a_t x_t}{b_u} \tag{35}$$

$$= \frac{b_t + \frac{\sum_{i\in\mathcal{K}} \lambda_i - k b_t}{k+1}}{b_u} \tag{36}$$

$$= \frac{b_t + \sum_{i\in\mathcal{K}} \lambda_i}{(k+1) b_u} \tag{37}$$

$$= \frac{b_t + k b_u}{(k+1) b_u} \leq 1 \tag{38}$$

Obviously, if $a_t = a_u = 0$ all the prices are equal. □

The example of Section 6 taken from [12] and given in Figure 5 shows that this bound is tight. At the Nash equilibrium, each player sends the flow $(0, k)$ resulting in the prices $(1, \frac{k}{k+1})$.

5 Price of flexibility

Assume that arc costs are fixed at the values of a laminar Nash equilibrium with c_t^N and $\partial_t^N = \frac{\partial c_t(x_t^N)}{\partial x_t} \geq 0$ are taken as data.

Theorem 3. *Consider a nonatomic congestion game with a laminar Nash equilibrium, the price of flexibility for an imposed small deviation in an arc is at least two times the first derivative of the corresponding arc cost at the Nash equilibrium without deviation.*

Proof of Theorem 3. Let us fix a player $i \in \mathcal{K}$ and a period $u \in \mathcal{T}$. At this equilibrium, player i has no incentive to deviate from the strategy $\mathbf{x}_i^N$. Assume we impose a small deviation Δ_u to player i in a unique arc u such that $x_{i,u} = x_{i,u}^N + \Delta_u$. Player i can solve the following optimization problem to modify its strategy:

$$\min_{\mathbf{x}_i} \quad \sum_{t\in\mathcal{T}} (c_t^N + \partial_t^N (x_{i,t} - x_{i,t}^N)) x_{i,t} \tag{39a}$$

$$\text{s.t.} \quad \sum_{t\in\mathcal{T}} x_{i,t} = D_i \qquad : \lambda_i \tag{39b}$$

$$x_{i,u} = x_{i,u}^N + \Delta_u \qquad : \mu_{i,u} \tag{39c}$$

Which can be reformulated by taking the new solution with respect to the Nash equilibrium by introducing the variables $\epsilon_{i,t}$ such that $x_{i,t} = x_{i,t}^N + \epsilon_{i,t}$. The optimization problem using the variables $\epsilon_{i,t}$ is

$$\min \quad \sum_{t\in\mathcal{T}} \left(\partial_t^N \epsilon_{i,t}^2 + (c_t^N + \partial_t^N x_{i,t}^N) \epsilon_{i,t}\right) \tag{40a}$$

$$\text{s.t.} \quad \sum_{t\in\mathcal{T}} \epsilon_{i,t} = 0 \qquad : \lambda_i \tag{40b}$$

$$\epsilon_{i,u} = \Delta_u \qquad : \mu_{i,u} \tag{40c}$$

Note that this problem is convex as $\partial_t^N \geq 0$. The Lagrangian reads

$$L_{i,u} = \sum_{t \in \mathcal{T}} \left(\partial_t^N \epsilon_{i,t}^2 + (c_t^N + \partial_t^N x_{i,t}^N)\epsilon_{i,t} \right) - \lambda_i \sum_{t \in \mathcal{T}} \epsilon_{i,t} - \mu_{i,u}(\epsilon_{i,u} - \Delta_u). \tag{41}$$

Canceling the derivative of the Lagrangian with respect to $\epsilon_{i,u}$ gives

$$\lambda_i + \mu_{i,u} = c_u^N + \partial_u^N x_{i,u}^N + 2\partial_u^N \Delta_u \tag{42}$$

$$= \lambda_i^N + 2\partial_u^N \Delta_u \tag{43}$$

Canceling the derivative of the Lagrangian with respect to $\epsilon_{i,t}$ with $t \neq u$ gives

$$\lambda_i = c_t^N + \partial_t^N x_{i,t}^N + 2\partial_t^N \epsilon_{i,t} \tag{44}$$

$$= \lambda_i^N + 2\partial_t^N \epsilon_{i,t}. \tag{45}$$

Lemma 1. *Assume that arc costs are fixed at the values of a laminar Nash equilibrium with c_t^N and $\partial_t^N = \frac{\partial c_t(x_t^N)}{\partial x_t} \geq 0$ are taken as data. Assume also an imposed deviation Δ_u in period u and writes the deviation in other periods t, $\epsilon_{i,t}$. Then $\epsilon_{i,t}$ is of opposite sign that Δ_u and such that $|\epsilon_{i,t}| \leq \Delta_u$, $\forall t \in \mathcal{T} \setminus \{u\}$.*

