Minimizing convex functions over integer points

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Playing with switches to approximate a function

Let $f : [0, T] \to \mathbb{R}$ be a given function *(signal)*
$g : \mathbb{R} \to \mathbb{R}$ be another given function *(filter)*
with $g(t) = 0$ for $t \leq 0$
$\delta := T/n$, where $n \in \mathbb{Z}_{++}$ *(time-step)*

We want to approximate f as well as possible by

$$\hat{f}_x(t) := \sum_{i=1}^{n} x_i g(t - (i-1)\delta), \text{ with } x_i \in \{0, 1\}.$$

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  $\Leftrightarrow \min\{x^T Q x + 2c^T x + d : x \in \{0, 1\}^n\}$

$Q_{ij} = \int_0^T g(t - (i-1)\delta) g(t - (j-1)\delta) dt$, $c_i = \int_0^T g(t - (i-1)\delta) f(t) dt$

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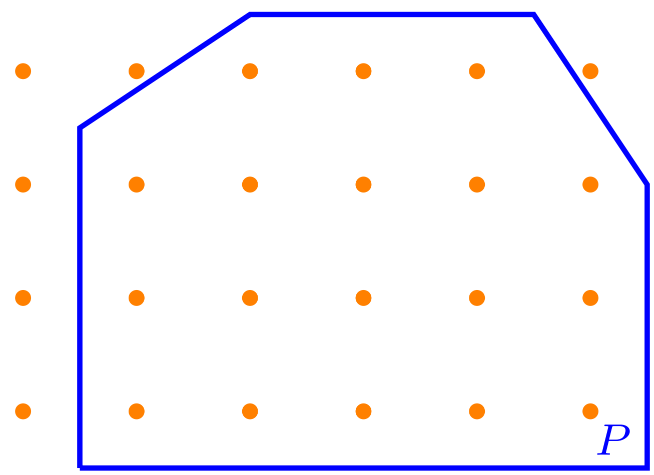
  $\Leftrightarrow \min\{x^T Q x + 2c^T x + d : x \in \{0,1\}^n\}$
- We have a convex objective with binary variables.

Our problem is very hard

$$f(x^*) := \min_{x\in\mathcal{F}} f(x)$$

- $\star$ $\mathcal{F} = P \cap \mathbb{Z}^n$, where P is a polytope;
- $\star$ $f : \mathbb{R}^n \to \mathbb{R}$ is convex.



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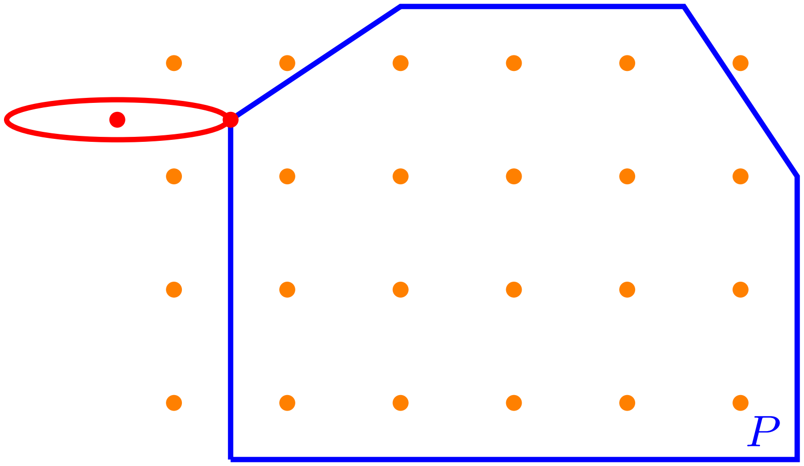

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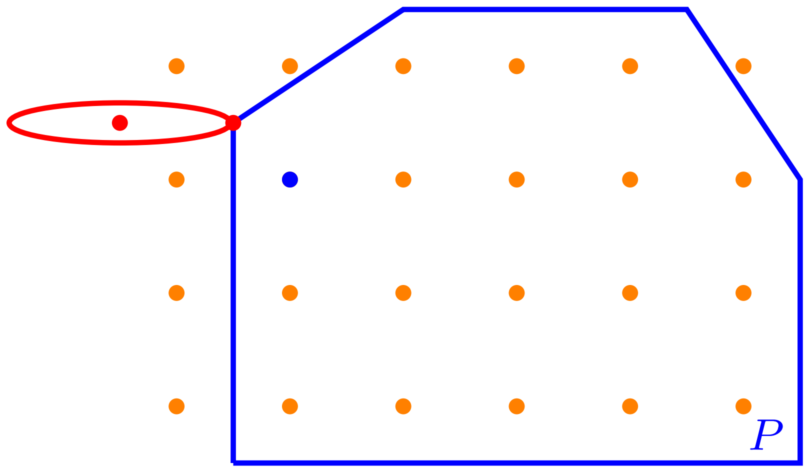

