Introduction to computability
Tutorial 11

Complexity

11 December 2018

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1. Let L_1 and L_2 be two languages that belong to P. Does the language $L_1.L_2$ belong to P as well?

2. Are the languages belonging to the class co-NP decidable?

3.

   - Show that the complement of a language belonging to P also belongs to P.
   - Using this result, show that if NP $\neq$ co-NP, then P $\neq$ NP.

Polynomial transformation

Consider the languages $L_1 \in \Sigma_1^*$ and $L_2 \in \Sigma_2^*$. A *polynomial transformation* from L_1 to L_2 (notation $L_1 \propto L_2$) is a function $f : \Sigma_1^* \to \Sigma_2^*$ that satisfies the following conditions:

1. it is computable in polynomial time,
2. $f(x) \in L_2$ if and only if $x \in L_1$.

Proving NP-completeness

To prove that $P_2 \in \text{NPC}$ knowing that $P_1 \in \text{NPC}$ one must establish that

1. $P_2 \in \text{NP}$,
2. $P_2 \in \text{NP-hard}$ by showing that $P_1 \propto P_2$.

4. Show that the dominating set problem (DS) is NP-complete.

Given a graph $G = (V, E)$ and an integer $j \leqslant |V|$, the problem is to determine if there exists a subset $V' \subseteq V$ such that $|V'| \leqslant j$ and such that for each vertex $u \in V \backslash V'$ there exists a vertex $v \in V'$ such that $(u, v) \in E$.

Hint: A polynomial transformation from VC to DS proves that DS is NP-hard. The idea of the transformation is to eliminate isolated vertices, to add to the initial graph one vertex per edge and to link each new vertex to the endpoints of the corresponding edge.

Solution

We need to show that

1. DS is NP
2. DS is NP-hard

1. DS is NP

Non-deterministic polynomial algorithm for DS

1. Generate non-deterministically a subset $V' \subseteq V$ s.t. $|V'| \leqslant j$.
2. Test if it is a dominating set.

The (non-deterministic) generation of V' and the testing can be done polynomially with respect to the number of vertices $|V|$.

2. DS is NP-hard

Hint: give a polynomial transformation from VC et DS (VC $\propto$ DS)

<u>Reminder:</u>

VC

Input
- Graph $G(V, E)$
- cst $j \leqslant |V|$

Problem: does there exist $V' \subseteq V$ s.t. $|V'| \leqslant j$ and s.t.
$\forall$ edge $(u, v) \in E$, $u \in V'$ or $v \in V'$

-> all edges have at least one endpoint in V'

DS

Input
- Graph $G(V, E)$
- cst $j \leqslant |V|$

Problem: does there exist $V' \subseteq V$ s.t. $|V'| \leqslant j$ and s.t.
$\forall$ vertex $u \in V \backslash V'$, $\exists v \in V'$ s.t. $(u, v) \in E$

-> all vertices are connected with V'

We want a function $f(G, j) = (G', j')$ that is polynomial and s.t.

$$(G, j) \in \mathrm{VC} \iff (G', j') \in \mathrm{DS}.$$

- G' is obtained by removing all isolated vertices from G (those that are endpoint of no edge) and by adding a new vertex v_{ij} for each edge (v_i, v_j) of G and by adding the edges (v_i, v_{ij}) and (v_{ij}, v_j).
- $j' = j$

f is a polynomial transformation

We need to show that

1. f is polynomial
2. $x \in \mathrm{VC} \iff f(x) \in \mathrm{DS}$

<u>1. f is polynomial</u>

We have that f is polynomial as it depends linearly on the number of edges and vertices in G.

2. $x \in \text{VC} \Rightarrow f(x) \in \text{DS}$

Let V_{VC} be the solution set of $VC(G, j)$, thus each edge has at least one endpoint in V_{VC}. We will show that V_{VC} is also a solution set of $DS(G', j')$ where $(G', j') = f(G, j)$.

Let $u \in G'$ such that $u \notin V_{VC}$

$u \in G$: Thus u is not an isolated vertex, as those are not in G'. Let (u, v) be an edge of G, then $v \in V_{VC}$, thus u is connected to V_{VC}.

$u \notin G$: Thus u is one of the new vertices v_{ij} where (v_i, v_j) is an edge of G. Therefore, either v_i or v_j belongs to V_{VC}. Hence $u = v_{ij}$ is connected to V_{VC} either via (v_i, v_{ij}) or via (v_{ij}, v_j).

2. $x \in \mathrm{VC} \Leftarrow f(x) \in \mathrm{DS}$

Let V_{DS} be the solution set of $DS(G', j')$ where $(G', j') = f(G, j)$. Thus each vertex of $G' \backslash V_{DS}$ is connected to V_{DS} with an arc. We can assume without loss of generality that all vertices of V_{DS} belong to G. (*) We will show that V_{DS} is also a solution set of $VC(G, j)$. Thus, we will show that each edge (u, v) of G has an endpoint in V_{DS}.

(*) If a vertex u of V_{DS} does not belong to G (that is $u = v_{ij}$, one of the new vertices), then $V_{DS} = (V_{DS} \cup \{v_i\}) \backslash \{u\}$. The new V_{DS} is still a solution of $DS(G', j')$. The only nodes that were affected are v_i, v_j and v_{ij}. As $v_i \in V_{DS}$, we only need to show that v_j and v_{ij} are connected with V_{DS}. This is the case as (v_i, v_{ij}) and (v_i, v_j) are edges of the graph G'.

Reminder: We will show that each edge (u, v) of G has an endpoint in V_{DS}.

Given an edge $(v_i, v_j) \in G$, we will show that $v_i \in V_{DS}$ or $v_j \in V_{DS}$. We know that $v_{ij} \notin V_{DS}$ and is therefore connected to V_{DS}, that is, there exists an edge (v_{ij}, u) such that $u \in V_{DS}$. By construction however, the only edges involving v_{ij} are (v_i, v_{ij}) and (v_{ij}, v_j), therefore either $v_i \in V_{DS}$ or $v_j \in V_{DS}$.

5. Show that PSPACE $\subseteq$ EXPTIME.

Bonus Exercise 12

Let L be the language over the alphabet $\{<, >, !\}$ generated by the following grammar where S is the start symbol

$$
\begin{aligned}
&S \rightarrow\ < S > \ \mid\ > S < \ \mid \varepsilon \\
&E \rightarrow\ !E \mid \varepsilon;
\end{aligned}
$$

1. Describe L.
2. Is L a regular language?
3. Is L a context-free language?