Proof of Lemma 1. Taking equation (45) for two arcs $t, v \in \mathcal{T} \setminus \{u\}$ yields $\partial_t^N \epsilon_{i,t} = \partial_v^N \epsilon_{i,v}$. As $\partial_t^N, \partial_v^N \geq 0$, $\epsilon_{i,t}$ and $\epsilon_{i,v}$ have the same sign. Using (40c) in (40b) gives $\sum_{t \in \mathcal{T} \setminus \{u\}} \epsilon_{i,t} = -\Delta_u$ which implies the lemma. □

Using Lemma 1 and (45) yields

$$\lambda_i \in \left[\lambda_i^N - 2\Delta_u \max_{t \in \mathcal{T} \setminus \{u\}} \partial_t^N, \lambda_i^N \right] \tag{46}$$

Injecting the latter in (43) yields

$$\mu_{i,u} \in \left[2\Delta_u \partial_u^N, 2\Delta_u \left(\partial_u^N + \max_{t \in \mathcal{T} \setminus \{u\}} \partial_t^N \right) \right] \tag{47}$$

which provides bounds on the flexibility price for each period u. Note that the dual variable $\mu_{i,u}$ is not directly dependent on the player's i data. Therefore, $\mu_{i,u}$ defines a single flexibility price in each arc u for any player i. Note that from (47), a marginal price of flexibility for arc u can easily be defined by dividing by Δ_u. Note also that the minimum bound is only dependent on the period under consideration. To match with the needs of simplicity for real life applications, we advise to take this minimum bound as the reference flexibility price. □

For instance, based on the market clearing of Figure 1b the price of flexibility for this period would be of 5.478 €cent/MWh which is three order of magnitude less than the energy price.

6 Price of anarchy

This section provides a bound on the price of anarchy for a nonatomic congestion game with laminar Nash equilibrium and affine cost functions $c_t(x_t) = a_t x_t + b_t$ and $a_t, b_t > 0$. The following lemma provides necessary and sufficient conditions on the Nash equilibrium in laminar flow.

Lemma 2. *In laminar flow, the quantities* $\mathbf{x}^N$ *are at Nash equilibrium if and only if,* $\forall t, u \in \mathcal{T}$,

$$\frac{k+1}{k} a_t x_t^N + b_t = \frac{k+1}{k} a_u x_u^N + b_u \tag{48}$$

where the last constraint is given by (7) in the affine case.

Proof of Lemma 2. Applying the optimality conditions (7) to the case of laminar flow and affine cost functions yields,

$$\lambda_i = a_t x_{i,t} + a_t x_t + b_t. \tag{49}$$

Summing over the players and dividing by k gives

$$1/k \sum_{i \in \mathcal{K}} \lambda_i = \frac{k+1}{k} a_t x_t^N + b_t \tag{50}$$

As the right member is independent of t, (50) may be applied to particular arcs t and u to obtain (48). □

A special case of Lemma 2 worth to be highlighted.

Corollary 3. *The optimal flows* x^* *in laminar flow with affine cost functions satisfies the following condition:* $\forall t, u \in \mathcal{T}$,

$$2a_t x_t^* + b_t = 2a_u x_u^* + b_u \tag{51}$$

The optimal flow corresponds to the case $k = 1$. Note that the case $k = +\infty$ is equivalent to say that the cost of every arc is equal at Nash equilibrium.

The following of the proof follows the same steps as in [4].

Lemma 3. *Note* $\mathbf{x}^N$ *the Nash equilibrium of a* k *players game in laminar flows and affine cost functions with a total flow of* D*. The flow* $\gamma \mathbf{x}^N$ *is optimal for the same game with a total flow of* γD *where* $\gamma = \frac{k+1}{2k}$.