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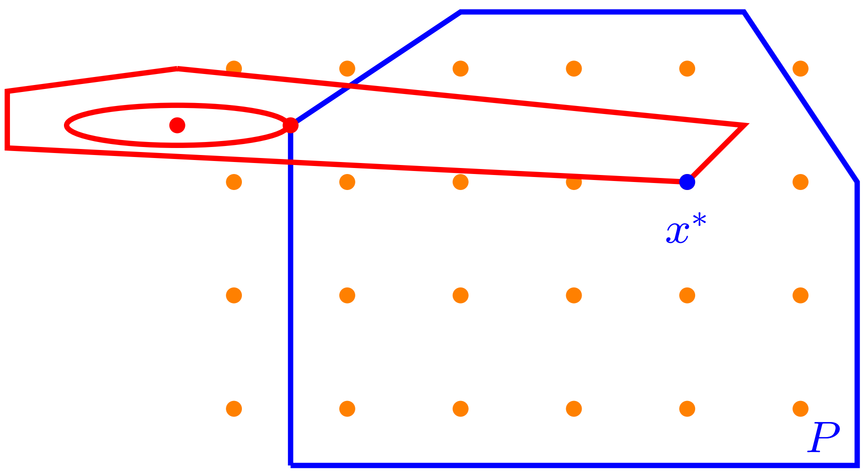

x^* can be arbitrarily far from the continuous minimum

A way to measure how hard our problem is

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We assume that f is strongly convex and L-smooth:
for every $x, y \in \mathbb{R}^n$:

$$\frac{l}{2}||x-y||_2^2 \leq f(y) - f(x) - f'(x)^T(y-x) \leq \frac{L}{2}||x-y||_2^2.$$

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- L/l bounds the asphericity of level sets.
  A trivial enumeration algorithm would take
  $\mathcal{O}(\min\{(L/l)^n, \text{diameter}(\mathcal{F})^n\})$ evaluations of f.
- In fact, the complexity of our methods depends on $L - l$.

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- ▶ Even with f linear, the problem is **NP-Hard**
- ▶ Even with f quadratic and $\mathcal{F} \subseteq \{0, 1\}^n$, the problem is **hard**, independently of the NP-Conjecture.

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- Even with f quadratic and $\mathcal{F} \subseteq \{0,1\}^n$,
  the problem is **hard**, independently of the NP-Conjecture.
  Consider $n = 4m$, $c \in \{2,3\}$, $\gamma = 5m - 1$,
  $\mathcal{F} \subseteq \{0,1\}^n$ be an independence system, and

$$\min\{f(x) = n^2(c^T x - \gamma)^2 + \mathbf{1}^T x : x \in \mathcal{F}\}.$$

  $\mathcal{F}$ is presented by an LP solver:
  taking as input $v \in \mathbb{R}^n$, returns $\arg\max_{x\in\mathcal{F}} v^T x$.

Note: We have $L/l \leq 9n^3 + 1$.

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Theorem 1 *Any method that solves less than* $\binom{2m}{m+1} \geq 2^m$ *LPs on* $\mathcal{F}$ *fails to find an* $\hat{x}$ *for which* $f(\hat{x}) - f(x^*) \leq n^2 - n$.

Very few methods exist for our problem

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1. **Branch and bound approach** *(Gupta, Ravindran, Leyffer)*
   Branching by fixing some variables.
   Lower bounds from continuous relaxation
   of all the other ones, with possibly extra cuts

2. **Outer Approximation** *(Duran, Grossman)*
   (assumes that we can solve the problem with f linear)
   f is replaced by a piecewise linear model,
   which is minimized in $\mathcal{F}$, then enriched.

Our methods use simple ideas from Convex Optimization

$$f(x^*) := \min_{x \in Q} f(x)$$

- $\star$ $Q \subseteq \mathbb{R}^n$ is convex and closed;
- $\star$ $f : \mathbb{R}^n \to \mathbb{R}$ is convex.

Subgradient method:

Select $x_0 \in Q$

for $k = 0, 1, 2, \ldots$

Choose $h_k > 0$, compute $f'(x_k) \in \partial f(x_k)$

Set $x_{k+1} := \pi_Q(x_k - h_k f'(x_k))$

$$= \arg\min_{y \in Q} \left\{ f'(x_k)(y - x_k) + \frac{1}{2h_k} ||y - x_k||_2^2 \right\}.$$

- Guaranteed decrease with $h_k = 1/L$, theoretically the best choice.

A substitute for the subgradient step

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Gradient method:

$$x_{k+1} := \arg\min_{y \in Q} \left\{ f'(x_k)(y - x_k) + \frac{L}{2} ||y - x_k||_2^2 \right\}.$$

Our strategy:

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▶ We allow for a step-size τ larger than $1/L$.

