Introduction to the Theory of Computation

Final exam

January 17th, 2018

Closed-book. Duration: 3h30.
Please answer each question on a separate sheet of paper with your name and section. ***Motivate all your answers and give sufficient details.***

1. a) Is the set of all closed intervals of $\mathbb{R}$ with integer bounds denumerable?

   b) Is the set whose elements are all finite and infinite unions of closed intervals of $\mathbb{R}$ with integer bounds denumerable?

2. Let G be the following context-free grammer over the alphabet $\Sigma = \{a, b, c\}$

$$\begin{aligned} S &\rightarrow\ XcX \\ X &\rightarrow\ YX \mid Xb \mid \varepsilon \\ Y &\rightarrow\ YY \mid a \mid \varepsilon \end{aligned}$$

   Let L be the language generated by G. Give

   a) regular expressions that describe the languages generated from the nonterminals X and Y;

   a) a regular expression that describes L;

   b) a regular grammar that generates L;

   c) a NFA that accepts L;

   d) and a DFA that accepts L.

3. a) Is the language $L = \{a^n b^m \mid n \text{ divides } m\}$ regular? Is this also the case if we invert the direction of the divisibility, i.e. consider the language $L' = \{a^n b^m \mid m \text{ divides } n\}$

   b) Is the union of two non-regular languages also a non-regular language?

4. a) Let $L = \{a^n b^* c^m \mid n, m \in \mathbb{N} \text{ and } n = 2m \text{ or } m = 2n\}$. Prove that the language L is context-free.

   b) Given a context-free grammar G, describe an algorithm that decides if a word w belongs to $L(G)$.

5. a) Give a Turing machine that computes the function $f(x) = 3x$. Consider that numbers are encoded using a single symbol, so that x is represented by x repetitions of this single symbol. To simplify manipulating numbers encoded in this way, we will also use another symbol to mark the beginning and the end of the number's encoding. If q_0 is an initial state and q_f a final state of the Turing machine, you must for instance have

$$q_0 \cdot \underline{0}110\# \vdash^* q_f \cdot 01111110\underline{\#}$$

   b) State the Turing-Church Thesis. What type of justification can be given for this thesis?

6. a) Two integers x and y are said to be relatively prime if their greatest common divisor is 1. Euler's funtion $\phi(x)$ counts the positive integers up to x that are relatively prime to x. Show that $\phi(x)$ is primitive recursive. Is $\phi(x)$ μ-recursive?
   For example, $\phi(5) = 4$ and $\phi(9) = 6$.

   b) Do there exist computable functions that are not primitive recursive?

7. a) Let M be a Turing Machine. Show that the problem that consists in deciding whether there exists a word w of even length such that M stops on w is undecidable.
   Hint: Use the empty-word halting problem.

   b) Do there exist languages that are not in the class R nor in the class RE? If so, give an example and prove that it doesn't belong to these classes.

8. a) Define *polynomial transformations.*

   b) Define the *Travelling Salesman Problem* (TS) and the *Hamiltonian Circuit Problem* (HC) and give a polynomial transformation from HC to TS. What can you conclude from this transformation with respect to the membership in P of these two problems?