Proof of Lemma 3. As $\mathbf{x}^N$ satisfies equation (48), the demand allocation γx^N satisfies equation (51). □

The following lemma is taken from [4] and adapted to our notations.

Lemma 4. *Suppose an instance of a total flow of* D *for which* $\mathbf{x}^*$ *is an optimal flow. Let* $l_t(x_t)$ *be the minimum marginal cost of increasing the flow in arc* t *with respect to* x_t*. Then, for any* $\delta \geq 0$*, a feasible flow for the same instance with of total flow* $(1+\delta)D$ *has cost at least*

$$C(\mathbf{x}^*) + \delta \sum_{t \in \mathcal{T}} l_t(x_t^*) x_t^* \tag{52}$$

Proof of Lemma 4. See Lemma 4.4 of article [4]. □

The main results can now be obtained using the previous lemmas.

Theorem 4. *The price of anarchy of a k-players nonatomic congestion with a laminar Nash equilibrium and affine cost functions with positive coefficients is at most*

$$\frac{4k^2}{(k+1)(3k-1)} \tag{53}$$

Proof of Theorem 4. The laminar Nash equilibrium flow $\mathbf{x}^N$ for the game of total demand D is such that the flow $\gamma\mathbf{x}^N$ with $\gamma = \frac{k+1}{2k}$ is optimal for the same game with a total demand of γD. The cost of the optimal flow $\mathbf{x}^*$ can be bounded with respect to $\gamma\mathbf{x}^N$ using Lemma 4:

$$\begin{aligned}
C(\mathbf{x}^*) &\geq C(\gamma\mathbf{x}^N) + \frac{1-\gamma}{\gamma}\sum_{t\in\mathcal{T}} l_t(\gamma x_t^N)\gamma x_t^N && (54)\\
&= \sum_{t\in\mathcal{T}} \left(a_t\gamma^2(x_t^N)^2 + b_t\gamma x_t^N\right) + (1-\gamma)\sum_{t\in\mathcal{T}}(2a_t\gamma x_t^N + b_t)x_t^N && (55)\\
&= \sum_{t\in\mathcal{T}} \left[a_t\left(\frac{k+1}{2k}\right)^2(x_t^N)^2 + b_t\frac{k+1}{2k}x_t^N + \frac{k-1}{2k}\left(2a_t\frac{k+1}{2k}(x_t^N)^2 + b_t x_t^N\right)\right] && (56)\\
4k^2C(\mathbf{x}^*) &\geq \sum_{t\in\mathcal{T}} \left[\left(3k^2+2k-1\right)a_t(x_t^N)^2 + 4k^2 b_t x_t^N\right] && (57)\\
4k^2C(\mathbf{x}^*) &\geq (3k^2+2k-1)C(x^N) = (k+1)(3k-1)C(x^N) && (58)
\end{aligned}$$

where the transition from (57) to (58) is given by $4k^2 \geq 3k^2 + 2k - 1$ for $k \in [1, +\infty[$. □

Note that for $k = 1$ we get that $C(\mathbf{x}^*) \leq C(\mathbf{x}^N)$ and for $k \to \infty$ the result tends to the price of anarchy of 4/3 found in [4]. In a two player game system with affine prices and positive coefficients, the price of anarchy is at most 16/15. Figure 5 shows an example taken from [12] of k players controlling a demand of k where the bound on the price of anarchy is tight. The optimum flow is $(\frac{k-1}{2}, \frac{k+1}{2})$ with a total cost of $\frac{3k-1}{4}$. At the Nash equilibrium, each retailer games $(0, 1)$ resulting in the prices $(1, \frac{k}{k+1})$ and a total cost of $\frac{k^2}{k+1}$.