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- We allow for a step-size τ larger than $1/L$.
- This quadratic problem can be very hard to solve exactly
- $x_{k+1} \in \mathcal{F}$ and $g_{x_k}(x_{k+1}) \leq (1 - \alpha) \min_{y \in \mathcal{F}} g_{x_k}(y) < 0$,

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If we don't move anymore, we are done

Select $x_0 \in \mathcal{F}$, $0 \leq \alpha < 1$, $l \leq \tau \leq L$, $N \in \mathbb{Z}_+$
for $k = 0, 1, \ldots, N-1$,
  Compute x_{k+1} such that
    $g_{x_k}(x_{k+1}) \leq (1-\alpha) \arg\min_{y \in \mathcal{F}} g_{x_k}(y)$
  if $x_{k+1} \in \{x_0, \ldots, x_k\}$
    STOP
return the best $\widehat{x} \in \{x_0, \ldots, x_N\}$.

Theorem 2 *If $\tau = 1/l$ and $x_{k+1} = x_k$, then $f(x_k) = f(x^*)$.*

Our method can solve the problem approximately

Select $x_0 \in \mathcal{F}$, $0 \leq \alpha < 1$, $1/L \leq \tau \leq 1/l$, $N \in \mathbb{Z}_+$
for $k = 0, 1, \ldots, N-1$,
  Compute x_{k+1} such that
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    STOP
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Theorem 3 *If* $\alpha > 0$, $\eta > 0$, $\delta_{\mathcal{F}} =$ *diameter*$(\mathcal{F})$ *and*

$$N := \left\lceil \frac{1}{\ln(1/\alpha)} \ln\left(\max\left\{1, \frac{f(x_0) - f(x^*)}{\eta}\right\}\right)\right\rceil,$$

then $f(\widehat{x}) - f(x^*) \leq \dfrac{L-l}{2(1-\alpha)}\delta_{\mathcal{F}}^2 + \eta.$

The subproblem can sometimes be solved

We know: quadratic problems on $\mathcal{F}$ can be **hard**

Theorem 4 (Heinz) *We can solve the subproblem exactly ($\alpha = 0$) in $s^{\mathcal{O}(1)} 2^{\mathcal{O}(n)}$ arithmetic operations, where s is its binary encoding length.*

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- If $\mathcal{F} \subseteq \{0, 1\}^n$, the subproblem is linear (as $x_i^2 = x_i$). We can solve it efficiently for classes of problems where IP is easy (e.g. matroid problems, matchings)
- Many quadratic problems admit a fast approximate method with guaranteed α. (e.g. binary knapsack problems)

A delicate issue: avoiding cycling

Guarantee: $f(\widehat{x}) - f(x^*) \le \frac{L-l}{1-\alpha}\delta_{\mathcal{F}}^2$
after a known number of steps.

Let's keep it running a bit more, hoping for the best.

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Remedy: Assume $f(x) \in \mathbb{Z}$ for $x \in \mathcal{F}$ and set $\tau = 1/l$.
Instead of computing $\min\{g_{x_k}(x) : x \in \mathcal{F}\}$, do:

$$\min\{g_{x_k}(x) : g_{x_i}(x) \leq -1 \text{ for } 0 \leq i \leq k,\ x \in \mathcal{F}\}$$

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- Computes x^* in $\mathcal{O}(c_f(L-l)\delta_{\mathcal{F}}^2 s^{\mathcal{O}(1)})$,

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  where $c_f = \max_{\alpha \in \mathbb{Z}} |\mathcal{F} \cap \{x : f(x) = \alpha\}|$.
- Modest, but can be better than an enumeration ($\mathcal{O}(\delta_{\mathcal{F}}^{n})$).

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- Never generates twice the same point
- Subproblems polynomially solvable
  for some well-structured binary problems
  (e.g. when $\mathcal{F} \subseteq \{0,1\}^n$ is a vectorial matroid)

Conclusion and outlook 1

- The field is almost new.
- Convex integer problems are intrinsically hard.
- Our methods provide polynomial algorithms for instances where quadratic problems can be easily minimized on their feasible set.
- We have complexity and accuracy guarantees when quadratic functions can only be solved approximately.
- The mixed-integer case remains largely unaddressed

Conclusion and outlook 2

- We are developing new methods based on the existence of a *level set oracle*:

  Fix $\alpha \geq 0$, $\delta \geq 0$. For every $x \in \mathbb{R}^n$,

  the oracle finds $\widehat{x} \in \mathbb{Z}^n$

  such that $f(\widehat{x}) \leq (1+\alpha)f(x) + \delta$,

  or declares it does not exists.
- They can be extended to general lattices.
- They allow us to solve 2 dim. problems polynomially in $\ln(L/l)$.
- They seem promising to attack mixed-integer convex problems.

Thank you