Improved bounds on crossing numbers of graphs via semidefinite programming

Etienne de Klerk[‡] and Dima Pasechnik

[‡]Tilburg University, The Netherlands

Francqui chair awarded to Yurii Nesterov, Liege, February 17th, 2012

Outline

- The (two page) crossing numbers of complete bipartite graphs.
- A nonconvex quadratic programming relaxation of the two page crossing number of $K_{m,n}$.
- A semidefinite programming relaxation of the quadratic program and its implications.

What is semidefinite programming?

Standard form problem

$$\min_{X \succeq 0} \langle A_0, X \rangle \ \text{ subject to } \ \langle A_k, X \rangle = b_k \ \ (k = 1, \ldots, m),$$

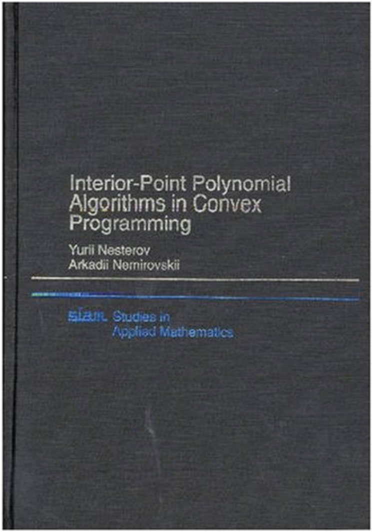
where the symmetric data matrices A_i $(i = 0, \ldots, m)$ are linearly independent.

- The inner product is the Euclidean one: $\langle A_0, X \rangle = \mathrm{trace}(A_0 X)$;
- $X \succeq 0$: X symmetric positive semi-definite.

Why is semidefinite programming interesting?

- Many applications in control theory, combinatorial optimization, structural design, electrical engineering, quantum computing, etc.
- There are polynomial-time interior-point algorithms available to solve these problems to any fixed accuracy.

How do we know this?



The authors ...

YURII NESTEROV
CATHOLIC UNIVERSITY OF LOUVAIN, BELGIUM

ARKADI NEMIROVSKII
GEORGIA TECH, USA

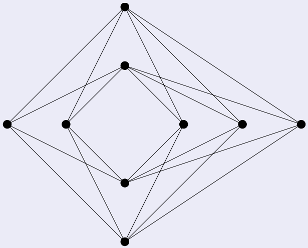
... at the HPOPT 2008 conference in Tilburg.

Crossing number of a graph

Definition

The crossing number $\mathrm{cr}(G)$ of a graph $G = (V, E)$ is the minimum number of edge crossings that can be achieved in a drawing of G in the plane.

Example: the complete bipartite graph



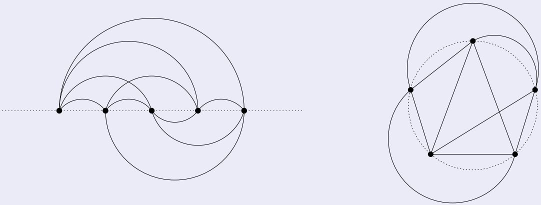
An optimal drawing of $K_{4,5}$ with $\mathrm{cr}(K_{4,5}) = 8$ edge crossings.

Two-page crossing number of a graph

Definition

In a two-page drawing of $G = (V, E)$ all vertices V must be drawn on a straight line (resp. circle) and all edges either above/below the line (resp. inside/outside the circle). The two-page crossing number $\nu_2(G)$ corresponds to two-page drawings of G.

Example: the complete graph K_5



Equivalent two-page drawings of K_5 with $\nu_2(K_5) = 1$ crossing.

Applications and complexity

- Crossing numbers are of interest for graph visualization, VLSI design, quantum dot cellular automata, ...
- It is NP-hard to compute $\mathrm{cr}(G)$ or $\nu_2(G)$ [Garey-Johnson (1982), Masuda et al. (1987)];
- The (two-page) crossing numbers of K_n and $K_{n,m}$ are only known for some special cases ...
- Crossing number of $K_{n,m}$ known as Turán brickyard problem — posed by Paul Turán in the 1940's.

