Introduction to the Theory of Computation

Final exam

25 August 2015

Closed-book. Duration: 3h30.
Please answer each question on a separate sheet with your name and section. ***Motivate all your answers and give sufficient details.***

1. a) Is the set containing all finite binary trees denumerable?

   b) Give a sufficient criterion for the complement of a denumerable set to be denumerable?

2. a) Let L be the language on the alphabet $\{a, b\}$ of the words that contain an even number of letters a and an odd number of letters b, thus

   $L = \{w \mid w \text{ contains an even number of } a\text{'s and an odd number of } b\text{'s}\}$

   Give a DFA that accepts L and a regular grammar that generates L^R.

   b) Prove that every nondeterministic finite automaton can be converted to an equivalent one that has a single accepting state.

3. a) Is the language $\{a^k b^{3k} c^n d^{3n} \mid k, n \in \mathbb{N}\}$ regular? Is it context-free?

   b) Let L_1 and L_2 be two regular languages over the alphabets Σ_1 and Σ_2 respectively. Is the language $L_1 \oplus L_2$ that contains all the words that belong only to one of the two languages always a regular language?

4. a) Is the language $L = \{a^i b^j c^k \mid i = j \text{ or } j = k \text{ where } i, j, k \geqslant 0\}$ context-free?

   b) Given a context-free language L, do there exist algorithms for checking if $L = \varnothing$ or $L = \Sigma^*$? If so, give the algorithm.

5. a) For a Turing machine M, define the notions of *accepted* and *decided language*. Give an example of a language that is accepted but not decided by a Turing machine and an example of a language that is decided but not accepted by a Turing machine, or explain why such an example does not exist.

b) Construct a Turing machine that computes the function $f(x) = x \operatorname{div} 2$ where div is the integer division and where x is encoded using a unary alphabet, so that x is represented by x repetitions of the single letter of the alphabet. For example, if the initial tape content is $\#11111\#$, the final tape content has to be $\#11\#$.

6. a) Let n, $m \in \mathbb{N}$ with $m \neq 0$. The function Divceil(n, m) computes the value of $\frac{n}{m}$ rounded to the smallest following integer, that is

$$\text{Divceil}(n, m) = \left\lceil \frac{n}{m} \right\rceil$$

Example: Divceil$(6, 3) = 2$ and Divceil$(8, 3) = 3$.
Is the function Divceil(n, m) primitive recusrive and / or μ-recursive?

b) Prove that there exist computable functions that are not primitive recursive.

7. a) Explain what the "reduction technique" is.

b) Let M be a Turing machine and x, y and z three words. Show that determining whether all the words in the language $L = xy^*z$ are accepted by M is undecidable.

8. a) Let $L \in NP$, is L decidable? Give a counter-example or prove your statement.

b) Represent *regular* and *context-free* languages as well as the classes of languages R, RE, P, NP and NPC in a Venn diagram showing inclusions between these classes. Which of these inclusions are not yet known to be proper?