Appendix

Lemma 5. *The inverse of $\mathbb{1}_k + \mathbb{I}_k$, where $\mathbb{I}_k$ is an identity matrix of dimension k and $\mathbb{1}_k$ a square matrix of ones of dimension k, is $\mathbb{I}_k - \frac{1}{k+1}\mathbb{1}_k$.*

Proof. The proof is obtained by showing than multiplying $\mathbb{1}_k + \mathbb{I}_k$ by the candidate inverse yields the identity matrix.

$$\begin{aligned}
(\mathbb{1}_k + \mathbb{I}_k)(\mathbb{I}_k - \frac{1}{k+1}\mathbb{1}_k) &= \mathbb{1}_k + \mathbb{I}_k - \frac{1}{k+1}\mathbb{1}_k\mathbb{1}_k - \frac{1}{k+1}\mathbb{1}_k && (59)\\
&= \mathbb{I}_k + \mathbb{1}_k(1 - \frac{k}{k+1} - \frac{1}{k+1}) && (60)\\
&= \mathbb{I}_k && (61)
\end{aligned}$$

where (60) is obtained using the fact that $\mathbb{1}_k\mathbb{1}_k = k\mathbb{1}_k$. □

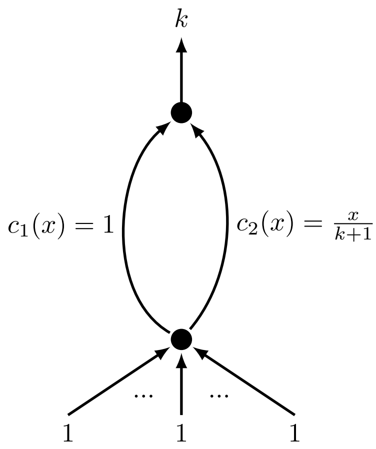


Figure 5: Example of nonatomic congestion game for which the bound on the price of anarchy and on the ratio between the maximum and minimum arc cost is tight.

Theorem 1. *Consider a nonatomic congestion game with affine cost functions. If the Nash equilibrium is laminar then the flow of each player is independent of other players.*

Proof of Theorem 1. In the affine case where $x_{i,t} > 0 \; \forall i \in \mathcal{K}, t \in \mathcal{T}$, the laminar equilibrium point can be computed by solving the following system of equations:

$$\sum_{t \in \mathcal{T}} x_{i,t} = D_i \qquad \forall i \in \mathcal{K} \tag{62a}$$

$$2a_t x_{i,t} + a_t \sum_{j \in \mathcal{K} \setminus \{i\}} x_{j,t} - \lambda_i = -b_t \qquad \forall i \in \mathcal{K}, t \in \mathcal{T} \tag{62b}$$

We solve this linear system (62) of the form $Ay = d$ where

$$y = \left(\mathbf{x}_1^T, \ldots, \mathbf{x}_k^T, \lambda_1, \ldots, \lambda_k\right)^T \tag{63}$$

$$d = \left(D_1, \ldots, D_k, -b_1, \ldots, -b_T, \ldots, -b_1, \ldots, -b_T\right)^T. \tag{64}$$

As an illustration, we provide an example of (62) with three arcs and two players. We have

$$d = \left(D_1 \, D_2 - b_1 - b_2 - b_3 - b_1 - b_2 - b_3\right)^T \tag{65}$$

$$A = \left(\begin{array}{ccc:ccc:cc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 2a_1 & 0 & 0 & a_1 & 0 & 0 & -1 & 0 \\ 0 & 2a_2 & 0 & 0 & a_2 & 0 & -1 & 0 \\ 0 & 0 & 2a_3 & 0 & 0 & a_3 & -1 & 0 \\ \hdashline a_1 & 0 & 0 & 2a_1 & 0 & 0 & 0 & -1 \\ 0 & a_2 & 0 & 0 & 2a_2 & 0 & 0 & -1 \\ 0 & 0 & a_3 & 0 & 0 & 2a_3 & 0 & -1 \end{array}\right). \tag{66}$$