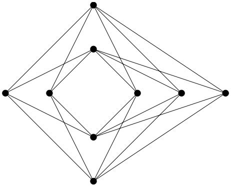
Erdös and Guy (1973):

"Almost all questions that one can ask about crossing numbers remain unsolved."

The Zarankiewicz conjecture

$K_{m,n}$ can be drawn in the plane with at most $Z(m, n)$ edges crossing, where

$$Z(m,n) = \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor.$$



A drawing of $K_{4,5}$ with $Z(4,5) = 8$ crossings.

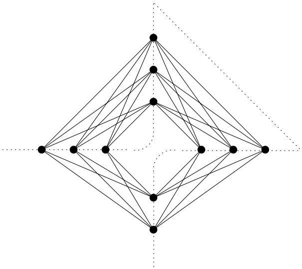
Zarankiewicz conjecture (1954)

$$\mathrm{cr}(K_{m,n}) \stackrel{?}{=} Z(m,n).$$

Known to be true for $\min\{m,n\} \le 6$ (Kleitman, 1970), and some special cases.

The 2-page Zarankiewicz conjecture

The Zarankiewicz drawing may be mapped to a 2-page drawing:



"Straighten the dotted line".

2-page Zarankiewicz conjecture

$$\nu_2(K_{m,n}) \stackrel{?}{=} Z(m,n).$$

Weaker conjecture since $\mathrm{cr}(G) \leq \nu_2(G)$.

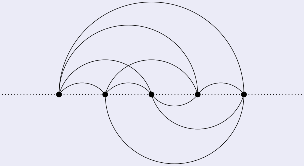
The (2-page) Harary-Hill conjecture

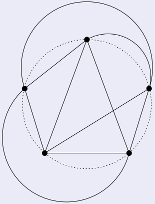
Conjecture (Harary-Hill (1963))

$$\mathrm{cr}(K_n) \stackrel{?}{=} \nu_2(K_n) \stackrel{?}{=} Z(n) := \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

NB: it is only known that $\mathrm{cr}(K_n) \le \nu_2(K_n) \le Z(n)$ in general.

Example: the complete graph K_5





Optimal two-page drawings of K_5 with $Z(5) = 1$ crossing.

Some known results

Theorem (De Klerk, Pasechnik, Schrijver (2007))

One has

$$1 \geq \lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{Z(n)} \geq 0.8594, \quad 1 \geq \lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq 0.8594 \textit{ if } m \geq 9,$$

Theorem (Pan and Richter (2007), Buchheim and Zheng (2007))

$$\mathrm{cr}(K_n) = Z(n) \quad \textit{if } n \leq 12, \quad \nu_2(K_n) = Z(n) \quad \textit{if } n \leq 14.$$

New results (this talk)

Theorem (De Klerk and Pasechnik (2011))

For the complete graph K_n, one has

$$1 \geq \lim_{n\to\infty} \frac{\nu_2(K_n)}{Z(n)} \geq 0.9253$$

and

$$\nu_2(K_n) = Z(n) \quad \text{if } n \leq 18 \text{ or } n \in \{20, 22\}.$$

For the complete bipartite graph $K_{m,n}$, one has

$$\lim_{n\to\infty} \frac{\nu_2(K_{m,n})}{Z(m,n)} = 1 \text{ if } m \in \{7, 8\}.$$

New results: outline of the proofs

For K_n:

- The problem of computing $\nu_2(K_n)$ has a formulation as a maximum cut problem (Buchheim and Zheng (2007));
- The new results for $\nu_2(K_n)$ follow by computing the Goemans-Williamson maximum cut bound for $n = 899$.
- The Goemans-Williamson bound is computed using semidefinite programming (SDP) software and using algebraic symmetry reduction.

