Means and medians in nonlinear spaces

Anne Collard and Rodolphe Sepulchre
Departement of Electrical Engineering and Computer Science,
University of Liège, B-4000 Liège, Belgium
Email: Anne.Collard@ulg.ac.be, R.Sepulchre@ulg.ac.be

1 Introduction

Until a few years ago, the majority of numerical techniques for optimization assumed an underlying Euclidean space. However, many of these computational problems are posed on non-Euclidean spaces. This motivates the development of new optimization methods that take into account the particular structure of the considered space. In this work, we focus on those spaces which can be seen as Riemannian manifolds. Some instances of these spaces are the space of rotation and the space of positive semidefinite matrices, for which many applications exist. An example of these applications is the new medical imaging technique called Diffusion Tensor Imaging. Some adapted tools are necessary to use this technique, as for instance methods for approximation, interpolation, filtering and estimation on this manifold. For this space as for others, one of the principal needs is to find robust statistical estimators of Riemannian data [1]. In this work, we present and study some known estimators for different spaces.

2 Statistical estimators

The mean is a natural statistical estimator of a set of data. However, as we can easily check in an Euclidean space, the mean is relatively sensible to outliers. The theory of robust estimation in Euclidean space has led to the development of numerous robust estimators, one of which is the geometric median. Since a (geodesic) distance function is chosen for a manifold, these notions of mean and geometric median on Euclidean space can easily be extended to this manifold.

The computation of these estimators implies to choose a distance function for the considered manifold and to implement an optimization algorithm on this manifold. These estimators are indeed defined as the minimum of a particular function.

Each manifold can be described in different ways. Depending upon the chosen metric, means and medians of a same set of data can be slightly different.

3 Means and medians on Riemannian manifolds

In this work, we consider four different manifolds: the group of rotations, the set of p-dimensional subspaces in $\mathbf{R}^n$, the set of positive definite matrices and the set of positive semidefinite matrices of fixed rank. For each of these, many representations are studied and the resulting differences in the means and medians are analyzed. For illustrating purposes, this work focus on three dimensional spaces.

For the space of 3D-rotations (usually called $SO(3)$), we choose to represent rotations as rotations matrices. Two different distances are considered: the chordal one [2] and the geodesic one.
The space of positive definite matrices is denoted by $S_+(3)$. We use two different representations of this space: the Log-Euclidean one [3] and the affine invariant one [4]. Relations between means computed with different metrics are studied.
The Grassman manifold is the set of p-dimensional linear subspaces of $\mathbf{R}^n$ [5]. The representation used in this work enables us to compute the mean of a set of elements of this manifold using a Newton algorithm.
The space of flat ellipsoids (positive semidefinite matrices of fixed rank) is the less studied of these manifolds. The mean between two flat ellipsoids can be computed by combining results of $S_+(3)$ with results of Grassman manifolds [6].

For each computed estimators, the methods are compared in respect to the robustness to outliers and to the computational cost of the methods.

References

[1] P.T Fletcher *et al*, *The geometric median on Riemannian manifolds with application to robust atlas estimation*, NeuroImage **45**, S143-S152 (2009)

[2] A. Sarlette, *Geometry and Symmetries in Coordination Control*, PhD Thesis, ULg (2009)

[3] V. Arsigny *et al*, *Geometric means in a novel vector space structure on symmetric positive-definite matrices*, SIAM J. on Matrix Anal. and Appl. **29**, 328-347 (2007)

[4] X. Pennec *et al*, *A Riemannian framework for tensor computing*, Internation Journal of Computer Vision **66**, 41-66 (2006)

[5] P.-A. Absil *et al*, *Riemannian geometry of Grassman manifolds with a view on algorithmic computation*, Acta Appl. Math. **80**, 199-220 (2004)

[6] S. Bonnabel and R. Sepulchre, *Riemannian metric and geometric mean for positive semidefinite matrices of fixed rank*, SIAM J. on Matrix Anal. and Appl. **31**, 1055-1070 (2009)