The horizontal line delimits the constraints (62a) which corresponds to the line indexes $m \leq k$. We define

$$\beta = \sum_{t\in\mathcal{T}} \prod_{v\in\mathcal{T}\setminus\{t\}} a_v \tag{67a}$$

$$\alpha_t = \frac{\prod_{v\in\mathcal{T}\setminus\{t\}} a_v}{\beta} \tag{67b}$$

$$\delta_{t,u} = \frac{\prod_{v\in\mathcal{T}\setminus\{t,u\}} a_v}{\beta(k+1)} = \delta_{u,t} \tag{67c}$$

$$\gamma_t = \sum_{u\in\mathcal{T}\setminus\{t\}} \delta_{t,u} \tag{67d}$$

$$\omega = \frac{\prod_{t\in\mathcal{T}} a_t}{\beta}. \tag{67e}$$

We claim that the inverse of the matrix A defined in (66) is given by

$$B = \left(\begin{array}{cc|ccc:ccc}
\alpha_1 & 0 & 2\gamma_1 & -2\delta_{1,2} & -2\delta_{1,3} & -\gamma_1 & \delta_{1,2} & \delta_{1,3} \\
\alpha_2 & 0 & -2\delta_{2,1} & 2\gamma_2 & -2\delta_{2,3} & \delta_{2,1} & -\gamma_2 & \delta_{2,3} \\
\alpha_3 & 0 & -2\delta_{3,1} & -2\delta_{3,2} & 2\gamma_3 & \delta_{3,1} & \delta_{3,2} & -\gamma_3 \\
\hdashline
0 & \alpha_1 & -\gamma_1 & \delta_{1,2} & \delta_{1,3} & 2\gamma_1 & -2\delta_{1,2} & -2\delta_{1,3} \\
0 & \alpha_2 & \delta_{2,1} & -\gamma_2 & \delta_{2,3} & -2\delta_{2,1} & 2\gamma_2 & -2\delta_{2,3} \\
0 & \alpha_3 & \delta_{3,1} & \delta_{3,2} & -\gamma_3 & -2\delta_{3,1} & -2\delta_{3,2} & 2\gamma_3 \\
\hdashline
2\omega & \omega & -\alpha_1 & -\alpha_2 & -\alpha_3 & 0 & 0 & 0 \\
\omega & 2\omega & 0 & 0 & 0 & -\alpha_1 & -\alpha_2 & -\alpha_3
\end{array}\right). \tag{68}$$

The vertical line delimits column indexes $n \leq k$. The analytical solution for $x_{i,t}$ can be obtained by taking the corresponding element of Bd. For instance, we have for the first player in the second arc

$$x_{1,2} = \alpha_2 D_1 + 2\delta_{2,1} b_1 - 2\gamma_2 b_2 + 2\delta_{2,3} b_3 - \delta_{2,1} b_1 + \gamma_2 b_2 - \delta_{2,3} b_3 \tag{69}$$

$$= \frac{3D_1 a_1 a_3 - b_2(a_1 + a_3) + b_1 a_3 + b_3 a_1}{3(a_1 a_2 + a_1 a_3 + a_2 a_3)}. \tag{70}$$

We now consider the general case of k players and T arcs and derive a complete description of A. Let us fix a row index m and a column index n. We define for $m > k$, indexes dependent on m

$$i(m) = \lfloor (m - 1 - k)/T \rfloor + 1 \tag{71}$$

$$t(m) = (m - 1 - k) \bmod T + 1 \tag{72}$$

which for the sake of conciseness are denoted i and t. Observe that i represents the player corresponding to the choice of the row m and t corresponds to the period. In the following, m is a row index and n a column index. The element (m, n) of a matrix A is denoted $A(m, n)$, its m^{th} row $A(m, :)$ and its n^{th} column $A(:, n)$. The non-zero elements of A are

$$A(m, n) = 1 \qquad \forall m \leq k, n \in \{T(i-1)+1, Ti\} \tag{73a}$$

$$A(m, (i-1)T + t) = 2a_t \qquad \forall m > k \tag{73b}$$

$$A(m, (l-1)T + t) = a_t \qquad \forall m > k, l \in \mathcal{K} \setminus \{i\} \tag{73c}$$

$$A(m, kT + i - 1) = -1 \qquad \forall m > k \tag{73d}$$

In order to define the elements of B, the candidate inverse matrix, we need two further sets of indices for columns $n > k$:

$$j(n) = \lfloor (n-1-k)/T \rfloor + 1 \tag{74}$$

$$u(n) = (n-1-k) \bmod T + 1 \tag{75}$$

which for the sake of conciseness are denoted j and u. Observe that j represents the player corresponding to the choice of the column n and u corresponds to the period. We define

