Introduction to the Theory of Computation

Final exam

7 January 2015

Closed-book. Duration: 3h30.
Please answer each question on a separate sheet with your name and section. ***Motivate all your answers and give sufficient details.***

1. a) Is an infinite subset of a set that is non-denumerable necessarily non-denumerable?

   b) Is the union of two denumerable sets necessarily denumerable?

   c) Show, using a cardinality argument, that there must exist uncomputable functions.

2. Let L be the language generated by the grammar

$$\begin{array}{lcl} S & \to & A \mid B \\ A & \to & YZYA \mid \varepsilon \\ B & \to & Xc \\ X & \to & aX \mid bX \mid \varepsilon \\ Y & \to & cY \mid \varepsilon \\ Z & \to & a \mid b \end{array}$$

   where S is the start symbol.

   a) For each non-terminal symbol, give a regular expression for the language generated from that symbol.

   b) Give a DFA that accepts L.

   c) Give a regular grammar that generates $\overline{L}$.

3. a) Is the language $\{a^k b^{3k} c^n d^{3n} | k, n \in \mathbb{N}\}$ regular?

   b) Let L_1 and L_2 be two regular languages over the alphabets Σ_1 and Σ_2 respectively. Is the language $L_1 \oplus L_2$ that contains all the words that belong only to one of the two languages always a regular language?

4. a) Is the language $L = \{a^i b^j c^k \mid j \geqslant i + k\}$ context-free?

   b) Given a context-free grammar G, describe an algorithm that decides if a word w belongs to $L(G)$.

5. a) For a Turing machine M, define the notions of *configuration*, *derivation*, *execution*, *accepted language* and *decided language*.

   b) Give an example of a language that is accepted but not decided by a Turing machine.

   c) – State the Turing-Church thesis. Why is it a thesis and not a theorem?
      – Imagine that one day the Turing-Church thesis is invalidated. What would be required to do this?

6. a) Define the notion of primitive recursive functions as well as the concepts used in this definition.

   b) The function SumDiv(n), where n is a strictly positive natural number, computes the sum of all proper positive divisors of n, that is, the sum of its positive divisors excluding the number itself. Show that this function is primitive recursive.
   Example: $\text{SumDiv}(6) = 1 + 2 + 3 = 6$.
   Hint: Use a function $\text{SumDivAux}(n, m)$ that computes the sum of all proper divisors of $n \leqslant m$.

   c) A perfect number is a strictly positive natural number that is equal to the sum of its proper positive divisors. Show that the predicate Perfect(n), that is true if and only if n is a perfect number, is a primitive recursive predicate.

7. a) Let M_1 and M_2 be two Turing machines that accept the languages L_1 and L_2 respectively. Show that determining whether there exists $w \in L_1$ such that M_2 stops on w is undecidable.

   b) Show that a language is accepted by a Turing machine if and only if it can be enumerated by an effective procedure.

8. a) Define the complexity class NP and all complexity measures used in this definition.

   b) Let $L \in NP$, is L decidable?

   c) State Cook's theorem. In the proof of Cook's theorem, which problem is encoded by a boolean formula?