For $K_{m,n}$:

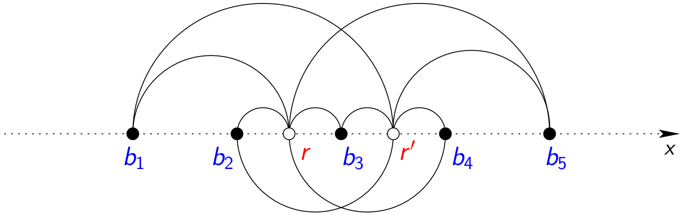
- We will formulate a (nonconvex) quadratic programming (QP) lower bound on $\nu_2(K_{m,n})$.
- Subsequently we compute an SDP lower bound on the QP bound for $m = 7$, again using algebraic symmetry reduction.

Drawings of $K_{m,n}$

Consider a drawing of $K_{m,n}$ with the n coclique colored red, and the m coclique blue.

Definition

Each red vertex r has a position $p(r) \in \{1, \ldots, m\}$ in the drawing, and a set of incident edges $U(r) \subseteq \{1, \ldots, m\}$ drawn in the upper half plane. We say r is of the type $(p(r), U(r))$. The set of all possible types is denoted by $\text{Types}(m)$, i.e. $|\text{Types}(m)| = m2^m$.



In the figure, r has type $(p(r), U(r)) = (2, \{1, 2, 3, 5\})$.

A quadratic programming relaxation of $\nu_2(K_{m,n})$

We define a $m2^m \times m2^m$ matrix Q with rows/colums indexed by Types(m).

Definition

Let $\sigma, \tau \in$ Types(m). Define $Q_{\tau,\sigma}$ as the number of unavoidable edge crossings in a 2-page drawing of $K_{2,m}$, where the vertices from the 2-coclique have type σ and τ respectively in the drawing.

Lemma

$$\nu_2(K_{m,n}) \geq \frac{n^2}{2}\left(\min_{x\in\Delta} x^T Qx\right) - \frac{m(m-1)n}{4}$$

where $\Delta = \left\{x \in \mathbb{R}^{m2^m} \;\middle|\; \sum_{\tau\in\mathrm{Types}(m)} x_\tau = 1,\ x_\tau \geq 0\right\}$ *is the standard simplex.*

- x_τ is the fraction of red vertices of type τ.
- This is a nonconvex quadratic program — we use a semidefinite programming relaxation (next slide).

A semidefinite programming relaxation of $\nu_2(K_{m,n})$ (ctd)

Standard semidefinite programming relaxation:

$$\min_{x\in\Delta} x^T Q x \quad \geq \quad \min\left\{\text{trace}(QX) \,\middle|\, \text{trace}(JX) = 1,\ X \succeq 0,\ X \geq 0\right\},$$

where J is the all-ones matrix and $X \geq 0$ means X is entrywise nonnegative.

- We may perform symmetry reduction using the structure of Q ...
- ... namely Q is a block matrix with $2m \times 2m$ circulant blocks (after reordering rows/columns).
- The reduced problem has $2m$ linear matrix inequalities involving $(2^{m-1}) \times (2^{m-1})$ matrices.

Computational results and implications

We could compute the SDP bound for $m = 7$ to obtain

$$\nu_2(K_{7,n}) \geq (9/4)n^2 - (21/2)n = Z(7,n) - O(n).$$

Since $\nu_2(K_{8,n}) \geq 8\nu_2(K_{7,n})/6$, we also get $\nu_2(K_{8,n}) \geq 3n^2 - 14n = Z(8,n) - O(n)$.

Corollary

$$\lim_{n\to\infty} \nu_2(K_{m,n})/Z(m,n) = 1 \text{ for } m = 7 \text{ and } 8.$$

In words, the 2-page Zarankiewicz conjecture is true asymptotically for $m = 7$ and 8.

Conclusion and summary

- We demonstrated improved asymptotic lower bounds on $\nu_2(K_n)$, $\nu_2(K_{7,n})$, and $\nu_2(K_{8,n})$.
- The proofs were computer-assisted, and the main tools were semidefinite programming (SDP) relaxations and symmetry reduction.
- The SDP relaxation was too large to solve for $\nu_2(K_{9,n})$ — challenge for SDP community.
- Preprint available at Optimization Online and arXiv.

And finally ...

Congratulations to Yurii!

Francqui Chair 2012.