$$B(m,n) = \alpha_{(m-1) \bmod T+1} \qquad \forall n \le k, \lfloor (m-1)/T \rfloor + 1 = n \tag{76a}$$

$$B(m,n) = 0 \qquad \forall n \le k, m \le kT : \lfloor (m-1)/T \rfloor + 1 \ne n \tag{76b}$$

$$B(m,n) = 2\omega \qquad \forall n \le k, m : m - kT = n \tag{76c}$$

$$B(m,n) = \omega \qquad \forall n \le k, m > kT : m - kT \ne n \tag{76d}$$

$$B(m,n) = k\gamma_u \qquad \forall n > k, m \le kT : i = j, t = u \tag{76e}$$

$$B(m,n) = -\gamma_u \qquad \forall n > k, m \le kT : i \ne j, t = u \tag{76f}$$

$$B(m,n) = -k\delta_{t,u} \qquad \forall n > k, m \le kT : i = j \tag{76g}$$

$$B(m,n) = \delta_{t,u} \qquad \forall n > k, m \le kT : i \ne j \tag{76h}$$

$$B(m,n) = -\alpha_u \qquad \forall n > k, m : m - kT = j \tag{76i}$$

$$B(m,n) = 0 \qquad \forall n > k, m > kT : m - kT \ne j \tag{76j}$$

We claim that B is the inverse of A. To prove this claim, we perform the inner product of rows of A with columns of B and show that we obtain the element of an identity matrix. The reader is advised to use the matrices of the example given in (66) and (68) as support.

$\mathbf{m = n \le k}$**:** In the example, this case corresponds to the inner product of row 1 of (66) and column 1 of (68).

$$A(m,:)B(:,n) = \sum_{v \in \mathcal{T}} (73a)(76a) \tag{77}$$

$$= \sum_{v \in \mathcal{T}} \alpha_v = 1 \tag{78}$$

$\mathbf{m = n \le k}$**:** In the example, this case corresponds to the inner product of row 1 of (66) and column 1 of (68).

$$A(m,:)B(:,n) = \sum_{v \in \mathcal{T}} (73a)(76a) \tag{79}$$

$$= \sum_{v \in \mathcal{T}} \alpha_v = 1 \tag{80}$$

$\mathbf{m, n \le k, m \ne n}$**:** This case corresponds to the inner product of row 1 of (66) and column 2 of (68).

$$A(m,:)B(:,n) = \sum_{v \in \mathcal{T}} (73a)(76b) = 0 \tag{81}$$

$\mathbf{m \le k, n > k}$:

- $m = j$: This case corresponds to the inner product of row 1 of (66) and column 3 of (68) with $j = 1$ and $u = 1$.

$$A(m,:)B(:,n) = \text{(73a)(76e)} + \sum_{v \in \mathcal{T} \setminus \{u\}} \text{(73a)(76g)} \tag{82}$$

$$= k\gamma_u - \sum_{v \in \mathcal{T} \setminus \{u\}} k\delta_{u,v} = 0 \tag{83}$$

- $m \neq j$: This case corresponds to the inner product of row 1 of (66) and column 6 of (68) with $j = 2$ and $u = 1$.

$$A(m,:)B(:,n) = \text{(73a)(76e)} + \sum_{v \in \mathcal{T} \setminus \{u\}} \text{(73a)(76h)} \tag{84}$$

$$= -\gamma_u + \sum_{v \in \mathcal{T} \setminus \{u\}} \delta_{u,v} = 0 \tag{85}$$

$\mathbf{m > k, n \le k}$:

- $n = i$: This case corresponds to the inner product of row 3 of (66) and column 1 of (68) with $i = 1$ and $t = 1$.

$$A(m,:)B(:,n) = \text{(73b)(76a)} + \sum_{l \in \mathcal{K} \setminus \{i\}} \text{(73c)(76b)} + \text{(73d)(76c)} \tag{86}$$

$$= 2a_t\alpha_t + 0 - 2\omega = 0 \tag{87}$$

- $n \neq i$: This case corresponds to the inner product of row 3 of (66) and column 2 of (68) with $i = 1$ and $t = 1$.

$$A(m,:)B(:,n) = \text{(73b)(76b)} + \text{(73c)(76a)} + \sum_{l \in \mathcal{K} \setminus \{i,j\}} \text{(73c)(76b)} + \text{(73d)(76d)} \tag{88}$$

$$= 0 + a_t\alpha_t + 0 - \omega = 0 \tag{89}$$

as $a_t\alpha_t = \omega$.

$\mathbf{m = n > k}$: This case corresponds to the inner product of row 3 of (66) and column 3 of (68) with $i = j = 1$ and $t = u = 1$. Note that $a_t\delta_{t,u} = \frac{\alpha_u}{k+1}$ and consequently $a_t\gamma_t = \frac{\sum_{u \in \mathcal{T} \setminus \{t\}} \alpha_u}{k+1}$. We have,

$$A(m,:)B(:,n) = \text{(73b)(76e)} + \sum_{l \in \mathcal{K} \setminus \{i\}} \text{(73c)(76f)} + \text{(73d)(76i)} \tag{90}$$

$$= 2a_t k\gamma_t - \sum_{l \in \mathcal{K} \setminus \{i\}} a_t\gamma_t + \alpha_t \tag{91}$$

$$= (k+1)a_t\gamma_t + \alpha_t \tag{92}$$

$$= \sum_{u \in \mathcal{T}} \alpha_u = 1 \tag{93}$$

$\mathbf{m, n > k, m \neq n}$:

- $t = u$ and $i \neq j$: This case corresponds to the inner product of row 1 of (66) and column 6 of (68) with $i = 1$, $j = 2$ and $t = u = 1$.

$$A(m,:)B(:,n) = \text{(73b)(76f)} + \text{(73c)(76e)} + \sum_{l \in \mathcal{K} \setminus \{i,j\}} \text{(73c)(76f)} + \text{(73d)(76j)} \tag{94}$$

$$= -2a_t\gamma_t + a_t k\gamma_t - \sum_{l \in \mathcal{K} \setminus \{i,j\}} a_t\gamma_t + 0 = 0 \tag{95}$$

- $t \neq u$ and $i = j$: This case corresponds to the inner product of row 1 of (66) and column 4 of (68) with $i = j = 1$, $t = 1$ and $u = 2$.

$$A(m,:)B(:,n) = \text{(73b)(76g)} + \sum_{l \in \mathcal{K} \setminus \{i\}} \text{(73c)(76h)} + \text{(73d)(76i)} \tag{96}$$

$$= -2a_t k\delta_{t,u} + \sum_{l \in \mathcal{K} \setminus \{i\}} a_t\delta_{t,u} + \alpha_u \tag{97}$$

$$= -(k+1)a_t\delta_{t,u} + \alpha_u = 0 \tag{98}$$

- $t \neq u$ and $i \neq j$: This case corresponds to the inner product of row 1 of (66) and column 7 of (68) with $i = 1$, $j = 2$, $t = 1$ and $u = 2$.

$$A(m,:)B(:,n) = \text{(73b)(76h)} + \text{(73c)(76g)} + \sum_{l \in \mathcal{K} \setminus \{i,j\}} \text{(73c)(76h)} + \text{(73d)(76j)} \tag{99}$$

$$= 2a_t\delta_{t,u} - a_t k\delta_{t,u} - \sum_{l \in \mathcal{K} \setminus \{i,j\}} a_t\delta_{t,u} + 0 = 0 \tag{100}$$

The analytical form of $x_{i,t}$ is obtained by taking the corresponding row of Bd and therefore

$$x_{i,t} = D_i\alpha_t - b_t\gamma_t - \sum_{u \in \mathcal{T} \setminus \{t\}} b_u\delta_{t,u} \tag{101}$$

$$= \frac{D_i(k+1)\prod_{v \in \mathcal{T} \setminus \{t\}} a_v - b_t \sum_{u \in \mathcal{T} \setminus \{t\}} \prod_{v \in \mathcal{T} \setminus \{t,u\}} + \sum_{u \in \mathcal{T} \setminus \{t\}} b_u \prod_{v \in \mathcal{T} \setminus \{t,u\}} a_v}{\beta(k+1)} \tag{102}$$

which is not dependent on the demand of other players than i. □

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